Resonances of large random samples

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The general picture

On $L^2(\mathbb{R}^d)$, consider $V_\omega$ a bounded random potential and the operator

$$H_\omega = -\Delta + V_\omega.$$ 

Well known spectral theory for $H_\omega$ (under assumptions on $V_\omega$):
- almost sure spectrum;
- edge of spectrum are localized (only (dense) eigenvalues);
- associated eigenvectors decay exponentially.

Large sample: pick $L \in \mathbb{N}$, $L \gg 1$ and set $H_{\omega,L} = -\Delta + V_\omega 1_{|x| \leq L}$.

Relatively compact perturbation of $-\Delta$:
- $\sigma_{\text{ess}}(H_{\omega,L}) = \sigma(-\Delta) = [0, +\infty)$;
- outside $[0, +\infty)$, $H_{\omega,L}$ only discrete eigenvalues.

Theorem

*The operator valued function $z \in \mathbb{C}^+ \mapsto (z - H_{\omega,L})^{-1}$ admits a meromorphic continuation from $\mathbb{C}^+$ to $\mathbb{C} \setminus (-\infty, 0]$ with values in the operators from $L^2_{\text{comp}}$ to $L^2_{\text{loc}}$.***
**Resonances**: poles of analytic continuation of resolvent of $H_{\omega,L}$, associated to finite dimensional resonant subspaces.

**Resonance width**: imaginary part of the pole.

**Well known**: resonance widths give “large” time behavior of $e^{-itH_{\omega,L}}$; smallest widths give leading order contribution.

**Goal**: “compute” the resonances: relate them (distribution, distribution of width, etc) to the spectral characteristics of $H_{\omega} = -\Delta + V_{\omega}$.

**Resonance free regions**

**Theorem (K.-Vogel)**

Fix $I$ a compact interval $(0, +\infty)$. Consider $V_h$ real valued bounded with support in fixed compact.

For some $C > 0$, on $\mathbb{R}^d$, for $h$ small enough, $-h^2 \Delta + V_h$ has no resonance in

$$\{\text{Re} z \in I, \text{Im} z \geq -e^{-Ch^{4/3}\log h}\}.$$ 

**The random case:**

**Corollary (K.-Vogel)**

Let $I$ be a compact interval $(0, +\infty)$. Then, for some $C > 0$ s.t., for $L$ large, $\omega$-a.s., one has no resonance in

$$\{\text{Re} z \in I, \text{Im} z \geq -e^{-CL^{4/3}\log L}\}.$$ 

**Description of the resonances closest to the real axis:**

Assumptions on $H_{\omega} = -\Delta + V_{\omega}$: pick $I$ compact interval in $(0, +\infty)$

- 1AD: $(H_{\omega})|_{\Lambda}$ and $(H_{\omega})|_{\Lambda'}$ independent if $d(\Lambda, \Lambda')$ large.
- Density of states: $N(E) := \lim_{L \to \infty} \frac{\text{tr}([1]_{-E,E}((H_{\omega})|_{\Lambda_L}))}{|\Lambda_L|}$ is continuous on $I$.
- Localization: for all $\xi \in (0, 1)$, one has

$$\sup_{L > 0} \sup_{\text{supp} f \subset I} \sup_{|f| \leq 1} \mathbb{E} \left( \sum_{\gamma \in \mathbb{Z}^d} e^{\gamma\xi} \|1_{\Lambda(0)}f((H_{\omega})|_{\Lambda_L})1_{\Lambda(\gamma)}\|_2 \right) < +\infty.$$ 

- Wegner: $\sup_E \mathbb{E}(\text{tr}([1]_{E,E+\epsilon}((H_{\omega})|_{\Lambda_L}))) \leq C e^{\alpha L^{\beta}}$.

**Theorem (K.-Vogel)**

Let $I$ be a compact interval $(0, +\infty)$ where assumptions satisfied. Then, $\omega$-a.s., for any $\kappa \in (0, 1)$,

$$\frac{1}{L^d} \# \left\{ z \text{ resonance of } H_{\omega,L} \text{ s.t. } \text{Re} z \in I, \text{Im} z \geq -e^{-L^\kappa} \right\} \to \int_I dN(E).$$
In dimension one

Consider

- $x \mapsto W(x)$, a 1-periodic potential,
- an Anderson type random potential $x \mapsto V_{\omega}(x) = \sum \omega_n v(x - n)$ with
  - $(\omega_n)_{n \in \mathbb{Z}}$ bdd i.i.d. real valued random variables with bdd density,
  - $v : \mathbb{R} \to \mathbb{R}^+$ cont. supported in $[0, 1]$.

Set $H_\omega = -\Delta + W + V_{\omega}$ on $L^2(\mathbb{R})$:
- let $\Sigma$ be almost sure spectrum of $-\Delta + W + V_{\omega}$ [Pastur];
- Anderson localization: only simple eigenvalues associated to exponentially decaying eigenfunctions [Damanik-Stolz, Kunz-Souillard, etc].

Define $\rho(E)$ be the Lyapunov exponent at energy $E$:

$$
\rho(E) := \lim_{x \to +\infty} \frac{\log(|u'(x)| + |u(x)|)}{x}
$$

where $-u'' + V_{\omega}u = Eu, u(0) = 0, u'(0) = 1$

The model: study resonances of $H_{\omega,L} = -\Delta + W + V_{\omega}1_{-L \leq x \leq L}$

Model (discrete analog) was studied:
- in physics: Titov - Fyodorov, Texier - Combet, etc on $l^2(\mathbb{N})$
- in mathematics: Kunz - Shapiro: resonances far away from the real axis when $L = +\infty$ (i.e. rand. pot. fills half-axis); Klopp.

