Sum Rules via Large Deviations: A short panoramic travel

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Spectral Theory of Quasi-Periodic and Random Operators - CRM

Introduction

There are two remarkable formulas, which use the relative entropy $\mathcal{K}$ of two probabilities P and Q

$$\mathcal{K}(P|Q) = \begin{cases} \int \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ +\infty & \text{otherwise}. \end{cases} \tag{1}$$

The first one is

Szegő-Verblunsky's formula

$$\mathcal{K}(\mathrm{UNIF}|\mu) = -\sum_j \log(1 - |\alpha_j|^2)$$

where the α_j's are the Verblunsky coefficients of the measure μ on the unit circle $\mathbb{T}$ and UNIF is the uniform probability on $\mathbb{T}$.

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Killip-Simon sum rule ('03)

$$\mathcal{K}(SC|\,\mu) + \sum_n \mathcal{F}(E_n^+) + \sum_n \mathcal{F}(E_n^-) = \sum_j G(a_j^2) + \frac{b_j^2}{2}\,,$$

when

$$\mathrm{Supp}(\mu) = [-2,2] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+}$$

where N^+ (resp. N^-) is 0, finite or infinite, $E_j^- \uparrow -2$ and $E_j^+ \downarrow 2$ are isolated points of the support , a_j's and b_j's are the Jacobi coefficients of μ,

$$\mathcal{F}(x) = \int_2^{|x|} \sqrt{t^2-4}\,dt \quad , \; G(x) = x - 1 - \log x\,,$$

and SC is the semi-circle distribution :

$$SC(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx\,.$$

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The common features of both formulas are

- their positivity
- the presence of a reference measure μ_0
- their structure ; the "measure side" is a discrepancy between μ and μ_0 and the "coefficient side" is a "discrepancy" between the coefficients of μ and the corresponding coefficients of μ_0.

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How to get a sum rule with a probabilistic method

- Consider a compactly supported measure μ as the spectral measure of some operator M in some class, at a vector e :

$$< e, M^k e >= \int_E x^k d\mu(x) \ , \ (k \geqslant 0)$$

  $E = \mathbb{T}$ or $\mathbb{R}$, M unitary or self-adjoint.
- Randomize in this class, a family of finite-dimensional operators $(M_n)_{n\geqslant 1}$ and their spectral measures $(\mu_n)_{n\geqslant 1}$ at $(e^{(n)})_{n\geqslant 1}$
- Consider the two encodings of the spectral measures μ_n
  1) the pair "locations, weights"

$$\mu_n = \sum_{k=1}^{n} w_k^{(n)} \delta_{\lambda_k^{(n)}}$$

  2) the recursion coefficients (Jacobi or Verblunsky) : $(a_k^{(n)}, b_k^{(n)})$ or $(\alpha_k^{(n)})$.

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- Prove two Large Deviation Principles :

$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{sp}(\mu)$$
$$\frac{1}{n}\log P(\mu_n \cong \mu) \cong -\mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

- Write equality of both rate functions :

$$\mathcal{I}_{sp}(\mu) = \mathcal{I}_{Jac}(a_1, b_1, a_2, \dots)$$

Notice the difference between the two measures

$$\mu_n = \mu_n^{SP} = \sum_{k=1}^{n} w_k \delta_{\lambda_k} \quad \text{(spectral measure)}$$

$$\mu_n^{ESD} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k} \quad \text{(empirical spectral distribution)}.$$

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Randomization for the KS/SR

- Suppose the distribution of M_n has the GUE-density

$$\mathcal{Z}_n^{-1} \exp -\frac{n}{2} \mathrm{tr} M^2$$

- Dumitriu-Edelman ('02) proved that the Jacobi parameters are independent and

$$b_k^{(n)} \sim \mathcal{N}(0; n^{-1}) \quad (1 \leqslant k \leqslant n),$$
$$(a_k^{(n)})^2 \sim \text{Gamma}\,(n-k; n^{-1}) \;\; (1 \leqslant k \leqslant n-1)\,.$$

Note that $b_k^{(n)} \to 0$, $a_k^{(n)} \to 1$, the Jacobi coefficients of SC.

