Random matrices, operators and analytic functions

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Circular $\beta$-ensemble

Eigenvalues of a uniform $n \times n$ unitary matrix:
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joint density: $\frac{1}{Z_n} \prod_{j<k} |e^{i\lambda_j} - e^{i\lambda_k}|^2$
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Circular $\beta$-ensemble: $\frac{1}{Z_{n,\beta}} \prod_{j < k} |e^{i\lambda_j} - e^{i\lambda_k}|^\beta$
Point process limit

\[ \{ e^{i \lambda_j^{(n)}}, \lambda_j^{(n)} \in (-\pi, \pi] \}, \ n \text{ points on the circle.} \]
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\( \beta = 1, 2, 4: \) determinantal/Pfaffian structure.

Scaling limit is the same as the bulk limit of GOE/GUE/GSE.

Dyson-Gaudin-Mehta

Limit is described via the joint intensities.

Killip-Stoiciu '06

Nakano '14, V-Virág '16: the limit is the same as the bulk limit of the Gaussian \( \beta \)-ensemble, Sine \( \beta \).
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Limit is described via the joint intensities.

**General \( \beta > 0 \) case:** Killip-Stoiciu ’06

Limit is described via the counting function (coupled system of SDEs).

Nakano ’14, V-Virág ’16: the limit is the same as the bulk limit of the Gaussian \( \beta \)-ensemble, \( \text{Sine}_\beta \)
Random operators

Dumitriu-Edelman '02:
tridiagonal representation for Gaussian and Laguerre $\beta$-ensembles

Edelman-Sutton '06:
random tridiagonal matrices $\Rightarrow$ random differential operators

Limit processes: spectra of random differential operators

Soft edge: Ramírez-Rider-Virág '06 (Gaussian, Laguerre)

$$A_\beta = -\frac{d^2}{dx^2} + x + 2\sqrt{\beta} dB$$

Hard edge: Ramírez-Rider '08 (Laguerre)

$$B_\beta, a = -e^{(a+1)x} + 2\sqrt{\beta} B(x) \frac{d}{dx}\left\{ e^{-ax} - 2\sqrt{\beta} B(x) \frac{d}{dx}\right\}$$

$B$: standard Brownian motion, $dB$: white noise

domain: $[0, \infty) \rightarrow \mathbb{R}, L^2$ and boundary conditions
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$$B_{\beta,a} = -e^{(a+1)x + \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \left\{ e^{-ax - \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \right\}$$

$B$: standard Brownian motion, $dB$: white noise
domain: $[0, \infty) \to \mathbb{R}$, $L^2$ and boundary conditions
The Sine$_\beta$ operator - operator in the bulk

There is a self-adjoint differential operator (Dirac-operator) $\tau$: $f \rightarrow 2\mathbb{R} - 1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t)$, $f: [0, 1) \rightarrow \mathbb{R}^2$. With spectrum given by the Sine$_\beta$ process. $R_t: [0, 1) \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$ is a simple function of a hyperbolic Brownian motion. Also: Several finite classical random matrix models, $\beta$-generalizations and scaling limits can be represented in this form.
The $\text{Sine}_\beta$ operator - operator in the bulk

**Thm** (V-Virág '16):
There is a self-adjoint differential operator (Dirac-operator)

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**Also:** Several finite classical random matrix models, $\beta$-generalizations and scaling limits can be represented in this form.
Dirac operators

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\( R_t \): positive definite matrix valued function
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**Ingredients:** a path \(x_t + iy_t : [0, 1) \rightarrow \mathbb{H}\) in the hyperbolic plane, two boundary points in \(\mathbb{H}\).

\[X_t = \frac{1}{\sqrt{y_t}} \begin{bmatrix} 1 & -x_t \\ 0 & y_t \end{bmatrix}, \quad R_t = X_t^T X_t.\]
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**Domain:** differentiability, \( L^2 \) and boundary conditions
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Two boundary points \( \sim \) boundary conditions for \( \tau \)
Dirac operators

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\[ R = X_t^T X_t, \quad X_t = \frac{1}{\sqrt{y_t}} \begin{bmatrix} 1 & -x_t \\ 0 & y_t \end{bmatrix}. \]

Claim: if \( x_t + iy_t \) does not converge too fast towards \( \partial \bar{\mathbb{H}} \) then \( \tau \) is a self-adjoint operator on the appropriate domain and its inverse is Hilbert-Schmidt in \( L^2_R \).
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The integral kernel in \( L^2_R \) is

\[ \mathcal{K}(s, t) = u_0 u_1^T \mathbf{1}(s < t) + u_1 u_0^T \mathbf{1}(s \geq t) \]

\( u_0, u_1 \): boundary conditions in \( \tau \)
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Conjugation with \( X^{-1} \): self-adjoint integral operator on \( L^2 \).
Random Dirac operators

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Examples:

- Sine_\beta (time-changed hyperbolic BM in \( \mathbb{H} \))
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- hard edge limits (time-changed BM with drift embedded in \( \mathbb{H} \))
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- limits of certain one dimensional random Schrödinger operators (hyperbolic BM up to a fixed time)
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- limits of certain one dimensional random Schrödinger operators (hyperbolic BM up to a fixed time)
- finite circular \( \beta \)-ensemble and circular Jacobi ensembles (random walk in \( \mathbb{H} \))
Dirac operators for unitary matrices

**Thm:** $V$ is an $n \times n$ unitary matrix with distinct eigenvalues $e^{i\lambda_j}$
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$\leadsto$ Dirac operator with spectrum $\{n\lambda_j + 2\pi kn, k \in \mathbb{Z}\}$
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Apply G-S to $e, Ve, \ldots, V^{n-1}e \rightsquigarrow \Phi_0(z), \ldots, \Phi_{n-1}(z)$ OPUC
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$\Phi_k^*(z) := z^k\overline{\Phi_k(1/\bar{z})}$ conjugate polynomials
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Szegő recursion:
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Szegő recursion:

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\begin{bmatrix}
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\end{bmatrix} =
\begin{bmatrix}
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-\alpha_k & 1
\end{bmatrix}
\begin{bmatrix}
z & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\Phi_k(z) \\
\Phi_k^*(z)
\end{bmatrix},
\begin{bmatrix}
\Phi_0(z) \\
\Phi_0^*(z)
\end{bmatrix} =
\begin{bmatrix}
1 \\
1
\end{bmatrix}
$$

$\alpha_k$: Verblunsky coefficients, $|\alpha_k| < 1$

Can be extended with a final step with $|\alpha_{n-1}| = 1$,

$\Phi_n(z)$: characteristic polynomial
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With \( z = e^{i\lambda/n} \) and a simple transformation of \( \begin{bmatrix} \Phi_k(z) \\ \Phi^*_k(z) \end{bmatrix} \) we can turn the Szegő recursion into the ev equation of a Dirac operator:

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2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t) = \lambda f(t)
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The function \(R_t\) and the corresponding path \(x_t + iy_t\) are constant on each \(\left[\frac{k}{n}, \frac{k+1}{n}\right]\). The path is built from the \(\alpha_k\).
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Similar construction starting from the recursion for \(\frac{\Phi_k(z)}{\Phi_k(1)}\). In that case the path itself satisfies a linear recursion.
Circular ensembles

**Thm** (Killip-Nenciu '04) If $V$ is a uniformly chosen $n \times n$ unitary matrix then the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are independent with nice distributions.
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Similar construction for the $\beta$ generalization.
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∽ Dirac operator representation for the finite circular $\beta$-ensembles ($x + iy$ is a random walk)
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$\rightsquigarrow$ Dirac operator representation for the finite circular $\beta$-ensembles ($x + iy$ is a random walk)

Construction of the hyperbolic RW: $b_0 = i, \ldots, b_{n-1} \in \mathbb{H}, b_n \in \partial \mathbb{H}$

Given $b_k$ we choose $b_{k+1}$ uniformly on a hyperbolic circle with random radius $\xi_k$. In the Poincaré disk with center $b_k$ we have $\xi_k^2 \sim \text{Beta}(1, \frac{\beta}{2}(n - k - 1))$. The last step is chosen uniformly on $\partial \mathbb{H}$ as viewed from $b_{n-1}$. 
Operator level bulk limit

The previous methods required the derivation of a one-parameter family of SDE system. Here we need to understand the limit of the integral kernel (convergence of a RW to a BM).
Operator level bulk limit

finite model
  ↓
  differential operator built from RW
  ↓
  integral operator built from RW
  ↓
  integral operator built from BM
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Operator level bulk limit

**Thm** (V-Virág, ‘17):
One can couple the finite $n$ circular $\beta$-ensembles to $\text{Sine}_\beta$ so that the corresponding operators are within $\log^3 n \cdot n^{-1/2}$ in H-S norm.
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$$\iint_0^1 \iint_0^1 \text{tr}((\mathcal{K} - \mathcal{K}_n)(\mathcal{K} - \mathcal{K}_n)^t)dx \, dy \leq \frac{\log^6 n}{n}.$$
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In this coupling

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\sum_k \left| \frac{1}{\lambda_{k,n}} - \frac{1}{\lambda_k} \right|^2 \leq \frac{\log^6 n}{n}
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Coupling bound for $\beta = 2$: Maples, Najnudel, Nikeghbali ’13
TV bounds on the counting functions ($\beta = 2$): Meckes, Meckes ’16
Limits of characteristic polynomials

Thm (Chhaibi, Najnudel, Nikeghbali '17): Label the points of $\text{Sine}_2$ as $\ldots < \lambda - 1 < \lambda_0 < \lambda_1 < \ldots$ Then

$$\xi(z) := (1 - z \lambda_0) \prod_{k=1}^{\infty} (1 - z \lambda_{-k})(1 - z \lambda_k)$$

defines a random entire function. Moreover, there is a coupling of the finite circular unitary ensembles to $\text{Sine}_2$ so that a.s.

