Sum Rules via Large Deviations: A short panoramic travel

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Spectral Theory of Quasi-Periodic and Random Operators - CRM
There are two remarkable formulas, which use the relative entropy $\mathcal{K}$ of two probabilities $P$ and $Q$

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\mathcal{K}(P|Q) = \begin{cases} 
\int \log \frac{dP}{dQ} \, dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\
+\infty & \text{otherwise}. 
\end{cases}
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The first one is

**Szegő-Verblunsky’s formula**

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\mathcal{K}\text{(UNIF}|\mu) = -\sum_j \log(1 - |\alpha_j|^2)
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where the $\alpha_j$’s are the Verblunsky coefficients of the measure $\mu$ on the unit circle $\mathbb{T}$ and UNIF is the uniform probability on $\mathbb{T}$. 
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**Killip-Simon sum rule (’03)**

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\mathcal{K}(\text{SC}\mid \mu) + \sum_{n} \mathcal{F}(E_n^+) + \sum_{n} \mathcal{F}(E_n^-) = \sum_{j} G(a_j^2) + \frac{b_j^2}{2},
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when

\[
\text{Supp}(\mu) = [-2, 2] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+}
\]

where \(N^+\) (resp. \(N^-\)) is 0, finite or infinite, \(E_j^- \uparrow -2\) and \(E_j^+ \downarrow 2\) are isolated points of the support, \(a_j\)'s and \(b_j\)'s are the Jacobi coefficients of \(\mu\),

\[
\mathcal{F}(x) = \int_{2}^{\left| x \right|} \sqrt{t^2 - 4} \, dt, \quad G(x) = x - 1 - \log x,
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and \(\text{SC}\) is the semi-circle distribution:

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The common features of both formulas are

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- the presence of a reference measure $\mu_0$
- their structure; the "measure side" is a discrepancy between $\mu$ and $\mu_0$ and the "coefficient side" is a "discrepancy" between the coefficients of $\mu$ and the corresponding coefficients of $\mu_0$. 
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How to get a sum rule with a probabilistic method

- Consider a compactly supported measure $\mu$ as the spectral measure of some operator $M$ in some class, at a vector $e$:

$$<e, M^k e> = \int_E x^k d\mu(x), \ (k \geq 0)$$

$E = \mathbb{T}$ or $\mathbb{R}$, $M$ unitary or self-adjoint.

- Randomize in this class, a family of finite-dimensional operators $(M_n)_{n \geq 1}$ and their spectral measures $(\mu_n)_{n \geq 1}$ at $(e^{(n)})_{n \geq 1}$

- Consider the two encodings of the spectral measures $\mu_n$
  1) the pair "locations, weights"

$$\mu_n = \sum_{k=1}^{n} w_k^{(n)} \delta_{\lambda_k^{(n)}}$$

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Prove two Large Deviation Principles:

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\frac{1}{n} \log P(\mu_n \approx \mu) \approx -I_{sp}(\mu)
\]

\[
\frac{1}{n} \log P(\mu_n \approx \mu) \approx -I_{Jac}(a_1, b_1, a_2, \ldots)
\]

Write equality of both rate functions:

\[
I_{sp}(\mu) = I_{Jac}(a_1, b_1, a_2, \ldots)
\]

Notice the difference between the two measures

\[
\mu_n = \mu_n^{sp} = \sum_{k=1}^{n} w_k \delta_{\lambda_k} \quad \text{(spectral measure)}
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\[
\mu_n^{ESD} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \quad \text{(empirical spectral distribution)}.
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Randomization for the KS/SR

- Suppose the distribution of $\mathcal{M}_n$ has the GUE-density

$$
\mathcal{Z}_n^{-1} \exp \left( - \frac{n}{2} \text{tr} \mathcal{M}^2 \right)
$$

- Dumitriu-Edelman ('02) proved that the Jacobi parameters are independent and

$$
b_k^{(n)} \sim \mathcal{N}(0; n^{-1}) \quad (1 \leq k \leq n),
$$

$$
(a_k^{(n)})^2 \sim \text{Gamma} \left( n - k; n^{-1} \right) \quad (1 \leq k \leq n - 1).
$$

Note that $b_k^{(n)} \to 0$, $a_k^{(n)} \to 1$, the Jacobi coefficients of SC.