Resonance free regions

**Theorem**

Let $I$ be a compact interval in $\Sigma_W := \sigma(-\Delta + W)$. Then, $\omega$-a.s., one has

- if $I \subset \mathbb{R} \setminus \Sigma$, $\exists C > 0$ s.t., for $L$ large, no resonance in $\{\text{Re}z \in I, \text{Im}z \geq -1/C\}$;
- for $I \subset \Sigma$, set $\rho_I := \max_{E \in I} \rho(E)$;
  for $L$ large, no resonance in $\{\text{Re}z \in I, \text{Im}z \geq -e^{-2\rho_I L(1+o(1))}\}$.

Description of the resonances closest to the real axis

**Theorem**

Let $I$ be a compact interval in $\Sigma_W \cap \Sigma$. Then, $\omega$-a.s.,

- for any $\kappa \in (0, 1)$,
  $$
  \frac{1}{L} \# \left\{ z \text{ resonance of } H_{\omega,L} \text{ s.t. } \text{Re}z \in I, \text{Im}z \geq -e^{-L^k} \right\} \to \int_I dN(E);
  $$
- for $c > 0$, $\omega$-a.s., one has
  $$
  \lim_{L \to +\infty} \frac{1}{L} \# \left\{ z \text{ resonances in } I + i(-\infty, -e^{-2cL}] \right\} = \int_I \min\left( \frac{c}{\rho(E)}, 1 \right) N(E) dE
  $$
The local behavior of the resonances
Let \( n(E) \) denote the density of states of \( H_\omega \) at energy \( E \) i.e. \( n(E) = N'(E) \) (exists a.e.).

Fix \( E_0 \in (0, +\infty) \cap \Sigma \) s.t. \( n(E_0) > 0 \).

Fix increasing sequence of scales \((\ell_L)_{L \to +\infty} \) s.t. \( \ell_L \leq 2L \) and \( \frac{\ell_L}{\log L} \to +\infty \).

**Rescaling the resonances:** for \((z^*_L(\omega))_i\) resonances of \(H_{\omega,L}\), define

\[
x_j = x^*_L(\omega) = (\text{Re} z^*_L(\omega) - E_0) \ell_L \quad \text{and} \quad y_j = y^*_L(\omega) = -\frac{1}{\ell_L} \log |\text{Im} z^*_L(\omega)|.
\]

Consider the two-dimensional point process \( \xi_L(\omega) = \sum \delta(x_j, y_j) \).

**Theorem**

*The point process \( \xi_L \) converges weakly to a Poisson process in \( \mathbb{R} \times [0, 1] \) with density \( n(E_0) \rho(E_0) \, dx \, dy \).*

Rescaled resonances:

The periodic case

Consider \( x \mapsto W(x) \) and \( x \mapsto V(x) \) \( 1 \)-periodic potentials.

Set \( H = -\Delta + W + V \) on \( L^2(\mathbb{R}) \).

Spectrum of \( -\Delta + W + V \): \( \Sigma \) is purely absolutely continuous spectrum [Floquet, Bloch, Gelfand, etc].

The model: study resonances of \( H_L = -\Delta + W + V1_{|x| \leq L} \).

Resonance free regions:

**Theorem**

*Let \( I \) be a compact interval in \( \sigma(-\Delta + W) \). Then,*

- if \( I \subset \mathbb{R} \setminus \Sigma \), then, there exists \( C > 0 \) such that, for \( L \) suff. large, no resonances in \( \{ \text{Re} z \in I, \text{Im} z \geq -1/C \} \);

- if \( I \subset \Sigma \), then, there exists \( C > 0 \) such that, for \( L \) suff. large, no resonances in \( \{ \text{Re} z \in I, \text{Im} z \geq -1/(CL) \} \);
Description of the resonances near the real axis

**Theorem**

There exist

- $\mathcal{D}$, a discrete (possibly empty) set of energies in $\sigma(-\Delta + W) \cap \Sigma$,
- $h$ real analytic and not vanishing on $\sigma(-\Delta + W) \setminus \mathcal{D}$

s.t., for $I \subset \Sigma \cap \sigma(-\Delta + W) \setminus \mathcal{D}$ compact interval, $\exists c_0 > 0$ s.t. for $L \in \mathbb{N}$ suff. large, one has

- many resonances in $I + i[-c_0, 0]$; more precisely,

$$\frac{\#\{z \in I + i[-c_0, 0], \ z \text{ resonance of } H_L\}}{2L} = \int_I dN(E) + o(1) \quad (1.1)$$

where $o(1) \to 0$ as $L \to +\infty$;

- let $(z_j)_j$ be resonances of $H_L$ in $I + i[-c_0, 0]$ ordered by increasing real part; then,

$$L \cdot \text{Re}(z_{j+1} - z_j) \asymp 1 \quad \text{and} \quad L \cdot \text{Im}z_j = h(\text{Re}z_j) + o(1), \quad (1.2)$$

estimate uniform for all resonances in $I + i[-c_0, 0]$.

- Rescaled resonances accumulate on a real analytic curve;
- Local (linear) density of resonances given by density of states of $H$.

**Figure:** Rescaled resonances away from $\mathcal{D}$.

**Figure:** Rescaled resonances near $\mathcal{D}$.