Random matrices, operators and analytic functions

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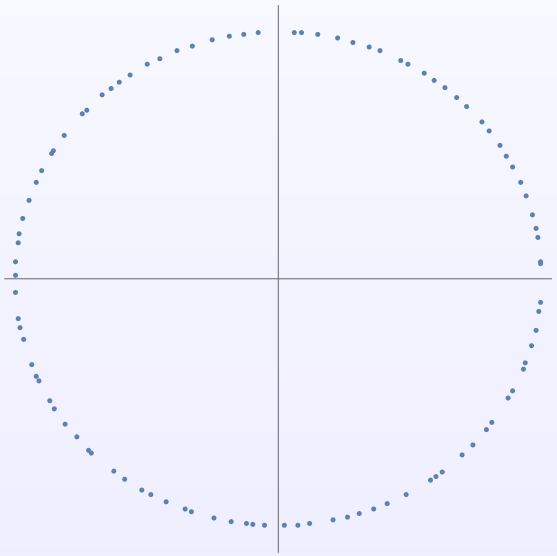
Circular β-ensemble

Eigenvalues of a uniform $n \times n$ unitary matrix:



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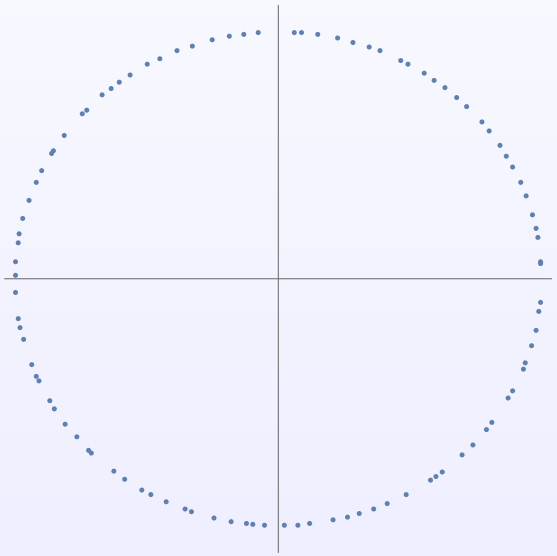

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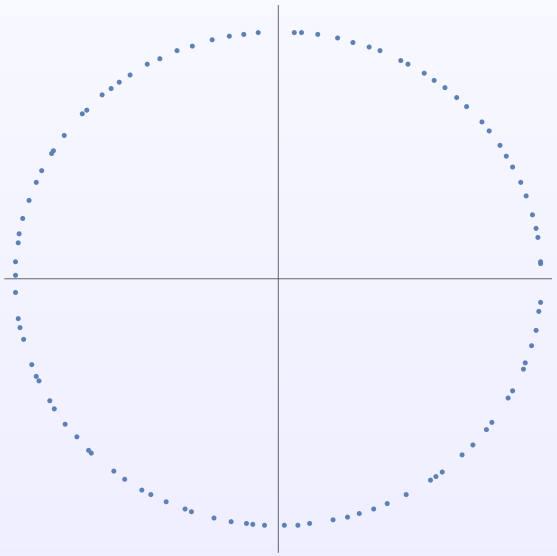

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Circular β-ensemble: $\frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\lambda_j} - e^{i\lambda_k}|^\beta$

Point process limit

$\{e^{i\lambda_j^{(n)}}, \lambda_j^{(n)} \in (-\pi, \pi]\}$, n points on the circle.

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Scaling limit is the same as the bulk limit of GOE/GUE/GSE.
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Limit is described via the joint intensities.

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General $\beta > 0$ case: Killip-Stoiciu '06
Limit is described via the counting function (coupled system of SDEs).