Theorem (GR '11)

$\mu_n^{SP}$ satisfies the LDP with speed $n$ and rate function

$$
\mathcal{J}_{Jac} = \sum_{1}^{\infty} \frac{1}{2} b_k^2 + \sum_{1}^{\infty} G(a_k^2), \quad G(x) = x - 1 - \log x.
$$
LDP for the "measure side", general potential (no gap)

- $\mathcal{M}_n$ random complex Hermitian $n \times n$ matrix with density

$$\left( Z_n^V \right)^{-1} \exp(-n \text{tr} V(\mathcal{M}))$$

- Potential $V : \mathbb{R} \rightarrow (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).

- $\mu_{n}^{\text{SP}} = \sum_{1}^{n} w_i \delta_{\lambda_i}$

  with $w_i = |U_{1,i}|^2$ for $U$ unitary matrix of eigenvectors.

- The joint density of eigenvalues is

$$\left( Z_n^V \right)^{-1} \prod_{i<j}(\lambda_i - \lambda_j)^2 \prod_{i} \exp(-n V(\lambda_i))$$

and $(w_1, \ldots, w_n)$ is uniformly distributed on the simplex, and independent of the eigenvalues.
Theorem (GNR ’16)

Under assumptions on $V$, the sequence of random spectral measures $\mu^{(n)}$ satisfies the LDP with speed $n$ with good rate function

$$ I_{sp}(\mu) = K(\mu_V \mid \mu) + \sum_k \mathcal{F}_V(E_k^+) + \sum_k \mathcal{F}_V(E_k^-) $$

for probability measures $\mu$ on $\mathbb{R}$ satisfying

$$ \text{Supp}(\Sigma) = [a_V, b_V] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+} $$

where $N^+$ (resp. $N^-$) is 0, finite or infinite, $E_j^- \uparrow a_V$ and $E_j^+ \downarrow b_V$ are isolated points of the support.
First point: Asymptotics of the empirical spectral distribution

The ESD

\[ \mu^\text{ESD}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \]

satisfies:

- \( \lim_n \frac{\mu^\text{ESD}_n}{n} = \mu_V \) in probability with \( \mu_V \) compactly supported by \([\alpha_V^-, \alpha_V^+]\) (equilibrium measure a.k.a. density of states)

- Ben Arous, Guionnet ('97): \( (\mu^\text{ESD}_n)_n \) satisfies the LDP with speed \( n^2 \) and rate function

\[ J^\text{ESD}(\mu) = \int V(x) \, d\mu(x) - \int \int \log |x - y| \, d\mu(x) \, d\mu(y) - c_V \]

- Ben Arous, Dembo, Guionnet ('01): The largest/smallest ev satisfies the LDP with speed \( n \) and rate \( \mathcal{F}_V^\pm \).
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Second point: decoupling

At scale $n$, the measure $\mu_{n}^{ESD}$ is quasi-deterministic, and the randomness comes essentially from the weights $w_k$'s.

The weights $w_k$ are not independent but

$$(w_1, \ldots, w_n) \overset{d}{=} \left( \frac{\gamma_1}{\gamma_1 + \cdots + \gamma_n}, \ldots, \frac{\gamma_n}{\gamma_1 + \cdots + \gamma_n} \right)$$

where the $\gamma_k$'s are independent, $\exp(1)$. So, we can write

$$\mu_{n}^{SP} \overset{d}{=} \tilde{\mu}_n \int d\tilde{\mu}_n \text{ with } \tilde{\mu}_n = \sum_{k=1}^{n} \gamma_k \delta_{\lambda_k},$$

study first $\tilde{\mu}_n$ and then make the "contraction".

