Quantum Diffusion in Fluctuating Media

Jeffrey Schenker

Supported by NSF grant DMS-1500386

CRM, Montréal
November 15, 2018

Joint work with Zak Tilocco and Shiwen Zhang

MICHIGAN STATE
UNIVERSITY

Outline

MICHIGAN STATE UNIVERSITY

Quantum Diffusion Conjecture

MICHIGAN STATE UNIVERSITY

Consider the Schrödinger equation

$$\mathrm{i}\partial_t \psi(x) = H_\omega \psi_t(x)$$

where H_ω is an RSO with a spatially homogeneous distribution. Under suitable hypotheses,[a] *a diffusive rescaling of the mean wave density $\mathbb{E}\left(|\psi_t(x)|^2\right)$ should approximately solve a heat equation at large time:*

$$\mathbb{E}\left(|\psi_{Tt}(\sqrt{T}x)|^2\right) \sim \int_{\sigma(H_\omega)} \frac{1}{(D(E)t)^{d/2}} e^{-\frac{1}{4D(E)t}|x|^2} \mathrm{d}\nu(E), \quad T \to \infty$$

with $D(E) > 0$ in the center of the band —$D(E) = 0$ at the edges due to Lifshitz tails localization.

[a] Weak disorder, dimension ≥ 3, ...

Diffusion produced by waving hands

MICHIGAN STATE UNIVERSITY

Consider $H_\omega = H_0 + \lambda V_\omega$, with H_0 giving ballistic propagation and V_ω a random "perturbation."

Diffusion produced by waving hands

MICHIGAN STATE UNIVERSITY

Consider $H_\omega = H_0 + \lambda V_\omega$, with H_0 giving ballistic propagation and V_ω a random "perturbation."

- In a multiple scattering picture

$$e^{-\mathrm{i}tH_\omega} = e^{-\mathrm{i}tH_0} - \mathrm{i}\lambda \int_0^t e^{-\mathrm{i}(t-s)H_0} V_\omega e^{-\mathrm{i}sH_0} ds + \cdots$$

  suggests a build up of random phases.

Diffusion produced by waving hands

MICHIGAN STATE UNIVERSITY

Consider $H_\omega = H_0 + \lambda V_\omega$, with H_0 giving ballistic propagation and V_ω a random "perturbation."

- In a multiple scattering picture

$$e^{-\mathrm{i}tH_\omega} = e^{-\mathrm{i}tH_0} - \mathrm{i}\lambda \int_0^t e^{-\mathrm{i}(t-s)H_0} V_\omega e^{-\mathrm{i}sH_0} ds + \cdots$$

  suggests a build up of random phases.
- build up of random phases $\implies$ loss of coherence

Diffusion produced by waving hands

MICHIGAN STATE UNIVERSITY

Consider $H_\omega = H_0 + \lambda V_\omega$, with H_0 giving ballistic propagation and V_ω a random "perturbation."

- In a multiple scattering picture

$$e^{-\mathrm{i}tH_\omega} = e^{-\mathrm{i}tH_0} - \mathrm{i}\lambda \int_0^t e^{-\mathrm{i}(t-s)H_0} V_\omega e^{-\mathrm{i}sH_0} ds + \cdots$$

  suggests a build up of random phases.
- build up of random phases $\implies$ loss of coherence
- loss of coherence $\implies$ interference effects can be ignored: densities from different scattering paths are additive, i.e., "classical" propagation.

Diffusion produced by waving hands

MICHIGAN STATE UNIVERSITY

Consider $H_\omega = H_0 + \lambda V_\omega$, with H_0 giving ballistic propagation and V_ω a random "perturbation."

- In a multiple scattering picture

$$e^{-\mathrm{i}tH_\omega} = e^{-\mathrm{i}tH_0} - \mathrm{i}\lambda \int_0^t e^{-\mathrm{i}(t-s)H_0} V_\omega e^{-\mathrm{i}sH_0} ds + \cdots$$

  suggests a build up of random phases.
- build up of random phases $\implies$ loss of coherence
- loss of coherence $\implies$ interference effects can be ignored: densities from different scattering paths are additive, i.e., "classical" propagation.
- classical random scattering $\implies$ diffusion (CLT?)

Problems

MICHIGAN STATE UNIVERSITY

There are mathematical difficulties with every step of the above argument. Despite experience and a rich physical theory, we are far from a good mathematical understanding of this phenomenon.

Problems

MICHIGAN STATE UNIVERSITY

There are mathematical difficulties with every step of the above argument. Despite experience and a rich physical theory, we are far from a good mathematical understanding of this phenomenon.

In particular, *recurrence* is a huge problem. And in $1D$ —and maybe $2D$ —it is an essential obstacle.

Problems

MICHIGAN STATE UNIVERSITY

There are mathematical difficulties with every step of the above argument. Despite experience and a rich physical theory, we are far from a good mathematical understanding of this phenomenon.

In particular, *recurrence* is a huge problem. And in $1D$ —and maybe $2D$ —it is an essential obstacle.

Best results so far: Erdös, Salmhofer, Yau, proved diffusion in a limit with the "disorder strength" $\lambda \propto T^{-\frac{1}{2}+\epsilon}$.

Outline

MICHIGAN STATE UNIVERSITY

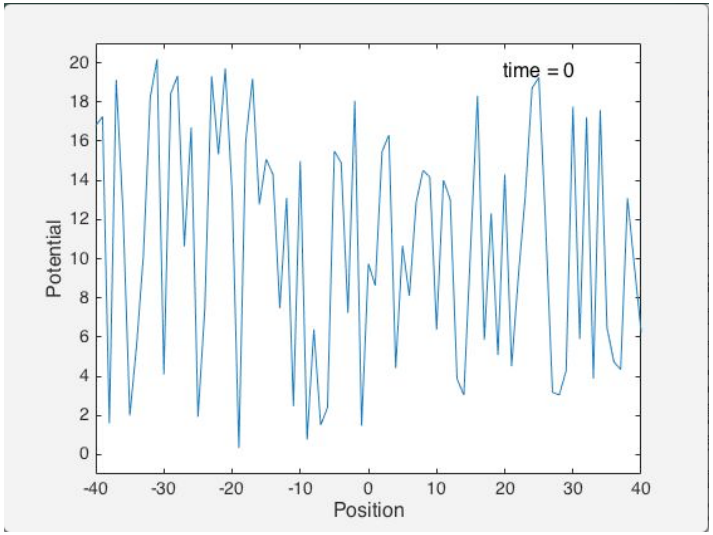
What if the potential fluctuates?

MICHIGAN STATE UNIVERSITY

What are the long time dynamics for solutions to

$$\partial_t U(t, t_0) \;=\; -\mathrm{i} H(t) U(t, t_0),$$

with $U(t_0, t_0) = I$ and $H(t) \;=\; H_0 + g \sum_x w(x,t) \left|x\right\rangle \left\langle x\right|$?



