Eigenvalue separation for quantum systems with discrete disorder

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Degeneracies (or near-degeneracies) create difficulties in understanding the behavior of quantum systems. In the case of disordered quantum systems such as the Anderson model, degeneracies provide avenues for long-range tunneling, and hence are a barrier to localization. I will discuss a method (the energy-following procedure) for tracking eigenvalues of the Anderson model Hamiltonian through a sequence of successive approximations. The rate at which eigenvalues separate in this procedure controls the statistics of separation of nearby eigenvalues. The method is a key part of the proof of localization for discrete disorder distributions. Reference: arXiv:1705:01916
Schur complement (or Feshbach map)

Let $H$ be a symmetric matrix in block form, $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $C = B^T$. Assume that $\|D - E\|^{-1} \leq \varepsilon^{-1}$, $\|B\| \leq \gamma$, $\|C\| \leq \gamma$. The Schur complement at energy $\lambda$ is:

$$F_\lambda \equiv A - B(D - \lambda)^{-1}C.$$

Let $\varepsilon$ and $\gamma/\varepsilon$ be small, and $|\lambda - E| \leq \varepsilon/2$. Then

(i) If $(F_\lambda - \lambda)\varphi = 0$, then $(H - \lambda)\begin{pmatrix} \varphi \\ -(D - \lambda)^{-1}C\varphi \end{pmatrix} = 0$, and all eigenvectors of $H$ with eigenvalue $\lambda$ are of this form.

(ii) $\|F_\lambda - F_E\| \leq 2\left(\frac{\gamma}{\varepsilon}\right)^2|\lambda - E|.$

(iii) The spectrum of $H$ in $[E - \varepsilon/2, E + \varepsilon/2]$ is in close agreement with that of $F_E$ in the following sense. If $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ are the eigenvalues of $H$ in $[E - \varepsilon/2, E + \varepsilon/2]$, then there are corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ of $F_E$, and $|\lambda_i - \lambda_i| \leq 2(\gamma/\varepsilon)^2|\lambda_i - E|$. 
Utility of the Schur complement

With the Schur complement, we are able to reduce the eigenvalue problem for a large matrix into an equivalent (and hopefully more tractable) one for a smaller matrix.

If one focuses on a window of energies in some neighborhood of $E$, the submatrix $D$ is gapped, in the sense that $\|(D - \lambda)^{-1}\| \leq 2/\varepsilon$ for $|\lambda - E| \leq \varepsilon/2$.

The degrees of freedom associated with $D$ are “eliminated,” and one can work with a “renormalized” operator $F_\lambda$ for $\lambda$ near $E$.

The fact that $F_\lambda$ depends on $\lambda$ means that we cannot find all of the eigenvalues in the interval by looking at one matrix. However, (ii) guarantees that the $\lambda$-dependence is mild, and then the closeness of the spectra of $H$ and that of $F_E$ is guaranteed by (iii).

In particular, if one finds a gap in the spectrum of $F_\lambda$ in the interval, then a similar gap occurs for $H$. This is a way to demonstrate non-degeneracy in the spectrum.
Application of iterated Schur complements to random Schrödinger

Consider the Anderson model in $\mathbb{Z}^d$:

$$H = -\gamma \Delta + v \text{ with } \gamma \ll 1.$$ 

The potentials $\{v_x\}_{x \in \mathbb{Z}^d}$ are iid random variables. We are particularly interested in the case of a discrete distribution; specifically we may give each $v_x$ a uniform distribution on $\{0, \frac{1}{N-1}, \frac{2}{N-1}, \ldots, 1\}$, with $N \gg 1$.

For $N = 2$, this is the Anderson-Bernoulli model, the “Ising model” of random Schrödinger operators.

We are interested in showing exponential decay of eigenfunctions (localization). We would also like to control the probability of degeneracies or near-degeneracies in the spectrum of $H$.

Related work: Bourgain, Kenig, Invent. Math. 2005: Localization for the continuum version of Anderson-Bernoulli ($N = 2, \mathbb{Z}^d \to \mathbb{R}^d$) for energies near the bottom of the spectrum, $E = 0$. 
Large $N$ result

Let $I_\delta(E)$ denote the interval $[E - \delta, E + \delta]$, and let $\mathcal{N}(I)$ denote the number of eigenvalues of $H$ in $I$.