Theorem (GR '11)

μ_n^{SP} *satisfies the LDP with speed* n *and rate function*

$$\mathcal{I}_{\mathrm{Jac}} = \sum_1^\infty \frac{1}{2} b_k^2 + \sum_1^\infty G(a_k^2)\,,\;\; G(x) = x - 1 - \log x\,.$$

LDP for the "measure side", general potential (no gap)

- M_n random complex Hermitian $n \times n$ matrix with density

$$(\mathcal{Z}_n^V)^{-1} \exp(-n\mathrm{tr}V(M))$$

- Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).
- $$\mu_n^{SP} = \sum_1^n w_i \delta_{\lambda_i}$$

  with $w_i = |U_{1,i}|^2$ for U unitary matrix of eigenvectors.
- The joint density of eigenvalues is

$$(Z_n^V)^{-1} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i^n \exp(-nV(\lambda_i))$$

  and $(w_1, \ldots, w_n)$ is uniformly distributed on the simplex, and independent of the eigenvalues.

Theorem (GNR '16)

Under assumptions on V*, the sequence of random spectral measures* $\mu^{(n)}$ *satisfies the LDP with speed* n *with good rate function*

$$\mathcal{I}_{sp}(\mu) = \mathcal{K}(\mu_V \,|\mu) + \sum_k \mathcal{F}_V(E_k^+) + \sum_k \mathcal{F}_V(E_k^-)$$

for probability measures μ *on* $\mathbb{R}$ *satisfying*

$$\textit{Supp}(\Sigma) = [a_V, b_V] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+}$$

where N^+ *(resp.* N^-*) is* 0*, finite or infinite,* $E_j^- \uparrow a_V$ *and* $E_j^+ \downarrow b_V$ *are isolated points of the support .*

First point : Asymptotics of the empirical spectral distribution

The ESD

$$\mu_n^{ESD} = \frac{1}{n}\sum_1^n \delta_{\lambda_i}$$

satisfies :

- $\lim_n \mu_n^{ESD} = \mu_V$ in probability with μ_V compactly supported by $[\alpha_V^-, \alpha_V^+]$ (equilibrium measure a.k.a. density of states)
- Ben Arous, Guionnet ('97) : $(\mu_n^{ESD})_n$ satisfies the LDP with speed n^2 and rate function

$$\mathcal{I}^{ESD}(\mu) = \int V(x)d\mu(x) - \int\int \log|x-y|d\mu(x)d\mu(y) - c_V$$

- Ben Arous, Dembo, Guionnet ('01) : The largest/smallest ev satisfies the LDP with speed n and rate $\mathcal{F}_V^{\pm}$.

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Second point : decoupling

At scale n, the measure μ_n^{ESD} is quasi-deterministic, and the randomness comes essentially from the weights w_k's.

The weights w_k are not independent but

$$(w_1, \dots, w_n) \overset{(d)}{=} \left(\frac{\gamma_1}{\gamma_1 + \cdots + \gamma_n}, \dots, \frac{\gamma_n}{\gamma_1 + \cdots + \gamma_n} \right)$$

where the γ_k's are independent, $\exp(1)$. So, we can write

$$\mu_n^{SP} \overset{(d)}{=} \frac{\tilde{\mu}_n}{\int d\tilde{\mu}_n} \text{ with } \tilde{\mu}_n = \sum_{k=1}^{n} \gamma_k \delta_{\lambda_k},$$

study first $\tilde{\mu}_n$ and then make the "contraction".

To go on, there are 2 methods : GNR / Breuer-Simon-Zeitouni (BSZ).

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GNR : $\mu \sim \{\int f d\mu; f \in \mathcal{C}_b\}$ (Laplace approach)

$$\mathbb{E}\left[\exp\left(n\int f d\tilde{\mu}_n\right)\right] = \mathbb{E}\left[\prod_{k=1}^{n}\exp(\gamma_k f(\lambda_k))\right]$$

$$= \mathbb{E}\left[\prod_{k=1}^{n}\exp(L\circ f(\lambda_k))\right] = \mathbb{E}\left[\exp\left(n\int (L\circ f) d\mu_n^{ESD}\right)\right]$$

with $L(x) = -\log(1-x)$. Then, it could be expected that

$$\frac{1}{n}\log\mathbb{E}\left[\exp\left(n\int f d\mu_n\right)\right] \to \int (L\circ f) d\mu_V$$

and then the LDP for $\tilde{\mu}_n$ would be

$$\tilde{I}(\mu) = \sup_f \int f d\mu - \int (L\circ f) d\mu_V = \mathcal{H}(\mu_V|\mu) + \int d\mu - 1$$

But there are contributions of the outliers, since their LDP is at speed n.

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$$= \mathbb{E}\left[\prod_{k=1}^{n}\exp(L\circ f(\lambda_k))\right] = \mathbb{E}\left[\exp\left(n\int (L\circ f) d\mu_n^{ESD}\right)\right]$$

with $L(x) = -\log(1-x)$. Then, it could be expected that

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and then the LDP for $\tilde{\mu}_n$ would be

$$\tilde{I}(\mu) = \sup_f \int f d\mu - \int (L\circ f) d\mu_V = \mathcal{H}(\mu_V|\mu) + \int d\mu - 1$$

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$$\mu \sim \{\mu(I_k^j);\ k \leqslant 2^j, j = 1, 2, \ldots\}$$

They replace measures by their "histogram versions" at increasing resolutions.