What kind of fluctuations?

MICHIGAN STATE UNIVERSITY

Throughout we take

$$w(x,t) \;=\; w(\tau_x \alpha_t)$$

with

What kind of fluctuations?

MICHIGAN STATE UNIVERSITY

Throughout we take

$$w(x,t) \;=\; w(\tau_x \alpha_t)$$

with

1. $t \mapsto \alpha_t$ a Markov process on a nice space Λ with a unique invariant probability measure ν and a "spectral gap."
   - Take the initial value α_0 distributed according to ν (for simplicity).

What kind of fluctuations?

MICHIGAN STATE UNIVERSITY

Throughout we take

$$w(x,t) \;=\; w(\tau_x \alpha_t)$$

with

1. $t \mapsto \alpha_t$ a Markov process on a nice space Λ with a unique invariant probability measure ν and a "spectral gap."
   - Take the initial value α_0 distributed according to ν (for simplicity).
2. $\{\tau_x\}_{x\in\mathbb{Z}^d}$ an action of $\mathbb{Z}^d$ on Λ as ν-measure preserving maps
   - assume the distribution of the Markov process is invariant under τ_x.

What kind of fluctuations?

MICHIGAN STATE UNIVERSITY

Throughout we take

$$w(x,t) \;=\; w(\tau_x \alpha_t)$$

with

1. $t \mapsto \alpha_t$ a Markov process on a nice space Λ with a unique invariant probability measure ν and a "spectral gap."
   - Take the initial value α_0 distributed according to ν (for simplicity).
2. $\{\tau_x\}_{x\in\mathbb{Z}^d}$ an action of $\mathbb{Z}^d$ on Λ as ν-measure preserving maps
   - assume the distribution of the Markov process is invariant under τ_x.
3. w a non-constant, mean-zero function on Λ such that

$$\lim_{x\to\infty} \int_\Lambda w(\tau_x\alpha) w(\alpha) \nu(\mathrm{d}\alpha) \;=\; 0.$$

Huh?

MICHIGAN STATE UNIVERSITY

If that was too much, think of the following:

Flip process

1. Initially $w(x, 0)$ are i.i.d. ± 1 with equal probability.

Huh?

MICHIGAN STATE UNIVERSITY

If that was too much, think of the following:

Flip process

1. Initially $w(x,0)$ are i.i.d. ± 1 with equal probability.
2. At each site $x \in \mathbb{Z}^d$ we have independent Poisson processes $t_1(x) < t_2(x) < \dots$, with mean waiting time τ.

Huh?

MICHIGAN STATE UNIVERSITY

If that was too much, think of the following:

Flip process

1. Initially $w(x,0)$ are i.i.d. ± 1 with equal probability.
2. At each site $x \in \mathbb{Z}^d$ we have independent Poisson processes $t_1(x) < t_2(x) < \dots$, with mean waiting time τ.
3. $w(x,t)$ is constant for $t \neq t_j(x)$ and flips at times $t = t_j(x)$.

Huh?

MICHIGAN STATE UNIVERSITY

If that was too much, think of the following:

Flip process

1. Initially $w(x,0)$ are i.i.d. ± 1 with equal probability.
2. At each site $x \in \mathbb{Z}^d$ we have independent Poisson processes $t_1(x) < t_2(x) < \dots$, with mean waiting time τ.
3. $w(x,t)$ is constant for $t \neq t_j(x)$ and flips at times $t = t_j(x)$.

Global resampling process

Take $w(x,0)$ i.i.d. and use a single Poisson process. Resample the whole distribution of w at each Poisson time.

Huh?

MICHIGAN STATE UNIVERSITY

If that was too much, think of the following:

Flip process

1. Initially $w(x,0)$ are i.i.d. ± 1 with equal probability.
2. At each site $x \in \mathbb{Z}^d$ we have independent Poisson processes $t_1(x) < t_2(x) < \dots$, with mean waiting time τ.
3. $w(x,t)$ is constant for $t \neq t_j(x)$ and flips at times $t = t_j(x)$.

Global resampling process

Take $w(x,0)$ i.i.d. and use a single Poisson process. Resample the whole distribution of w at each Poisson time.

These are just a couple of simple examples. There are many fancier ones that you can use (if you want).

What sort of hopping?

MICHIGAN STATE UNIVERSITY

Our methods apply to quite general hopping

$$T\psi(x) \;=\; \sum_y h(x-y)\psi(y)$$

such that

1. $\sum_y |y|^2 h(y) < \infty$, so that $\widehat{h}(k) = \sum_y h(y)\mathrm{e}^{-\mathrm{i}k\cdot y}$ is a C^2 function.

What sort of hopping?

MICHIGAN STATE UNIVERSITY

Our methods apply to quite general hopping

$$T\psi(x) \;=\; \sum_y h(x-y)\psi(y)$$

such that

1. $\sum_y |y|^2 h(y) < \infty$, so that $\widehat{h}(k) = \sum_y h(y)\mathrm{e}^{-\mathrm{i}k\cdot y}$ is a C^2 function.
2. h is *non-degenerate*: $\{y \mid h(y) \neq 0\}$ generates $\mathbb{Z}^d$.

What sort of hopping?

MICHIGAN STATE UNIVERSITY

Our methods apply to quite general hopping

$$T\psi(x) \;=\; \sum_y h(x-y)\psi(y)$$

such that

1. $\sum_y |y|^2 h(y) < \infty$, so that $\widehat{h}(k) = \sum_y h(y)\mathrm{e}^{-\mathrm{i}k\cdot y}$ is a C^2 function.
2. h is *non-degenerate*: $\{y \mid h(y) \neq 0\}$ generates $\mathbb{Z}^d$.

- T has some ac spectrum, with ballistic transport. It may have infinitely degenerate imbedded eigenvalues (if $\widehat{h}$ is constant on some open sets of positive measure).

What sort of hopping?

MICHIGAN STATE UNIVERSITY

Our methods apply to quite general hopping

$$T\psi(x) \;=\; \sum_y h(x-y)\psi(y)$$

such that

1. $\sum_y |y|^2 h(y) < \infty$, so that $\widehat{h}(k) = \sum_y h(y)\mathrm{e}^{-\mathrm{i}k\cdot y}$ is a C^2 function.
2. h is *non-degenerate*: $\{y \mid h(y) \neq 0\}$ generates $\mathbb{Z}^d$.

- T has some ac spectrum, with ballistic transport. It may have infinitely degenerate imbedded eigenvalues (if $\widehat{h}$ is constant on some open sets of positive measure).
- If h is finite range, or exponentially bounded, then the spectrum is pure a.c. since $\widehat{h}$ is analytic.