Nakano '14, V-Virág '16: the limit is the same as the bulk limit of the Gaussian β-ensemble, Sine_β

Random operators

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Dumitriu-Edelman '02:
tridiagonal representation for Gaussian and Laguerre β-ensembles

Edelman-Sutton '06:
random tridiagonal matrices $\rightsquigarrow$ random differential operators

Limit processes: spectra of random differential operators

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Soft edge: Ramírez-Rider-Virág '06 (Gaussian, Laguerre)

$$\mathcal{A}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} dB$$

Hard edge: Ramírez-Rider '08 (Laguerre)

$$\mathcal{B}_{\beta,a} = -e^{(a+1)x + \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \left\{ e^{-ax - \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \right\}$$

B: standard Brownian motion, dB: white noise

domain: $[0, \infty) \to \mathbb{R}$, L^2 and boundary conditions

The Sine$_\beta$ operator - operator in the bulk

The Sine_β operator - operator in the bulk

Thm (V-Virág '16):
There is a self-adjoint differential operator (Dirac-operator)

$$\tau : f \to 2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \qquad f : [0,1) \to \mathbb{R}^2.$$

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Also: Several finite classical random matrix models, β-generalizations and scaling limits can be represented in this form.

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Ingredients: a path $x_t + iy_t : [0,1) \to \mathbb{H}$ in the hyperbolic plane, two boundary points in $\mathbb{H}$.

$$X_t = \frac{1}{\sqrt{y_t}} \begin{bmatrix} 1 & -x_t \\ 0 & y_t \end{bmatrix}, \qquad R_t = X_t^T X_t.$$

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Two boundary points $\rightsquigarrow$ boundary conditions for τ

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The integral kernel in L^2_R is

$$\mathcal{K}(s,t) = u_0 u_1^T \mathbf{1}(s < t) + u_1 u_0^T \mathbf{1}(s \geq t)$$

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Conjugation with X^{-1}: self-adjoint integral operator on L^2.

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- limits of certain one dimensional random Schrödinger operators (hyperbolic BM up to a fixed time)
- finite circular β-ensemble and circular Jacobi ensembles (random walk in $\mathbb{H}$)

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α_k: Verblunsky coefficients, $|\alpha_k| < 1$

Can be extended with a final step with $|\alpha_{n-1}| = 1$,

$\Phi_n(z)$: characteristic polynomial

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With $z = e^{i\lambda/n}$ and a simple transformation of $\begin{bmatrix} \Phi_k(z) \\ \Phi^*_k(z) \end{bmatrix}$ we can turn the Szegő recursion into the ev equation of a Dirac operator:

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Similar construction starting from the recursion for $\frac{\Phi_k(z)}{\Phi_k(1)}$. In that case the path itself satisfies a linear recursion.

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Thm(Killip-Nenciu '04) If V is a uniformly chosen $n \times n$ unitary matrix then the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are independent with nice distributions.

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$\rightsquigarrow$ Dirac operator representation for the finite circular β-ensembles ($x + iy$ is a random walk)

Construction of the hyperbolic RW: $b_0 = i, \ldots, b_{n-1} \in \mathbb{H}$, $b_n \in \partial\mathbb{H}$

Given b_k we choose b_{k+1} uniformly on a hyperbolic circle with random radius ξ_k. In the Poincaré disk with center b_k we have $\xi_k^2 \sim \text{Beta}(1, \frac{\beta}{2}(n-k-1))$. The last step is chosen uniformly on $\partial\mathbb{H}$ as viewed from b_{n-1}.

Operator level bulk limit

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finite model

$\downarrow$

differential operator built from RW

$\downarrow$

integral operator built from RW

$\downarrow$

integral operator built from BM

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The previous methods required the derivation of a one-parameter family of SDE system.

Here we need to understand the limit of the integral kernel (convergence of a RW to a BM)

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Thm (V-Virág, '17):
One can couple the finite n circular β-ensembles to Sine_β so that the corresponding operators are within $\log^3 n \cdot n^{-1/2}$ in H-S norm.