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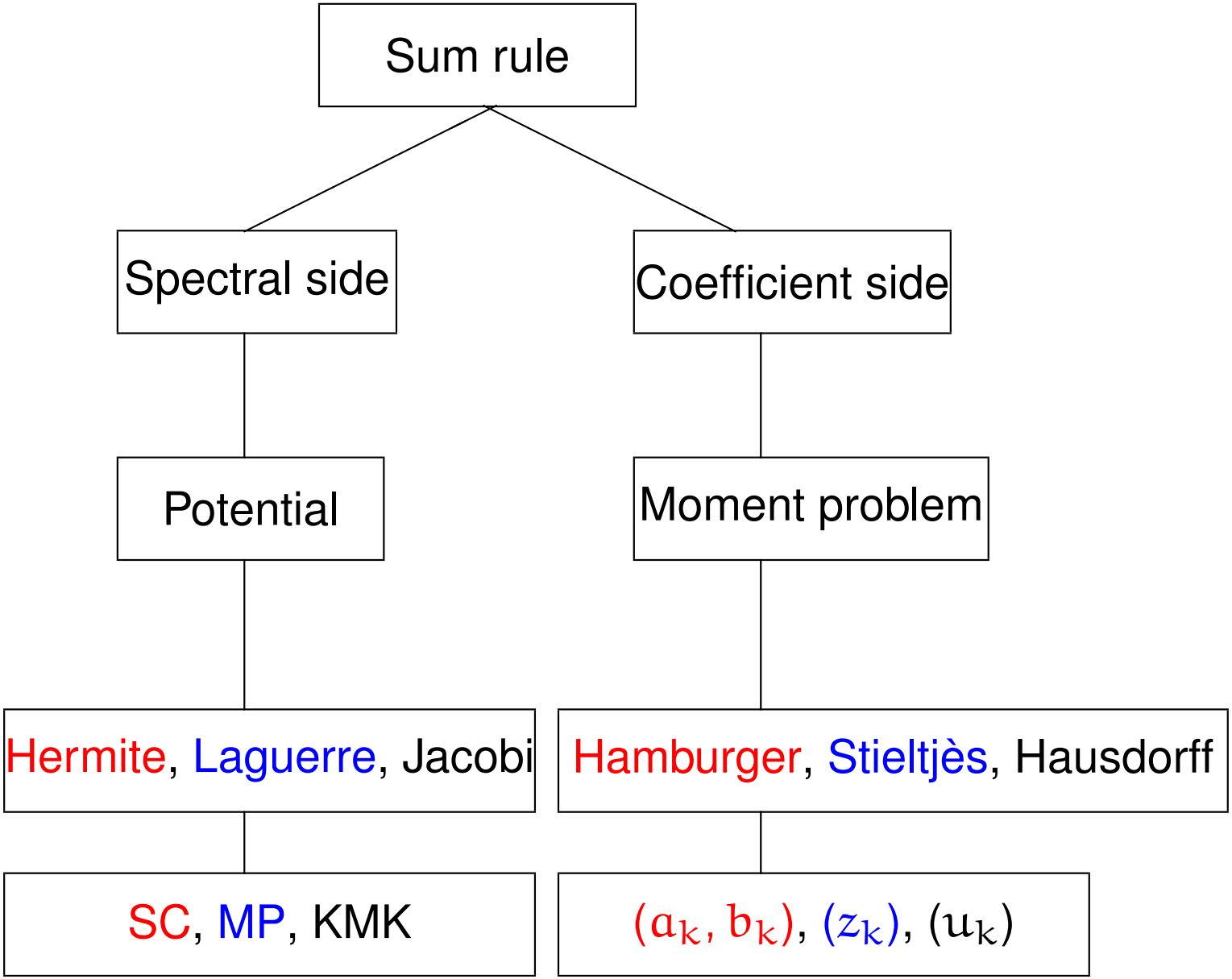
Sum rule
Spectral side
Coefficient side
Potential
Moment problem
Hermite, Laguerre, Jacobi
Hamburger, Stieltjès, Hausdorff
SC, MP, KMK
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New sum rules : Laguerre case

- The Laguerre ensemble :

$$V(x) = cx - (c-1)\log x\ ,\ x > 0\ ,\ c \geqslant 1$$

(distribution of $X = YY^\dagger$ when Y is a Gaussian $n \times cn$ matrix).