Diffusion

MICHIGAN STATE UNIVERSITY

Theorem (Kang and S. 2009)

If $\mathrm{i}\partial_t \psi_t(x) \;=\; T\psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_x x_i x_j \mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \mathbf{D}(g)_{i,j}$$

Diffusion

MICHIGAN STATE UNIVERSITY

Theorem (Kang and S. 2009)

If $\mathrm{i}\partial_t \psi_t(x) \;=\; T\psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_x x_i x_j \mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \mathbf{D}(g)_{i,j}$$

and

$$\lim_{T\to\infty} \sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x} \mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

Diffusion

MICHIGAN STATE UNIVERSITY

Theorem (Kang and S. 2009)

If $\mathrm{i}\partial_t \psi_t(x) \;=\; T\psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_x x_i x_j \mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \mathbf{D}(g)_{i,j}$$

and

$$\lim_{T\to\infty} \sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x} \mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

where $\mathbf{D}(g)$ *is a* $d \times d$ *positive definite matrix,*

Diffusion

MICHIGAN STATE UNIVERSITY

Theorem (Kang and S. 2009)

If $\mathrm{i}\partial_t \psi_t(x) \;=\; T\psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_x x_i x_j \mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \mathbf{D}(g)_{i,j}$$

and

$$\lim_{T\to\infty} \sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x} \mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

where $\mathbf{D}(g)$ *is a* $d \times d$ *positive definite matrix, and*

$$\mathbf{D}(g) \;=\; \frac{1}{g^2}\mathbf{D}_0 + O(1)$$

as $g \to 0$*.*

Comments

MICHIGAN STATE UNIVERSITY

1. A full central limit theorem follows from the result for the Fourier transform:

$$\lim_{T\to\infty} \sum_{x} f(x/\sqrt{t})\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; C_d \int_{\mathbb{R}^d} f(\boldsymbol{r})\mathrm{e}^{-\frac{1}{2}\,\mathrm{tr}(\mathbf{D}(g)^{-1}\boldsymbol{r}\otimes\boldsymbol{r})}\mathrm{d}\boldsymbol{r}.$$

Comments

MICHIGAN STATE UNIVERSITY

1. A full central limit theorem follows from the result for the Fourier transform:

$$\lim_{T\to\infty} \sum_{x} f(x/\sqrt{t})\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; C_d \int_{\mathbb{R}^d} f(\boldsymbol{r})\mathrm{e}^{-\frac{1}{2}\operatorname{tr}(\mathbf{D}(g)^{-1}\boldsymbol{r}\otimes\boldsymbol{r})}\mathrm{d}\boldsymbol{r}.$$

2. If T commutes with lattice rotations, then $\mathbf{D}(g) \;=\; D(g)\mathbb{1}$, where $D(g) = \frac{1}{d}\operatorname{tr}\mathbf{D}(g)$ is the *diffusion constant*.

Comments

MICHIGAN STATE UNIVERSITY

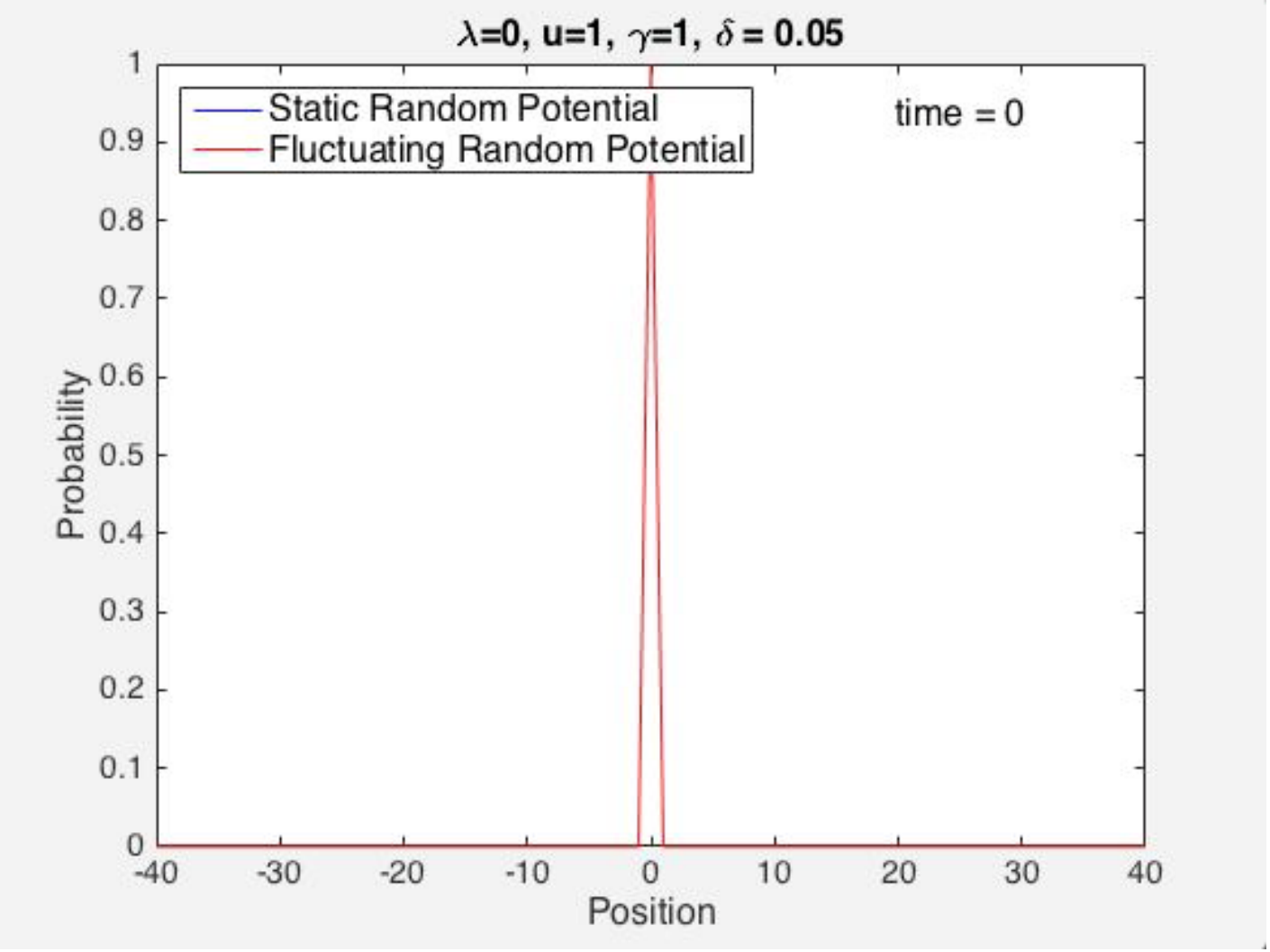
1. A full central limit theorem follows from the result for the Fourier transform:

$$\lim_{T\to\infty} \sum_x f(x/\sqrt{t})\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; C_d \int_{\mathbb{R}^d} f(\boldsymbol{r})\mathrm{e}^{-\frac{1}{2}\operatorname{tr}(\mathbf{D}(g)^{-1}\boldsymbol{r}\otimes\boldsymbol{r})}\mathrm{d}\boldsymbol{r}.$$

2. If T commutes with lattice rotations, then $\mathbf{D}(g) \;=\; D(g)\mathbb{1}$, where $D(g) = \frac{1}{d}\operatorname{tr}\mathbf{D}(g)$ is the *diffusion constant*.
3. For the global resampling process there is an explicit formula for $\mathbf{D}_0$ in terms of the Green's functions of T, suggesting that the asymptotics have to do with AC spectrum and ballistic transport.