To go on, there are 2 methods: GNR / Breuer-Simon-Zeitouni (BSZ).
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GNR : $\mu \sim \{ \int f \, d\mu ; f \in C_b \}$ (Laplace approach)

\[
E \left[ \exp \left( n \int f \, d\tilde{\mu}_n \right) \right] = E \left[ \prod_{k=1}^{n} \exp(\gamma_k f(\lambda_k)) \right] = E \left[ \prod_{k=1}^{n} \exp(L \circ f(\lambda_k)) \right] = E \left[ \exp \left( n \int (L \circ f) \, d\mu_n^{\text{ESD}} \right) \right]
\]

with $L(x) = -\log(1 - x)$. Then, it could be expected that

\[
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and then the LDP for $\tilde{\mu}_n$ would be

\[
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\]

But there are contributions of the outliers, since their LDP is at speed $n$. 
GNR : $\mu \sim \{ \int f \, d\mu ; f \in \mathcal{C}_b \}$ (Laplace approach)

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They replace measures by their "histogram versions" at increasing resolutions.

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New sum rules need LDPs for the coefficient side.

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Sum rule

Spectral side

Potential

Hermite, Laguerre, Jacobi

SC, MP, KMK

Coefficient side

Moment problem

Hamburger, Stieltjès, Hausdorff

(a_k, b_k), (z_k), (u_k)
New sum rules : Laguerre case

- The Laguerre ensemble:
  \[ V(x) = cx - (c - 1) \log x, \quad x > 0, \quad c \geq 1 \]
  (distribution of \( X = YY^\dagger \) when \( Y \) is a Gaussian \( n \times c n \) matrix).

- Equilibrium measure:
  \[
  MP_c(dx) = \frac{c \sqrt{(\tau^+ - x)(x - \tau^-)}}{2\pi x} \mathbb{1}_{(\tau^-, \tau^+)}(x) dx, \quad c^\pm = \frac{1}{c} (\sqrt{c} \pm 1)^2
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- For \( \mu \) with \( \text{Supp}(\mu) \subset [0, +\infty[ \), there exist parameters \( z_j, j \geq 0 \) such that
  \[
  b_n = z_{2n-2} + z_{2n-1}, \quad \text{and} \quad a_n^2 = z_{2n-1}z_{2n}.
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- The \( MP_c \) distribution corresponds to
  \[
  a_k^2 = c^{-1} \quad (k \geq 1), \quad b_1 = 1, \quad b_k = 1 + c^{-1} \quad (k \geq 2)
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In the random model (Dumitriu-Edelman), we have

\[ Z_{2k-1} \overset{\text{law}}{=} \frac{1}{cn} \Gamma(n - k), \quad Z_{2k} \overset{\text{law}}{=} \frac{1}{cn} \Gamma(cn - k - 1) \]

with \( \frac{m}{N} \to c^{-1} \). The sum rule is then (GNR ’16)

\[
K(MP_c | \mu) + \sum_{n=1}^{N^+} \mathcal{F}_L^+(\lambda_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_L^-(\lambda_n^-) = \sum_{k=1}^{\infty} cG(z_{2k-1}) + G(cz_{2k})
\]

where

\[
\mathcal{F}_L^+(x) = \int_{c^+}^{x} \frac{c \sqrt{(t-c^-)(t-c^+)}}{t} \, dt \quad \text{if} \ x \geq c^+,
\]

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Let $\mu$ be a p.m. on $[0, \infty)$ with coefficients $z_k > 0$, then

1. $\sum_{k=1}^{\infty}(z_{2k-1} - 1)^2 + (z_{2k} - c^{-1})^2 < \infty$ if and only if
2. $\text{supp} \quad \mu = [c^-, c^+] \cup \{\lambda_i^-\}_{i=1}^{N^-} \cup \{\lambda_i^+\}_{i=1}^{N^+}$
3. $\sum_{i=1}^{N^-}(c^- - \lambda_i^-)^{3/2} + \sum_{i=1}^{N^+}(\lambda_i^+ - c^+)^{3/2} < \infty$ and if $N^- > 0$, then $\lambda_1^- > 0$,
4. the decomposition $d\mu(x) = f(x) \, dx + d\mu_s(x)$ satisfies

$$\int_{c^-}^{c^+} \frac{\sqrt{c^+-x}(x-c^-)}{x} \log f(x) \, dx > -\infty.$$
New sum rules : Jacobi case