- Equilibrium measure :

$$\mathrm{MP}_c(dx) = \frac{c\sqrt{(\tau^+ - x)(x - \tau^-)}}{2\pi x}\ \mathbb{1}_{(c^-,c^+)}(x)dx\ \ ,c^\pm = \frac{1}{c}(\sqrt{c} \pm 1)^2$$

- For μ with $\mathrm{Supp}(\mu) \subset [0, +\infty[$, there exist parameters $z_j, j \geqslant 0$ such that

$$b_n = z_{2n-2} + z_{2n-1},\ \text{and}\ a_n^2 = z_{2n-1}z_{2n}.$$

- The MP_c distribution corresponds to

$$a_k^2 = c^{-1}\ (k \geqslant 1)\ \ ,\ \ b_1 = 1\ ,\ b_k = 1 + c^{-1}\ \ (k \geqslant 2)$$

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$$Z_{2k-1} \overset{\text{law}}{=} \frac{1}{cn}\text{Gamma}(n-k)\ ,\ Z_{2k} \overset{\text{law}}{=} \frac{1}{cn}\text{Gamma}(cn-k-1)$$

with $m/N \to c^{-1}$. The sum rule is then (GNR '16)

$$K(MP_c|\mu) + \sum_{n=1}^{N^+} \mathcal{F}_L^+(\lambda_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_L^-(\lambda_n^-) = \sum_{k=1}^{\infty} cG(z_{2k-1}) + G(cz_{2k})$$

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"Gem"

Let μ be a p.m. on $[0, \infty)$ with coefficients $z_k > 0$, then

- $\sum_{k=1}^{\infty}(z_{2k-1} - 1)^2 + (z_{2k} - c^{-1})^2 < \infty$ if and only if
- supp $(\mu) = [c^-, c^+] \cup \{\lambda_i^-\}_{i=1}^{N^-} \cup \{\lambda_i^+\}_{i=1}^{N^+}$
- $\sum_{i=1}^{N^-}(c^- - \lambda_i^-)^{3/2} + \sum_{i=1}^{N^+}(\lambda_i^+ - c^+)^{3/2} < \infty$ and if $N^- > 0$, then $\lambda_1^- > 0$,
- the decomposition $d\mu(x) = f(x)dx + d\mu_s(x)$ satisfies

$$\int_{c^-}^{c^+} \frac{\sqrt{c^+ - x)(x - c^-)}}{x} \log f(x)dx > -\infty\,.$$

New sum rules : Jacobi case

- The Jacobi ensemble :

$$V(x) = -\kappa_1 \log(2+x) - \kappa_2 \log(2-x) \ (\kappa_1, \kappa_2 \geqslant 0)$$

- Equilibrium measure :

$$\mathrm{KMK}_{\kappa_1,\kappa_2}(dx) = \frac{(2+\kappa_1+\kappa_2)}{2\pi} \frac{\sqrt{(u^+ - x)(x - u^-)}}{(2-x)(2+x)} 1_{(u^-,u^+)}(x) dx$$

- The Jacobi parameters of a measure on $[-2, 2]$ may be represented as (Geronimus relations)

$$b_{k+1} = (1-\alpha_{2k-1})\alpha_{2k} - (1+\alpha_{2k+1})\alpha_{2k-2}$$
$$a_{k+1} = \sqrt{(1-\alpha_{2k-1})(1-\alpha_{2k}^2)(1+\alpha_{2k+1})}$$

where $|\alpha_j| < 1$. The α_j's are independent and beta-distributed (Killip-Nenciu '05).

The parameters of KMK are

$$\alpha_{2k} \equiv \alpha_{even} := \frac{\kappa_1 - \kappa_2}{2 + \kappa_1 + \kappa_2}\ ,\ \alpha_{2k+1} \equiv \alpha_{odd} := -\frac{\kappa_1 + \kappa_2}{2 + \kappa_1 + \kappa_2}$$

LDP for the "coefficient side"

Theorem (GR '11)

μ_n satisfies the LDP at speed n and rate function

$$\mathcal{I}_{coeff} = \sum_{k=0}^{\infty} H_1(\alpha_{2k+1}) + H_2(\alpha_{2k})$$

$$H_1(\alpha) = -(1 + \kappa_1 + \kappa_2) \log \frac{1 - \alpha}{1 - \alpha_{odd}} - \log \frac{1 + \alpha}{1 + \alpha_{odd}}$$

$$H_2(\alpha) = -(1 + \kappa_1) \log \frac{1 + \alpha}{1 + \alpha_{even}} - (1 + \kappa_2) \log \frac{1 - \alpha}{1 - \alpha_{even}}$$

Sum rule GNR '16 and gem.