Fast diffusion

MICHIGAN STATE UNIVERSITY



Diffusion for localized states

MICHIGAN STATE UNIVERSITY

Theorem (S. 2015)

Let H_ω be a RSO $\mathrm{i}\partial_t \psi_t(x) \;=\; H_\omega \psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$. *Then*

$$\lim_{T\to\infty} \sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x} \mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

Diffusion for localized states

MICHIGAN STATE UNIVERSITY

Theorem (S. 2015)

Let H_ω be a RSO $\mathrm{i}\partial_t \psi_t(x) \;=\; H_\omega \psi_t(x) + gW(x,t)\psi_t(x)$ with, say, $\psi_0(x) = \delta_0(x)$. Then

$$\lim_{T\to\infty} \sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x} \mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

where $\mathbf{D}(g)$ a $d \times d$ positive definite matrix.

Diffusion for localized states

MICHIGAN STATE UNIVERSITY

Theorem (S. 2015)

Let H_ω be a RSO $\mathrm{i}\partial_t\psi_t(x) \;=\; H_\omega\psi_t(x) + gW(x,t)\psi_t(x)$ *with, say,* $\psi_0(x) = \delta_0(x)$. *Then*

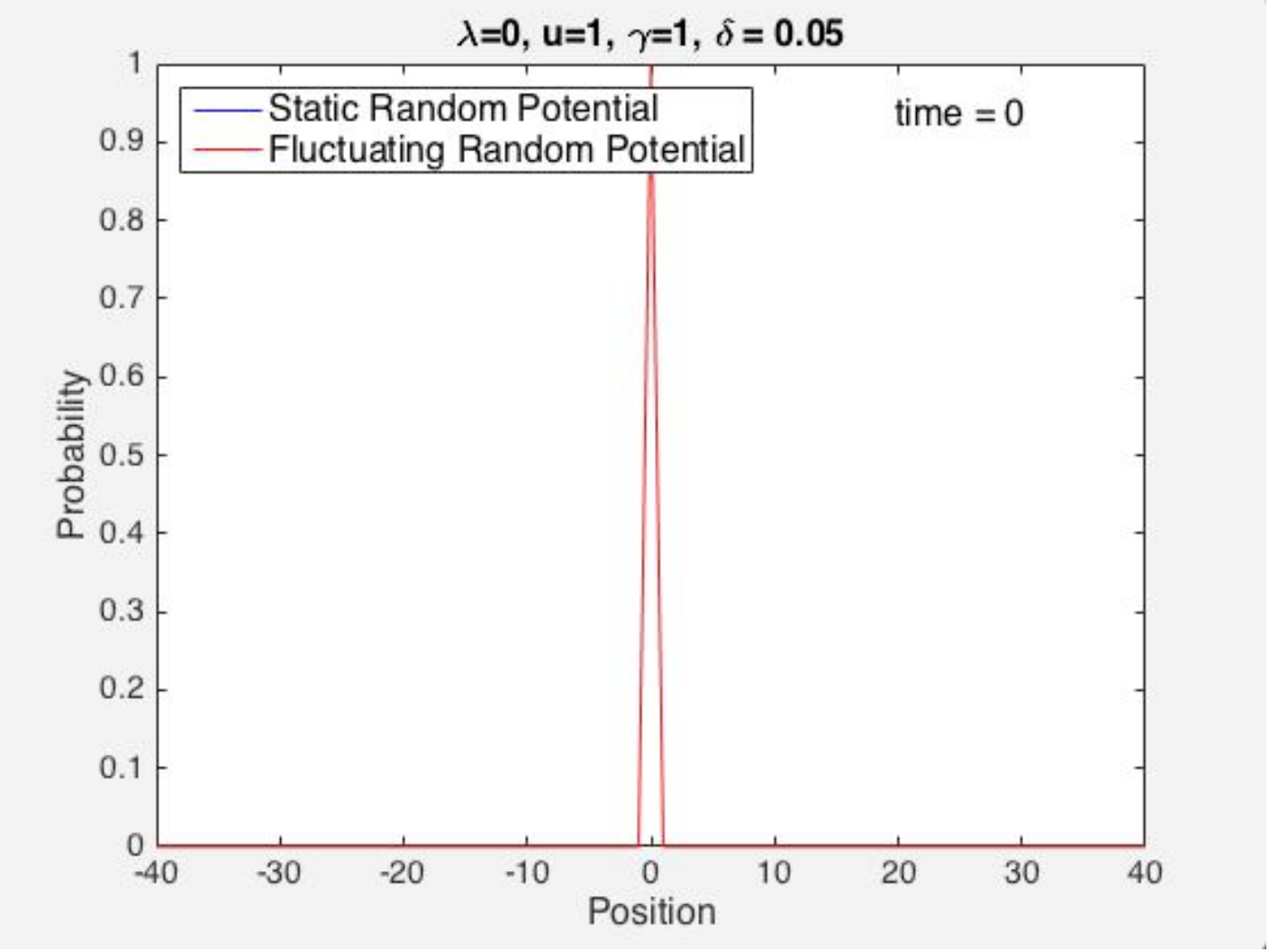
$$\lim_{T\to\infty}\sum_x \mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{T}}k\cdot x}\mathbb{E}\left(|\psi_{Tt}(x)|\right)^2 \;=\; \mathrm{e}^{-t\,\mathrm{tr}(\mathbf{D}(g)k\otimes k)},$$

where $\mathbf{D}(g)$ *a* $d\times d$ *positive definite matrix. If* H_ω *has exponential dynamical localization then*