- The Jacobi ensemble:

\[ V(x) = -\kappa_1 \log(2 + x) - \kappa_2 \log(2 - x) \quad (\kappa_1, \kappa_2 \geq 0) \]

- Equilibrium measure:

\[ \text{KMK}_{\kappa_1, \kappa_2}(dx) = \frac{(2 + \kappa_1 + \kappa_2)}{2\pi} \frac{\sqrt{(u^+ - x)(x - u^-)}}{(2 - x)(2 + x)} 1_{(u^-, u^+)}(x) \, dx \]

- The Jacobi parameters of a measure on \([-2, 2]\) may be represented as (Geronimus relations)

\[
\begin{align*}
b_{k+1} &= (1 - \alpha_{2k-1})\alpha_{2k} - (1 + \alpha_{2k+1})\alpha_{2k-2} \\
a_{k+1} &= \sqrt{(1 - \alpha_{2k-1})(1 - \alpha_{2k}^2)(1 + \alpha_{2k+1})}
\end{align*}
\]

where \(|\alpha_j| < 1\). The \(\alpha_j\)'s are independent and beta-distributed (Killip-Nenciu '05).
The parameters of KMK are

\[ \alpha_{2k} \equiv \alpha_{\text{even}} := \frac{\kappa_1 - \kappa_2}{2 + \kappa_1 + \kappa_2}, \quad \alpha_{2k+1} \equiv \alpha_{\text{odd}} := -\frac{\kappa_1 + \kappa_2}{2 + \kappa_1 + \kappa_2} \]

LDP for the "coefficient side"

**Theorem (GR ’11)**

\( \mu_n \) satisfies the LDP at speed \( n \) and rate function

\[ J_\text{coeff} = \sum_{k=0}^{\infty} H_1(\alpha_{2k+1}) + H_2(\alpha_{2k}) \]

\[ H_1(\alpha) = -(1 + \kappa_1 + \kappa_2) \log \frac{1 - \alpha}{1 - \alpha_{\text{odd}}} - \log \frac{1 + \alpha}{1 + \alpha_{\text{odd}}} \]

\[ H_2(\alpha) = -(1 + \kappa_1) \log \frac{1 + \alpha}{1 + \alpha_{\text{even}}} - (1 + \kappa_2) \log \frac{1 - \alpha}{1 - \alpha_{\text{even}}} \]

Sum rule GNR ’16 and gem.
Coming back to the moment problem:

The parameters in the Laguerre ensemble have a geometrical interpretation as canonical moments. When we fix $m_1, \ldots, m_j$, the $(j+1)$th moment lives in an interval $[m_{j+1}^-, \infty)$ and

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Matrix sum rule for the Jacobi ensemble

Definition
If \( p \) is some fixed integer, a \( p \times p \) matrix-valued probability measure \( \Sigma \) on \( \mathbb{R} \) (resp. \( \mathbb{T} \)) is a \( p \times p \) matrix of signed complex measures, such that \( \Sigma(A) \) is a Hermitian positive matrix for every Borel set \( A \), and \( \Sigma(\mathbb{R}) = 1 \) (resp. \( \Sigma(\mathbb{T}) = 1 \)).

MOPRL
To each \( p \times p \) matrix-measure \( \Sigma \) we associate a pseudo-scalar products
\[
\langle\langle F, G \rangle\rangle_R = \int_{\mathbb{R}} F(z)^\dagger d\Sigma(z) G(z),
\]
a sequence of right monic matrix orthogonal polynomials, and then a sequence of matrix orthonormal polynomials. They satisfy a three-term recurrence relation
\[
x p_n^R(x) = p_{n+1}^R(x) A_{n+1}^\dagger + p_n^R(x) B_{n+1} + p_{n-1}^R(x) A_n
\] (2)
with \( B_k \) Hermitian and \( A_k \) non singular (\( p \times p \) matrices).
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To each $p \times p$ matrix-measure $\Sigma$ we associate a pseudo-scalar products