**Theorem.** Choose a sufficiently large $p$. Then for $N$ sufficiently large (depending on $p$) and $\gamma$ sufficiently small (depending on $N$),

$$\mathbb{E} \mathcal{N}(I_\delta(E)) \leq |\Lambda|(\log \gamma \delta)^{-p},$$

and

$$P\left( \min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta \right) \leq |\Lambda|^2(\log \gamma \delta)^{-p}.$$

for any rectangle $\Lambda$ and any $\delta \in [\gamma \text{diam}(\Lambda)/2, 1]$.

Also, $\sum_\alpha |\psi_\alpha(x)\psi_\alpha(y)|$ decays exponentially, and its disorder average is bounded by $C|x - y|^{-p}$. 
Resonant set and block decomposition of $H$

Given an energy $E$, we expect eigenfunctions whose eigenvalues are close to $E$ will be localized near sites where $v_x - E$ is small. Eigenfunctions should decay rapidly away from such sites.

**Resonant Set:** Let $\varepsilon = 1/N$ and $\gamma \leq \varepsilon^{20}$. The resonant step for the first step is $R^{(1)} = \{i \in \Lambda : |v_i - E| \leq \varepsilon\}$. The nonresonant set is $R^{(1)c} = \Lambda \setminus R^{(1)}$.

**Block decomposition:** $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

where $A$, $D$ are $H$ restricted to $R^{(1)}$, $R^{(1)c}$, respectively, and $B$, $C$ contain the nearest-neighbor terms from the Laplacian that connect the two regions.

As $D$ operates on the subspace associated with nonresonant sites, where $|v_i - E| \leq \varepsilon$, one has that $\| (D - \lambda)^{-1} \| \leq 3/\varepsilon$ for $|\lambda - E| \leq \varepsilon/2$.

**Consequence:** If $\psi$ is an eigenvector of $H$ with eigenvalue $\lambda \in I_{\varepsilon/2}(E)$ then $\psi = \begin{pmatrix} \varphi \\ -(D - \lambda)^{-1}\varphi \end{pmatrix}$ with $\varphi$ an eigenvector of $F_\lambda = A - B(D - \lambda)^{-1}C$.

Also, $(D - \lambda)^{-1}$ decays exponentially, so $\psi(x)$ decays rapidly with $\text{dist}(x, R^{(1)})$. 
Connected components of the resonant set

1. Decompose the resonant set $R^{(1)}$ into connected blocks or supersites $B$ using proximity conditions.
Accurate diagonalization as in quasidegenerate perturbation theory

2. Obtain a very accurate approximation to the energy levels associated with each block $B$. We need to do much better than the values of $v_i$ for $i \in B$ because in the next step, the energy window will be shrunk considerably.

To do this, let $\bar{B}$ be a neighborhood of $B$ of width $L_1$. Here we introduce a sequence of length scales $L_k = L_0 2^k$ with $L_0$ a moderately large number. Then replace:

$$F_{\lambda} \rightarrow \tilde{F}_{\lambda}^{(1)} = A - B(D - \lambda)^{-1}C$$

where the matrix $D$ is now restricted to $\bar{B} \setminus B$, instead of $\Lambda \setminus B$. In terms of a random-walk expansion for $(D - \lambda)^{-1}(x, y)$, take only walks that remain inside $\bar{B}$.

Then we have that $F_{\lambda} = \bigoplus_{\alpha} \tilde{F}(B_{\alpha}) + O(\gamma^{2L_1})$.

This is analogous to the statement $H = \text{diag}(v_1, \ldots, v_{|\Lambda|}) + O(\gamma)$.

Weyl’s inequality: spectrum moves no more than the norm of the perturbation.
Recall the length scales $L_k = L_0 2^k$.

Energy window width in step $k$: $\varepsilon_k = \gamma^{L_k}$.