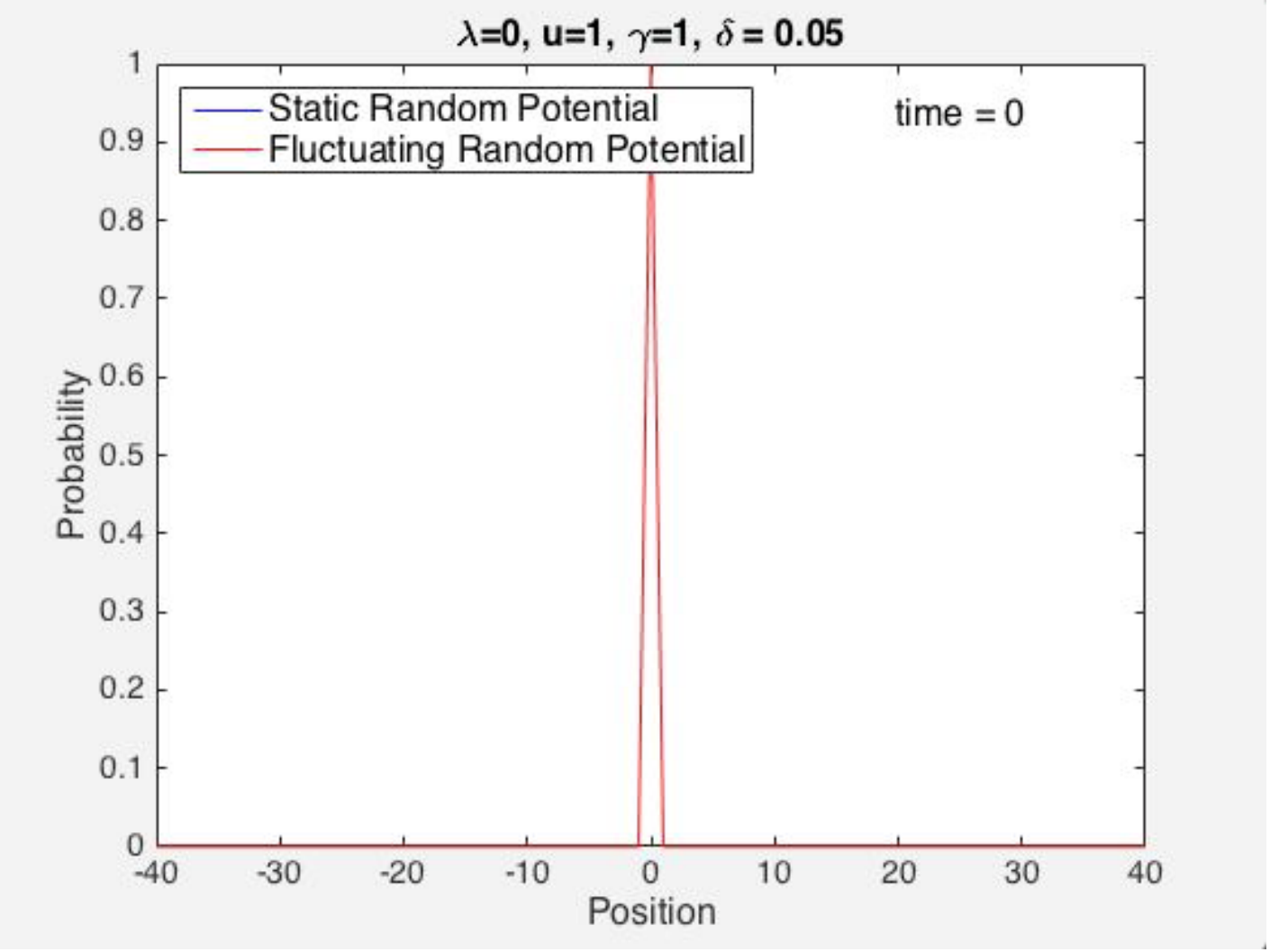
$$\mathbf{D}(g) \;=\; \mathbf{F}_0 g^2 + o(g^2).$$

Slow diffusion

MICHIGAN STATE UNIVERSITY



Slow diffusion



Conjecture

MICHIGAN STATE UNIVERSITY

Conjecture

Let H_0 be an ergodic Schrödinger operator and suppose that the dynamics $\mathrm{e}^{-\mathrm{i}tH_0}$ havea transport exponent ν:

$$\sum_x x_j^2 \left|\psi_t(x)\right|^2 \;\sim\; C_j t^{2\nu}$$

with $0 < C_j < \infty$. Then solutions to $\mathrm{i}\partial_t(x) \;=\; H_0\psi_t(x) + gW(x,t)\psi_t(x)$ are diffusive with diffusion matrix

$$\mathbf{D}(g) \;=\; g^{2-4\nu}\mathbf{D}_0 + o(g^{2-4\nu}).$$

Periodic Schrödinger operators

MICHIGAN STATE UNIVERSITY

Theorem (S., Tilocco, Zhang 2018, in preparation)

Let $H_0 = T + U$ where $U(x)$ is a periodic potential, and suppose that H_0 has ballistic transport.[a] *Then diffusion holds for solutions to* $\mathrm{i}\partial_t(x) \;=\; H_0\psi_t(x) + gW(x,t)\psi_t(x)$ *with*

$$\mathbf{D}(g) \;=\; \frac{1}{g^2}\mathbf{D}_0 + O(1)$$

[a] This is automatic if T is finite range, or exponentially bounded, but may not hold in full generality for polynomially decaying hopping.

Periodic Schrödinger operators

MICHIGAN STATE UNIVERSITY

Theorem (S., Tilocco, Zhang 2018, in preparation)

Let $H_0 = T + U$ where $U(x)$ is a periodic potential, and suppose that H_0 has ballistic transport.[a] *Then diffusion holds for solutions to* $\mathrm{i}\partial_t(x) \;=\; H_0\psi_t(x) + gW(x,t)\psi_t(x)$ *with*