Coming back to the moment problem :

The parameters in the Laguerre ensemble have a geometrical interpretation as canonical moments. When we fix $m_1, \ldots, m_j$, the $(j+1)$th moment lives in an interval $[m_{j+1}^-, \infty)$ and

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2) LDP for the "measure side" for multi-cut a.k.a. finite gap model
3) LDP for the coefficient side in the general one-cut model

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1) Szegő-Verblunsky sum rule (GNR '11, BSZ '16)
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Matrix sum rule for the Jacobi ensemble

Definition

If p is some fixed integer, a $p \times p$ matrix(-valued) probability neasure Σ on $\mathbb{R}$ (resp. $\mathbb{T}$) is a $p \times p$ matrix of signed complex measures, such that $\Sigma(A)$ is a Hermitian positive matrix for every Borel set A, and $\Sigma(\mathbb{R}) = 1$ (resp. $\Sigma(\mathbb{T}) = 1$).

MOPRL

To each $p \times p$ matrix-mesure Σ we associate a pseudo-scalar products

$$\langle\langle F, G \rangle\rangle_R = \int_{\mathbb{R}} F(z)^\dagger d\Sigma(z) G(z) \ ,$$

a sequence of right monic matrix orthogonal polynomials, and then a sequence of matrix orthonormal polynomials. They satisfy a three-term recurrence relation

$$x p_n^R(x) \quad = \quad p_{n+1}^R(x) A_{n+1}^\dagger + p_n^R(x) B_{n+1} + p_{n-1}^R(x) A_n \qquad (2)$$

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with B_k Hermitian and A_k non singular ($p \times p$ matrices).

If M is self-adjoint $N \times N$, the spectral theorem gives

$$M = \int_{\mathbb{R}} \lambda dE_\lambda$$

so that for every integer $p \leqslant N$ there exists a unique $p \times p$ matrix measure

$$\Sigma = (\Sigma_{ij})_{1 \leqslant i,j \leqslant p}\rangle \ , \ d\Sigma_{ij}(\lambda) = \langle dE_\lambda e_i, e_j\rangle \ (1 \leqslant i, j \leqslant p), k \in \mathbb{N} \, ,$$

so that

$$\langle M^k e_i, e_j\rangle = \int_{\mathbb{R}} x^k d\Sigma_{i,j}(x) \, .$$

Previously, M was chosen randomly in a classical ensemble of self-adjoint operators (Hermite, Laguerre, Jacobi).
$\Rightarrow$ OPRL, random Jacobi coefficients
Actually

$$\Sigma_p(dx) = \sum_{j=1}^{N} u_j u_j^\dagger \delta_{\lambda_j}$$

(u_j is the jth p-truncated column of the unitary matrix diagonalizing M).

Properties of the weights

Lemma

If U *is Haar distributed in* $\mathbb{U}(N)$ *then*

$$(u_1 u_1^\dagger, \ldots, u_N u_N^\dagger) \overset{(d)}{=} (h^{-1/2} v_1 v_1^\dagger h^{-1/2}, \ldots, h^{-1/2} v_N v_N^\dagger h^{-1/2})$$

where the v_j*'s are i.i.d. random gaussian vectors of size* p *and*

$$h = \sum_{j=1}^{N} v_j v_j^\dagger \, .$$

In the GUE or in the Laguerre ensemble, all the results on distributions are extended rather easily.

In the Jacobi ensemble, there is an issue.

- In the scalar case, as Killip and Nenciu ('05) we lift up the measure (supported by n points) on $[0, 1]$ or $[-2, 2]$ into a symmetric measure (supported by $2n$ points) on $\mathbb{T}$, by inverse Szegő mapping. It is the spectral measure of an element of $\mathbb{SO}(2n)$.
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In algebra, when one says $\mathrm{a} = \mathrm{b}$*, it is a tautology and so uninteresting ; while in analysis, when one says* $\mathrm{a} = \mathrm{b}$*, it is two deep inequalities.*
(attributed to S. Bochner)

If one only proves $\mathrm{a} = \mathrm{b}$ *by showing* $\mathrm{a} \leqslant \mathrm{b}$ *and* $\mathrm{b} \leqslant \mathrm{a}$*, one has not understood the true reason why* $\mathrm{a} = \mathrm{b}$*.*
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Our contribution could be :

If a and b are two positive functionals, when one says $\mathrm{a} = \mathrm{b}$, a probabilist may think that a and b could be two rate functions of the same large deviation principle under two different encodings.

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THANK YOU FOR YOUR ATTENTION!