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$$\sum_k \left| \frac{1}{\lambda_{k,n}} - \frac{1}{\lambda_k} \right|^2 \leq \frac{\log^6 n}{n}$$

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Coupling bound for $\beta = 2$: Maples, Najnudel, Nikeghbali '13
TV bounds on the counting functions ($\beta = 2$): Meckes, Meckes '16

Limits of characteristic polynomials

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Thm(Chhaibi, Najnudel, Nikeghbali '17): Label the points of Sine_2 as $\ldots < \lambda_{-1} < \lambda_0 < 0 < \lambda_1 < \ldots$ Then

$$\xi(z) := (1 - \tfrac{z}{\lambda_0}) \prod_{k=1}^{\infty} \left(1 - \tfrac{z}{\lambda_{-k}}\right) \left(1 - \tfrac{z}{\lambda_k}\right)$$

defines a random entire function.

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Moreover, there is a coupling of the finite circular unitary ensembles to Sine_2 so that a.s.

$$\frac{p_n\left(e^{i\frac{z}{n}}\right)}{p_n(1)} \to e^{i\frac{z}{2}} \cdot \xi(z)$$

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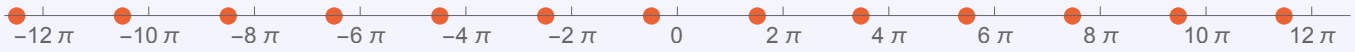
general β?

$\beta = \infty$ case

The finite ensemble is just n equally spaced points on the circle, rotated with a uniform angle. The scaling limit is $2\pi\mathbb{Z} + 2\pi U[0,1]$.



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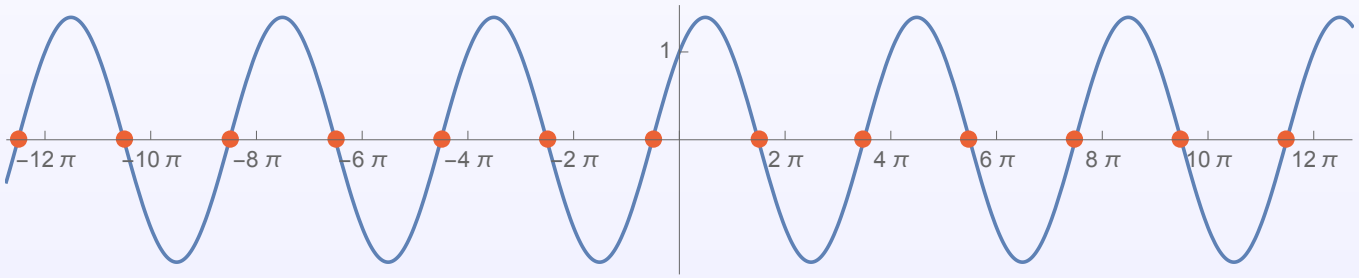

The limit is $\sin(z/2)$ with a random shift. After normalization:

$$\cos(z/2) + q\sin(z/2), \qquad q \sim \text{Cauchy}$$

Aizenmann-Warzel '15: On the ubiquity of the Cauchy distribution in spectral problems

Entire function from the random operator

$$\tau : f \to 2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \qquad f : [0,1) \to \mathbb{R}^2.$$

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$\sum_k \frac{1}{\lambda_k^2} < \infty$ holds a.s.$\rightsquigarrow$ $\det_2(I - z\tau^{-1})$ is well defined

$$\det_2(I - z\tau^{-1}) = \prod_k (1 - z\lambda_k^{-1}) e^{z\lambda_k^{-1}}$$

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Thm(V., Virág): The scaling limit of the normalized characteristic polynomials for circular β-ensembles is given by

$$e^{i\frac{z}{2}} \cdot \det_2(I - z\tau^{-1})e^{-z\cdot\text{"}\operatorname{Tr} \tau^{-1}\text{"}}$$

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For 'nice' R_t: z is an e.v. if the solution of the 'shooting problem'

$$\partial_t f(t,z) = -\frac{z}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R_t f(t,z), \qquad f(0,z) = u_0$$

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$\rightsquigarrow$ another characterization of the limiting analytic function.

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The resulting function can be written as $A + qB$ where

A, B are random entire functions that are real on $\mathbb{R}$, and q is an independent Cauchy.