Resonant set $R^{(k)}$ consists of blocks still resonant to $E$ within $\varepsilon_k$.

New proximity conditions: $k$th-step blocks separated by a distance $L_k^{3/2}$.

$k$th-step block $B_k$ has a neighborhood $\bar{B}_k$ of width $L_k$ (to approximate its spectrum and determine if it remains resonant).

Block decomposition of the $(k-1)^{st}$ Schur complement leads to the $k$th Schur complement:

$$F_{\lambda}^{(k-1)} = \begin{pmatrix} A^{(k)} & B^{(k)} \\ C^{(k)} & D^{(k)} \end{pmatrix},$$

$$F_{\lambda}^{(k)} = A^{(k)} - B^{(k)}(D^{(k)} - \lambda)^{-1} C^{(k)}.$$
Energy-following procedure

Define a sequence of energies $E_1, E_2, \ldots$.

$E_1 = v_i$ for some $i \in \Lambda$.

The site $i$ joins up with other nearby sites $j$ such that $|v_j - E_1| \leq \varepsilon_1$.

The result is the block $B_1$ containing $i$.

Perform the Schur complement and localize to the neighborhood $\bar{B}_1$.

Find the eigenvalues of $H_{\bar{B}_1}$ in the window $I_{\varepsilon_1/2}(E_1)$ by solving $\lambda \in \text{spec } \tilde{\mathcal{F}}^{(1)}_{\lambda}(B_1)$.

These are the choices for $E_2$ (group choices if within $\varepsilon_2$).
Energy-following procedure

\[ \sum \text{over choices for } E_{k+1} \text{ so that } E_{k+1} - \text{neighborhoods cover spectrum} \]
Blocks may combine as proximity conditions extended

If a nearby block has spectrum within $\varepsilon_2$ of $E_2$, then the blocks combine to form $B_2$. In $H_{B_2}$, the spectra are combined and perturbed by interactions (tunneling effects). The procedure continues by following groups of eigenvalues in $\varepsilon_3$-neighborhoods.

\[
\begin{pmatrix}
- & - & - \\
- & - & - \\
- & - & - \\
\end{pmatrix}
\]

Spectra combine

and split up in subsequent steps
Controlling the energy-following procedure

One can show that every eigenvalue of $H$ can be obtained through a sequence of choices $E_1, E_2, \ldots$ arising in this procedure.

The tree of choices for $\{E_k\}$ and the possibilities for the expanding sequence of blocks $\{B_k\}$ are controlled inside the expectation $\mathbb{E}$ over the potentials.

The key estimate is the small probability of remaining resonant to $E_k$ when expanding the neighborhood $\bar{B}_k \to \bar{B}_{k+1}$.

This leads to control of quantities such as the expected number of eigenvalues that have another eigenvalue within $\delta$.

By following eigenvalues in a controllable way, we avoid blind search as in the Klein-Molchanov proof of simplicity of the spectrum (which uses Minami to produce level-separation, but only works for sufficiently regular potential distributions).
Estimates for the multi-scale percolation problem

To obtain localization, we need to demonstrate that the resonant blocks (percolation clusters) do not percolate. Connections between blocks are made at a distance $L_k^{3/2}$, but a compensating effect is the fact that blocks become less and less likely to remain resonant.

Roughly speaking, the expected size of percolation clusters will stay bounded uniformly in $k$ if the probability of a block remaining resonant is bounded by $L_k^{-p}$ with $p > d$.

For a continuous distribution of potentials, the probability would go like $\varepsilon_k = \gamma^{L_k}$, by the Wegner estimate.

For a discrete distribution, narrowing the spectral window does not automatically lead to a reduction in probability.

Therefore, we need to demonstrate eigenvalue movement and separation (degeneracy breaking) based on the effect of potentials in the annuli $\bar{B}_k \setminus \bar{B}_{k-1}$. 
Eigenfunctions of $-\gamma \Delta + v$ cannot grow faster than exponentially on $\mathbb{Z}^d$

Kagome lattice: Compactly supported e-fns

$\mathbb{Z}^d$: normalized e-fn $\geq \gamma^r$ somewhere on $\partial B_r$
This implies the existence of influential sites where an eigenfunction has a lower bound $\sim \gamma^{L_k}$

Use randomness at one site in each annulus of radius $L_k$: the most influential one.