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a sequence of right monic matrix orthogonal polynomials, and then a sequence of matrix orthonormal polynomials. They satisfy a three-term recurrence relation

$$xp_n^R(x) = p_{n+1}^R(x) A_{n+1}^\dagger + p_n^R(x) B_{n+1} + p_{n-1}^R(x) A_n$$

with $B_k$ Hermitian and $A_k$ non singular ($p \times p$ matrices).
If \( M \) is self-adjoint \( N \times N \), the spectral theorem gives

\[
M = \int_{\mathbb{R}} \lambda dE_{\lambda}
\]

so that for every integer \( p \leq N \) there exists a unique \( p \times p \) matrix measure

\[
\Sigma = (\Sigma_{ij})_{1 \leq i, j \leq p}, \quad d\Sigma_{ij}(\lambda) = \langle dE_{\lambda} e_i, e_j \rangle \quad (1 \leq i, j \leq p), \quad k \in \mathbb{N},
\]

so that

\[
\langle M^k e_i, e_j \rangle = \int_{\mathbb{R}} x^k d\Sigma_{i,j}(x).
\]

Previously, \( M \) was chosen randomly in a classical ensemble of self-adjoint operators (Hermite, Laguerre, Jacobi).

\( \Rightarrow \) OPRL, random Jacobi coefficients

Actually

\[
\Sigma_p(dx) = \sum_{j=1}^{N} u_j u_j^\dagger \delta_{\lambda_j}
\]

(\( u_j \) is the jth \( p \)-truncated column of the unitary matrix diagonalizing \( M \)).
Properties of the weights

Lemma

If $U$ is Haar distributed in $\mathbb{U}(N)$ then

$$(u_1u_1^\dagger, \ldots, u_Nu_N^\dagger) \overset{(d)}{=} (h^{-1/2}v_1v_1^\dagger h^{-1/2}, \ldots, h^{-1/2}v_Nv_N^\dagger h^{-1/2})$$

where the $v_j$'s are i.i.d. random gaussian vectors of size $p$ and

$$h = \sum_{j=1}^{N} v_jv_j^\dagger.$$
In the GUE or in the Laguerre ensemble, all the results on distributions are extended rather easily.

In the Jacobi ensemble, there is an issue.

▶ In the scalar case, as Killip and Nenciu (’05) we lift up the measure (supported by $n$ points) on $[0, 1]$ or $[-2, 2]$ into a symmetric measure (supported by $2n$ points) on $\mathbb{T}$, by inverse Szegő mapping. It is the spectral measure of an element of $SO(2n)$.

▶ In the matrix case ($p > 1$), an element of $SO(2np)$ has a spectral measure at $(e_1, \ldots, e_p)$ which is not symmetric. To compute the distribution of some random coefficients, we use the canonical moments approach.
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Bibliography

Spectral theory
B. Simon, *Szegö’s Theorem and Its Descendants* (2011)

Large deviations
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Bibliography

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Large deviations
Papers  (LD = large deviations, SR = sum rules)

F. Gamboa, J. Nagel, A. Rouault
▶ LD for random spectral measures and SR, *AMRX* (2011)
▶ Operator-valued spectral measures and LD, *JSPI* (2014)
▶ SR and LD for spectral measures on the unit circle, *Random Matrices Th. and Appl.* (2017)

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From Barry Simon’s book "Szegő’s Theorem and Its Descendants", Chapter 2:

In algebra, when one says $a = b$, it is a tautology and so uninteresting; while in analysis, when one says $a = b$, it is two deep inequalities. (attributed to S. Bochner)

If one only proves $a = b$ by showing $a \leq b$ and $b \leq a$, one has not understood the true reason why $a = b$. (attributed to E. Noether)

Our contribution could be:

If $a$ and $b$ are two positive functionals, when one says $a = b$, a probabilist may think that $a$ and $b$ could be two rate functions of the same large deviation principle under two different encodings.
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THANK YOU FOR YOUR ATTENTION!