$$p_n(e^{i z_n}) \overset{p_n}{\to} e^{i z_2} \cdot \xi(z)$$

$p_n$: characteristic polynomial of the size $n$ ensemble.
Limits of characteristic polynomials

**Thm** (Chhaibi, Najnudel, Nikeghbali ’17): Label the points of $\text{Sine}_2$ as $\ldots < \lambda_{-1} < \lambda_0 < 0 < \lambda_1 < \ldots$ Then

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**general $\beta$?**
The finite ensemble is just \( n \) equally spaced points on the circle, rotated with a uniform angle. The scaling limit is \( 2\pi \mathbb{Z} + 2\pi U[0, 1] \).
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The limit is $\sin(z/2)$ with a random shift. After normalization:

$$\cos(z/2) + q \sin(z/2), \quad q \sim \text{Cauchy}$$

Aizenmann-Warzel ‘15: On the ubiquity of the Cauchy distribution in spectral problems
Entire function from the random operator

\[ \tau : f \rightarrow 2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \quad f : [0, 1) \rightarrow \mathbb{R}^2. \]
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\[ \sum_k \frac{1}{\lambda_k^2} < \infty \text{ holds a.s.} \Rightarrow \det_2(I - z\tau^{-1}) \text{ is well defined} \]

\[ \det_2(I - z\tau^{-1}) = \prod_k (1 - z\lambda_k^{-1})e^{z\lambda_k^{-1}} \]
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\[ \det(I - z\tau^{-1}) = \det_2(I - z\tau^{-1})e^{-z\text{Tr}\tau^{-1}} \]
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In our case $\text{Tr}\tau^{-1}$ is not defined, but the principal value sum exists:

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**Thm(V., Virág):** The scaling limit of the normalized characteristic polynomials for circular \( \beta \)-ensembles is given by

\[ e^{i\frac{z}{2}} \cdot \det_2(I-z\tau^{-1})e^{-z\cdot"\text{Tr} \tau^{-1}"} \]
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$$f(0) \parallel u_0, 'f(1) \parallel u_1'.$$

For ‘nice’ $R_t$: $z$ is an e.v. if the solution of the ‘shooting problem’

$$\partial_t f(t, z) = -\frac{z}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R_t f(t, z), \quad f(0, z) = u_0$$

satisfies $f(1, z) \parallel u_1$. 
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\[ \tau : f \rightarrow 2R_{t}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \quad f : [0, 1) \rightarrow \mathbb{R}^2. \]

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Thus \( g(z) = u_1^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f(1, z) \) would give an appropriate function.
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\( \rightsquigarrow \) another characterization of the limiting analytic function.
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The resulting function can be written as $A + qB$ where $A, B$ are random entire functions that are real on $\mathbb{R}$, and $q$ is an independent Cauchy.
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This is the analogue of the $\beta = \infty$ case! $A$ and $B$ for general $\beta$ are the ‘randomized’ versions of $\cos$ and $\sin$. 

\[ \frac{dE_t}{dt} = -i\beta \gamma z E_t(z) ds - \beta \lambda z \partial_z E_t(z) ds + \bar{E}_t(\bar{z}) - E_t(z) z dW, \]
\[ E_0(z) = 1 \]

$W$ is a complex BM.

The SDE system for $\partial_n z E_t(0)$, $n = 1, 2, ...$ can be solved explicitly.
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Using the scale invariance of the hyperbolic BM we can find an SPDE so that its stationary solution is $E = A - iB$:

$$dE_t = -i\frac{\beta}{8}zE_t(z)ds - \frac{\beta}{4}z\partial_z E_t(z)ds + \frac{\bar{E}_t(\bar{z}) - E_t(z)}{2i}dW, \quad E_0(z) = 1$$

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Moments of products of ratios

Borodin-Strahov ‘06: Limit of $E \left[ \prod_{j=1}^{k} \frac{\tilde{p}_n(z_j)}{\tilde{p}_n(w_j)} \right]$ for various classical random matrix models.  
($z_j, w_j \in \mathbb{C}$, $k$ fixed, $\tilde{p}_n$ is the scaled ch. polynomial in the ‘bulk’)
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If $\text{Im } w_j < 0$ for all $j = 1, \ldots, k$ then the limit simplifies to

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Q: Is this true for all $\beta > 0$?
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Thm(V.-Virág): The conjectured moment formula for \( \text{Im } w_j < 0 \) holds for the limiting analytic function for all \( \beta > 0 \).
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Outline: In the \( \text{Im} \, w_j < 0 \) case \( E \left[ \prod_{j=1}^{k} \frac{A(z_j)+qB(z_j)}{A(w_j)+qB(w_j)} \right] \) can be expressed using \( A - iB \). The expectation can now be evaluated using the SPDE representation for \( A - iB \).