It splits an isolated eigenvalue into $N$ values with spacing $\sim \gamma^{L_k}$.

The probability that the eigenvalue sits in an interval of size $\gamma^{L_k}$ is $N^{-k} = (2^{-k})^{\log_2 N} \sim L_k^{-p}$, where $p = \log_2 N$. This is summable decay for $p > d$, which means resonances don’t percolate.

If not isolated: the potential drives a rank 1 perturbation, which splits an $n$-fold (near) degeneracy to an $(n-1)$-fold degeneracy, with probability $\gtrsim 1 - 1/N$. 
Result: Cantor-like splitting of eigenvalue distribution on a logarithmic length scale

This leads to log-Hölder continuity of the density of states:

$$\mathbb{E} \mathcal{N}(I_{\delta}(E)) \leq |\Lambda| (\log \gamma \delta)^{-p}.$$ 

Large $p$, in particular $p > d \implies$ separation of $\delta$ resonances by a distance $L \sim (\log_{\gamma} \delta)^{p/d}$, so $\gamma^L \ll \delta$.

Thus the interaction strength is typically much smaller than $\delta$, which implies localization.
Unique continuation in $\mathbb{R}^d$

**Usual unique continuation principle:**
If $\psi(x)$ is a harmonic function that vanishes on an open set, then $\psi(x) \equiv 0$ everywhere.

**Quantitative unique continuation** (Bourgain-Kenig 2005):
If $\psi(x)$ is an eigenfunction of $-\Delta + \nu$ then

$$\int_{B_1(y)} |\psi(x)|^2 dx \geq \exp(-|y|^{4/3} \log |y|).$$

The possibility of faster-than-exponential growth is realized for complex $\nu$ examples (Meshkov 1992).

Longstanding conjecture by Landis (from the 60’s) that this does not happen for real $\nu$. 
Weak form of unique continuation in $\mathbb{Z}^d$ for solutions to $(-\Delta + \nu)\psi = 0$

Although eigenfunctions cannot decay uniformly faster than exponentially on the lattice, they can vanish on large sets.

Consider, for example, this solution to $(-\Delta - 4)\psi = 0$:

$$
\begin{align*}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{align*}
$$

In a box $X$, $|\psi_X(x)| \geq \exp(-c|z|)$ for at least one $x \in \partial X$ in a $45^\circ$ “light cone” of $R$ (where $\psi$ is known to be nonzero).
Some recent results in dimension 2


Localization near the edge for Anderson-Bernoulli in $\mathbb{Z}^2$: Ding, Smart, arXiv:1809.09041.
H = −Δ + ν with ν ∈ {0, 1} on ℤ^d.

A plethora of degeneracies have the potential for spoiling localization.

To combat the problem, take E > 0 small (Lifshitz tail regime).

To get spectrum near E one needs a region of size ℓ_0 ∼ E^{−1/2} with mostly 0’s.

Large deviation arguments \implies probability \sim \exp(−E^{d/2}) \implies separation of resonant spots as in the large N case.
Issues with $N = 2$

1. Recall that in the large $N$ case, we used the potential at one site per annulus (because of the “Cantor” condition: we need the upper bound for next-generation effects $\gamma^85L_{k+1} = \gamma^{1.7L_k}$ to fit inside the lower bound on the current effect $\gamma^{L_k}$.)

We reach length scale $L_k$ after $k$ eigenvalue splittings (using the potential at one site per annulus), and then each atom has probability $\sim N^{-k} = 2^{-kp} = L_k^{-p}$, where $p = \log_2 N$.

To prevent percolation of resonances we need $p > d$. But $L_k^{-p}$ is no longer integrable when $N = 2$.

So we need to find a way to use many sites per annulus.

2. Decay length is many lattice steps $\Rightarrow$ it is difficult to pin down the location of a site where the eigenfunction is guaranteed to be big (the location should not depend on the randomness in the “new” annulus).
To be continued...