$$\mathbf{D}(g) \;=\; \frac{1}{g^2}\mathbf{D}_0 + O(1)$$

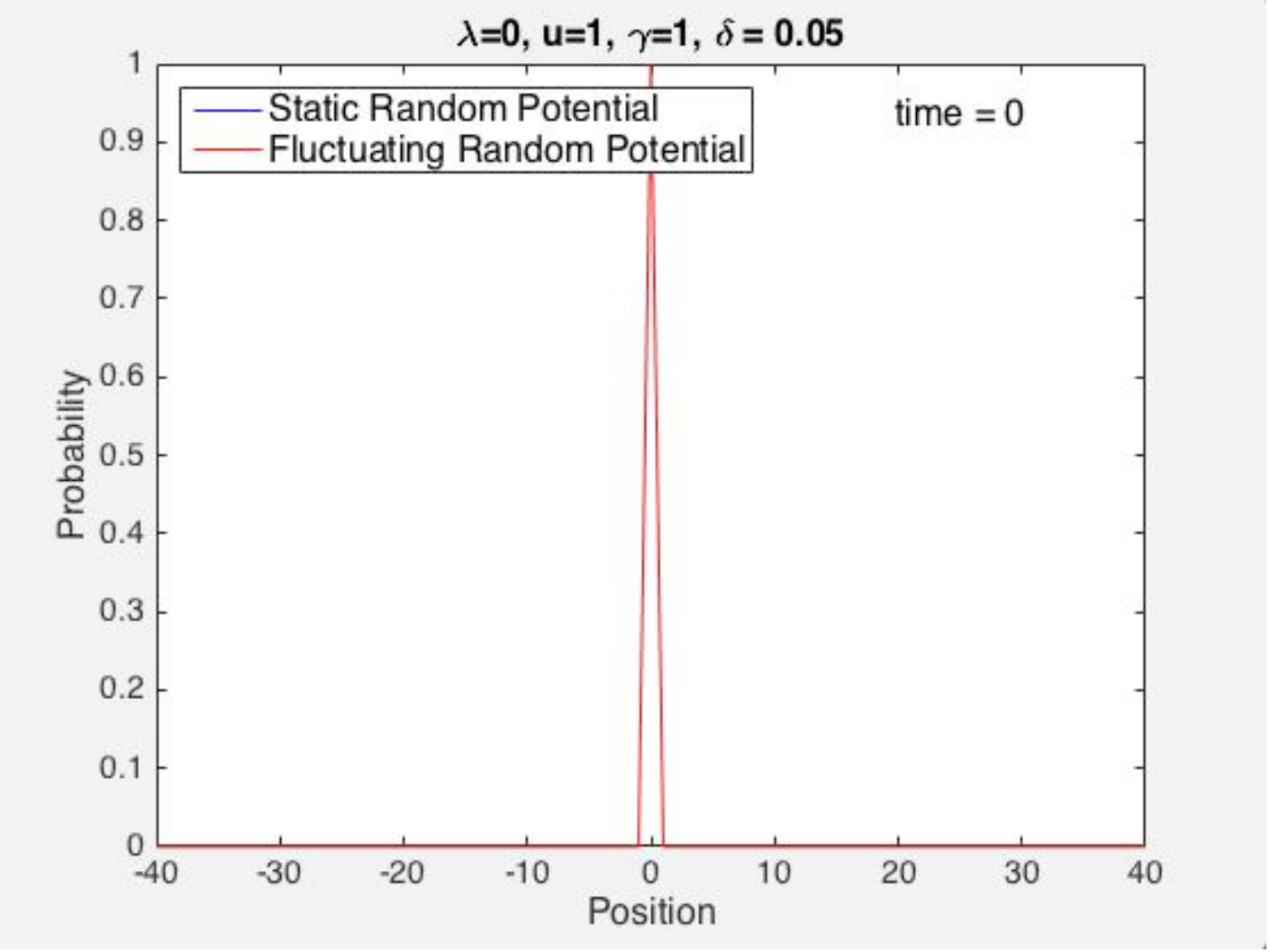
[a] This is automatic if T is finite range, or exponentially bounded, but may not hold in full generality for polynomially decaying hopping.

A formal calculation suggests that

$$\mathbf{D}(g) \;=\; \frac{1}{g^2}\mathbf{D}_0 + \mathbf{D}_1 + o(1).$$

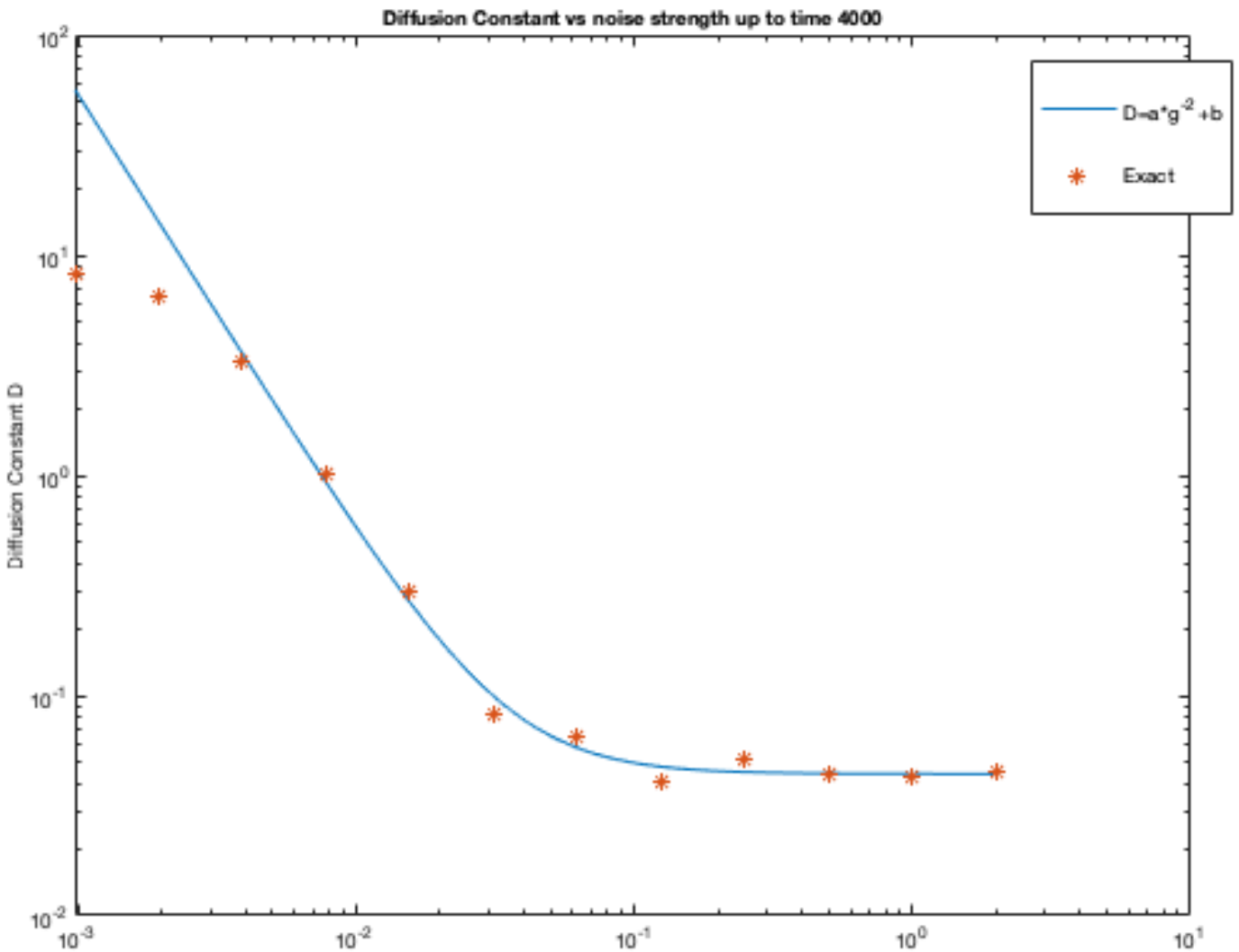
Slow/Fast diffusion

MICHIGAN STATE UNIVERSITY



D versus g

MICHIGAN STATE UNIVERSITY



Literature.