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Using the scale invariance of the hyperbolic BM we can find an SPDE so that its stationary solution is $E = A - iB$:

$$dE_t = -i\frac{\beta}{8} z E_t(z) ds - \frac{\beta}{4} z \partial_z E_t(z) ds + \frac{\bar{E}_t(\bar{z}) - E_t(z)}{2i} dW, \quad E_0(z) = 1$$

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The SDE system for $\partial_z^n E_t(0), n = 1, 2, \ldots$ can be solved explicitly.

Moments of products of ratios

Borodin-Strahov '06: Limit of $E\left[\prod_{j=1}^{k} \frac{\tilde{p}_n(z_j)}{\tilde{p}_n(w_j)}\right]$ for various classical random matrix models.
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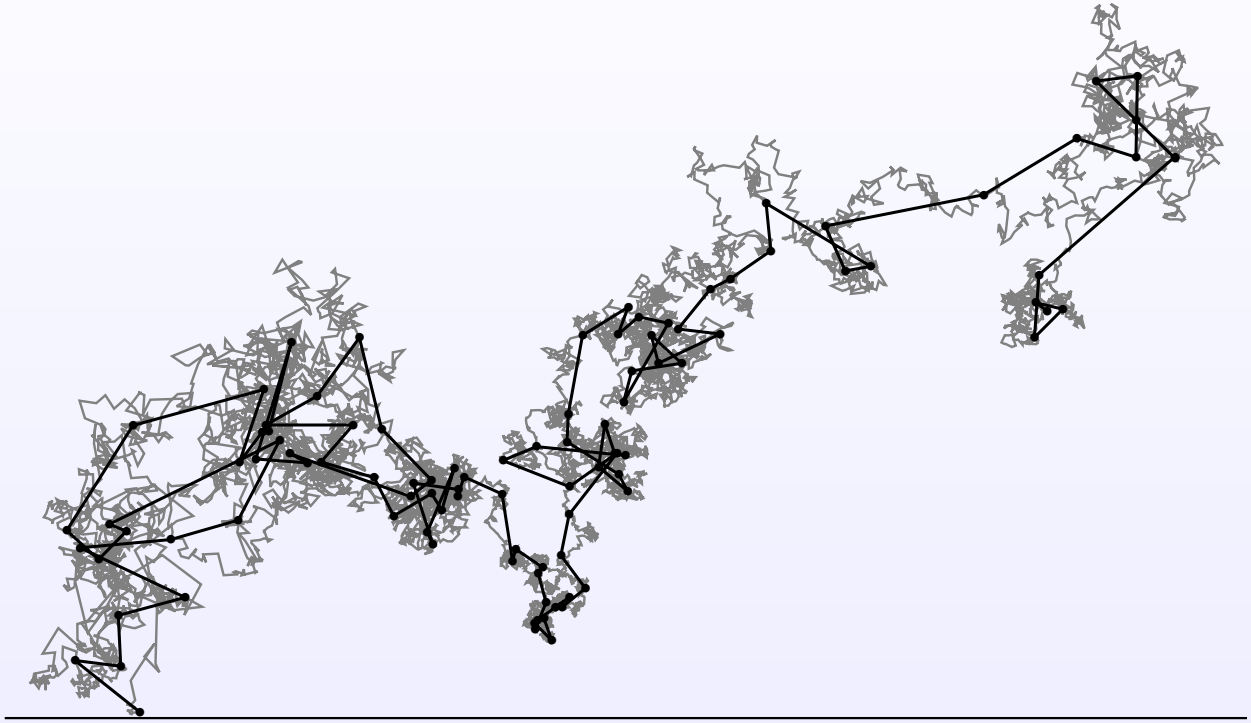
Outline: In the $\operatorname{Im} w_j < 0$ case $E\left[\prod_{j=1}^{k} \frac{A(z_j)+qB(z_j)}{A(w_j)+qB(w_j)}\right]$ can be expressed using $A - iB$. The expectation can now be evaluated using the SPDE representation for $A - iB$.



THANK YOU!