MICHIGAN STATE UNIVERSITY

- Ovchinnikov and Erikhman (JETP 1974)
- Pillet (CMP 1985)
- Tcheremchantsev (CMP 1997, CMP 1998)
- Fischer, Leshke, Müller (Ann. Phys. 1998)

Literature.

MICHIGAN STATE UNIVERSITY

- Ovchinnikov and Erikhman (JETP 1974)
- Pillet (CMP 1985)
- Tcheremchantsev (CMP 1997, CMP 1998)
- Fischer, Leshke, Müller (Ann. Phys. 1998)
- Kang & S. (JSP 2009); Hamza, Kang & S. (LMP 95 2010); S. (CMP 2015)

Literature.

MICHIGAN STATE UNIVERSITY

- Ovchinnikov and Erikhman (JETP 1974)
- Pillet (CMP 1985)
- Tcheremchantsev (CMP 1997, CMP 1998)
- Fischer, Leshke, Müller (Ann. Phys. 1998)
- Kang & S. (JSP 2009); Hamza, Kang & S. (LMP 95 2010); S. (CMP 2015)
- Hislop, Kirkpatrick, Olla, & S. (2018).

Outline

MICHIGAN STATE UNIVERSITY

Augmented space formalism

MICHIGAN STATE UNIVERSITY

- We use the "Augmented space formalism" with disorder variables in the Hilbert space:

$$\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \langle \delta_x \times \delta_x \times 1|\, \mathrm{e}^{-t\mathcal{G}}\, |\delta_0 \times \delta_0 \times 1\rangle_{\mathcal{H}\times\mathcal{H}\times L^2(\Lambda)}$$

  - $\mathcal{H} = \ell^2(\mathbb{Z}^d)$
  - $\mathcal{G} = \mathrm{i}\,[H, \cdot] + B$ **(non self adjoint)**,
  - $H = H_0 + gw(\tau_x\alpha)$ **(No time dependence!)**
  - $B =$ Markov process generator.

Augmented space formalism

MICHIGAN STATE UNIVERSITY

- We use the "Augmented space formalism" with disorder variables in the Hilbert space:

$$\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \langle \delta_x \times \delta_x \times 1|\, \mathrm{e}^{-t\mathcal{G}}\, |\delta_0 \times \delta_0 \times 1\rangle_{\mathcal{H}\times\mathcal{H}\times L^2(\Lambda)}$$

  - $\mathcal{H} = \ell^2(\mathbb{Z}^d)$
  - $\mathcal{G} = \mathrm{i}\,[H, \cdot] + B$ **(non self adjoint)**,
  - $H = H_0 + gw(\tau_x\alpha)$ **(No time dependence!)**
  - $B =$ Markov process generator.
- Generator $\mathcal{G}$ commutes with translations $\delta_x \times \delta_y \times f(\omega) \mapsto \delta_{x+\xi} \times \delta_{y+\xi} \times f(S_\xi\omega)$.

Augmented space formalism

MICHIGAN STATE UNIVERSITY

- We use the "Augmented space formalism" with disorder variables in the Hilbert space:

$$\mathbb{E}\left(|\psi_t(x)|^2\right) \;=\; \langle \delta_x \times \delta_x \times 1|\, \mathrm{e}^{-t\mathcal{G}}\, |\delta_0 \times \delta_0 \times 1\rangle_{\mathcal{H}\times\mathcal{H}\times L^2(\Lambda)}$$

  - $\mathcal{H} = \ell^2(\mathbb{Z}^d)$
  - $\mathcal{G} = \mathrm{i}\,[H, \cdot] + B$ **(non self adjoint)**,
  - $H = H_0 + gw(\tau_x\alpha)$ **(No time dependence!)**
  - $B =$ Markov process generator.
- Generator $\mathcal{G}$ commutes with translations $\delta_x \times \delta_y \times f(\omega) \mapsto \delta_{x+\xi} \times \delta_{y+\xi} \times f(S_\xi\omega)$.
- In the periodic case, this is a subgroup of possible translations.

Sketch of the Proof

MICHIGAN STATE UNIVERSITY

- Make a Bloch-Floquet transform

$$\sum_{x\in p\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x}\mathbb{E}\left(|\psi_t(x)|^2\right)$$
$$= \sum_{y\in P} \langle \delta_0 \times \delta_y \times 1| \, \mathrm{e}^{-t\widehat{\mathcal{G}}_k} \, |\delta_0 \times \delta_0 \times 1\rangle_{\mathcal{H}\times\ell^2(P)\times L^2(\Lambda)} \, ,$$

  where
  - P is the periodicity cell for the periodic potential,
  - $\widehat{\mathcal{G}}_k$ is an explicit fiber operator (non SA), with hopping in disorder space.
- $\widehat{\mathcal{G}}_0$ has a simple zero eigenvector (conservation of probability) and a *spectral gap*.

Sketch of the proof

MICHIGAN STATE UNIVERSITY

Basic perturbation theory shows that $\widehat{\mathcal{G}}_k$ has a simple near zero eigenvector $E(k)$ and a spectral gap.

Sketch of the proof

MICHIGAN STATE UNIVERSITY

Basic perturbation theory shows that $\widehat{\mathcal{G}}_k$ has a simple near zero eigenvector $E(k)$ and a spectral gap.

$$\nabla E(0) = 0$$

$$\mathbf{D}_{i,j} = \left. \frac{\partial^2}{\partial k_i \partial k_j} E(k) \right|_{k=0} . \quad \square$$

Sketch of the proof

MICHIGAN STATE UNIVERSITY

Basic perturbation theory shows that $\widehat{\mathcal{G}}_k$ has a simple near zero eigenvector $E(k)$ and a spectral gap.

$$\nabla E(0) = 0$$

$$\mathbf{D}_{i,j} = \frac{\partial^2}{\partial k_i \partial k_j} E(k)\Big|_{k=0} . \quad \square$$

Proof for the random operator case is a bit different. There is no gap, but we can use the resolvent $(\eta + \widehat{\mathcal{G}}_k)^{-1}$ to prove the result.

Sketch of the proof

MICHIGAN STATE UNIVERSITY

Basic perturbation theory shows that $\widehat{\mathcal{G}}_k$ has a simple near zero eigenvector $E(k)$ and a spectral gap.

$$\nabla E(0) = 0$$

$$\mathbf{D}_{i,j} = \frac{\partial^2}{\partial k_i \partial k_j} E(k)\Big|_{k=0} . \quad \square$$

Proof for the random operator case is a bit different. There is no gap, but we can use the resolvent $(\eta + \widehat{\mathcal{G}}_k)^{-1}$ to prove the result.

Estimating $\mathbf{D}$ at small g takes more work.

Outline

MICHIGAN STATE UNIVERSITY

Open Quantum System

MICHIGAN STATE UNIVERSITY

Diffusion for a quantum particle interacting with a thermal bath?

Open Quantum System

MICHIGAN STATE UNIVERSITY

Diffusion for a quantum particle interacting with a thermal bath?

- Vast physics literature going back to Mott.

Open Quantum System

MICHIGAN STATE UNIVERSITY

Diffusion for a quantum particle interacting with a thermal bath?

- Vast physics literature going back to Mott.
- Most work relies on Fermi Golden Rule (quantum Markov formalism).

Open Quantum System

MICHIGAN STATE UNIVERSITY

Diffusion for a quantum particle interacting with a thermal bath?

- Vast physics literature going back to Mott.
- Most work relies on Fermi Golden Rule (quantum Markov formalism).
- Some recent mathematical physics literature:
  - D. Spehner and J. Bellissard (JSP 2001);
  - G. Androulakis, J. Bellissard, and C. Sadel. (JSP 2012)
  - W. De Roeck and J. Fröhlich (CMP 2011);
  - Fröhich and S. (JMP 2016);

Open Quantum System in the Markov Approximation

MICHIGAN STATE UNIVERSITY

Theorem (Fröhlich, S. 2016)

$$\partial_t \rho_t \;=\; -\mathrm{i}\,[H_\omega, \rho_t] + u\mathcal{L}(\rho_t),$$

with $H_\omega = T + \lambda V_\omega$ *and a suitable*[a] *Lindbladian* $\mathcal{L}$. *Then*

$$D \;=\; \lim_{t\to\infty} \frac{1}{t} \sum_x |x|^2 \mathbb{E}\left(\langle x|\, \rho_t\, |x\rangle\right)$$

exists and satisfies $0 < D < \infty$.

Open Quantum System in the Markov Approximation

MICHIGAN STATE UNIVERSITY

Theorem (Fröhlich, S. 2016)

$$\partial_t \rho_t \;=\; -\mathrm{i}\,[H_\omega, \rho_t] + u\mathcal{L}(\rho_t),$$

with $H_\omega = T + \lambda V_\omega$ *and a suitable*[a] *Lindbladian* $\mathcal{L}$. *Then*

$$D \;=\; \lim_{t\to\infty} \frac{1}{t} \sum_x |x|^2 \mathbb{E}\left(\langle x| \rho_t |x\rangle\right)$$

exists and satisfies $0 < D < \infty$.

1. *If* $\lambda = 0$ *(no disorder), then* $D = \frac{C}{u}$ *for all* $u > 0$.
2. *If* H_ω *exhibits localization, then*

$$D \;=\; \Delta u + o(u)$$

where $0 < \Delta < (\textit{loc. length})^2$.

The Lindblad generator

MICHIGAN STATE UNIVERSITY

Describes a hopping process for the particle momentum:

$$\rho^W(X,\boldsymbol{p}) \;=\; \sum_{\xi} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\xi} \left\langle \frac{X+\xi}{2} \middle| \rho \middle| \frac{X-\xi}{2} \right\rangle,$$

$$\mathcal{L}\rho^W(X,\boldsymbol{p}) \;=\; \int \widehat{r}(\boldsymbol{p},\boldsymbol{q}) \left[\rho^W(X,\boldsymbol{q}) - \rho^W(X,\boldsymbol{p})\right] \mathrm{d}q,$$

$$\widehat{r}(\boldsymbol{p},\boldsymbol{q}) \;= \widehat{r}(\boldsymbol{q},\boldsymbol{p}),$$

and

$$\iint \widehat{r}(\boldsymbol{p},\boldsymbol{q}) \left|f(\boldsymbol{p}) - f(\boldsymbol{q})\right|^2 \mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{q} \;\geq\; c \iint \left|f(\boldsymbol{p}) - f(\boldsymbol{q})\right|^2 \mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{q}.$$

Outline

MICHIGAN STATE UNIVERSITY

Summary

MICHIGAN STATE UNIVERSITY

- Diffusion is universal in the presence of time dependent fluctuations. (Quantitative analysis)

Summary

MICHIGAN STATE UNIVERSITY

- Diffusion is universal in the presence of time dependent fluctuations. (Quantitative analysis)
- The speed of diffusion for weak noise carries information about the transport properties of the underlying time independent system. (Qualitative analysis)

Open Problems

MICHIGAN STATE UNIVERSITY

1. More general operators with ballistic transport: almost-periodic, quasi-periodic.

Open Problems

MICHIGAN STATE UNIVERSITY

1. More general operators with ballistic transport: almost-periodic, quasi-periodic.
2. Operators H_0 with anomalous diffusion.

Open Problems

MICHIGAN STATE UNIVERSITY

1. More general operators with ballistic transport: almost-periodic, quasi-periodic.
2. Operators H_0 with anomalous diffusion.
3. Handling more natural Lindblad generators.

Open Problems

MICHIGAN STATE UNIVERSITY

1. More general operators with ballistic transport: almost-periodic, quasi-periodic.
2. Operators H_0 with anomalous diffusion.
3. Handling more natural Lindblad generators.
4. Fluctuations around the limit of the mean. For example what can we say about
$$\mathbb{E}\left(\langle\psi_t|\, X^2\, |\psi_t\rangle - \langle\psi_t|\, X\, |\psi_t\rangle^2\right)?$$

Open Problems

MICHIGAN STATE UNIVERSITY

1. More general operators with ballistic transport: almost-periodic, quasi-periodic.
2. Operators H_0 with anomalous diffusion.
3. Handling more natural Lindblad generators.
4. Fluctuations around the limit of the mean. For example what can we say about
$$\mathbb{E}\left(\langle\psi_t|\, X^2\, |\psi_t\rangle - \langle\psi_t|\, X\, |\psi_t\rangle^2\right)?$$
5. Does the wave function self-average?
   - If not, does it make sense to look at the limit in distribution.

Hard problems

MICHIGAN STATE UNIVERSITY

1. Diffusion in an open quantum system without the Markov approximation.
   - How to *prove* that decoherence emerges and memory in the bath decays?

Hard problems

MICHIGAN STATE UNIVERSITY

1. Diffusion in an open quantum system without the Markov approximation.
   - How to *prove* that decoherence emerges and memory in the bath decays?
2. Diffusion for weak disorder (*without fluctuations*)
   - *Recurrence* is the problem.
   - Can fluctuations help?

MICHIGAN STATE UNIVERSITY

Thank you!