Forward-Backward random process for the spectrum of 1D Anderson model (another proof and quantitative results)

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Outline

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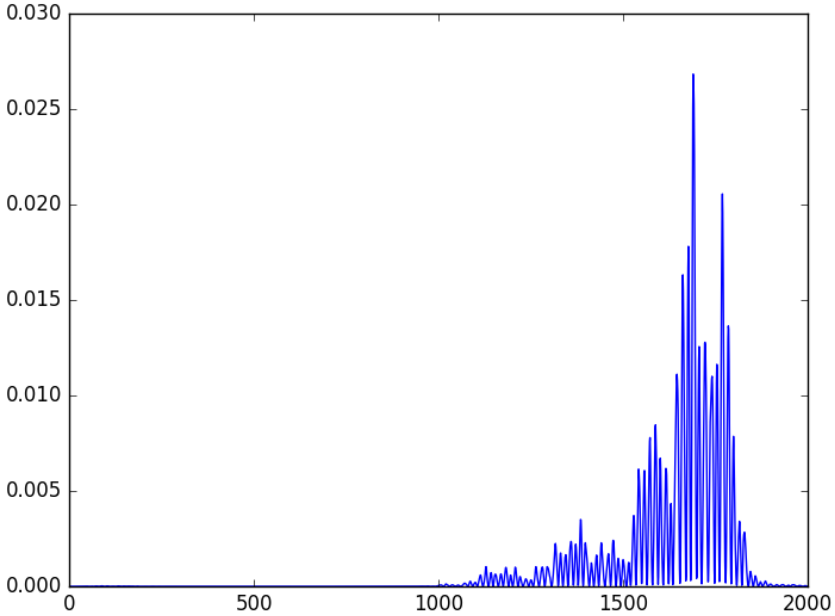
One dimensional critical Anderson model

The critical 1D Anderson model

On $[0, N]$

$$H^N = -\Delta + \frac{1}{\sqrt{N}} V_i^N$$

where V_i^N arer iid of same law as V with $\mathbb{E}(V^2) = \sigma^2$



One dimensional critical Anderson model

$$S(x) = \exp(B_x - \frac{1}{2}|x|)$$

We pick randomly an eigenvalue $\mu^N \in \sigma(H^N)$ and write $N|\psi_{\mu^N}(\lfloor Nt \rfloor)|^2$ the corresponding rescaled eigenvector.

Theorem [Rifkind,Virag]

As $N \to \infty$, we have the convergence in law

$$(\mu^N, N|\psi_{\mu^N}(\lfloor Nt \rfloor)|^2) \rightharpoonup (\mu, S(\tau(\mu)[x-u]))$$

where μ a random variable on $[-2,2]$ with density $\rho(\mu) = \frac{1}{\sqrt{1-\frac{\mu^2}{4}}}$, u is uniform on $[0,1]$ and S as defined above (all the three are independent). And $\tau(\mu) = \frac{\sigma^2}{2(1-\frac{\mu^2}{4})}$.

Outline

The set up (the non critical case).

the model

On $[1, N] \cap \mathbb{Z}$

$$H = -\Delta + V_\omega$$

with V_ω iid random variable with a continuous law.

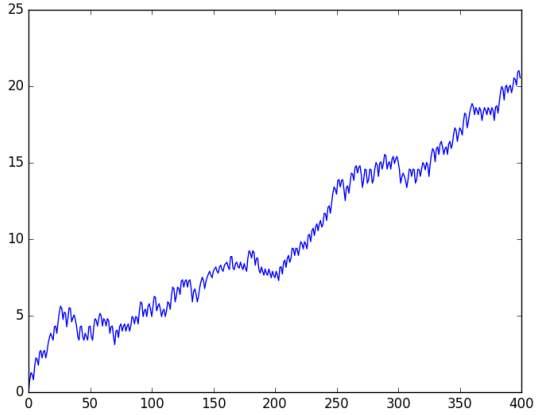
The function u_n is an eigenvector of H with eigenvalue λ iff

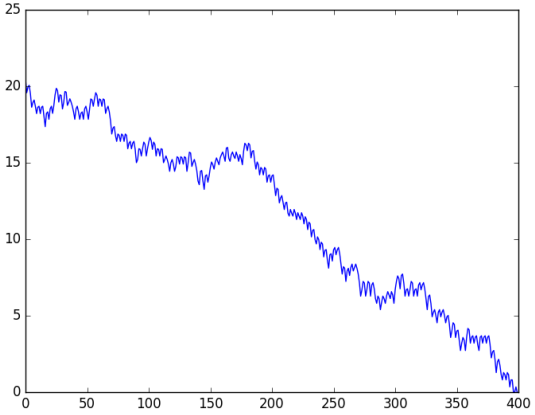
$$\begin{aligned}
& u_{n+1} = (v_n(\omega) - \lambda) u_n - u_{n-1} \\
& \Leftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} v_n(\omega) - \lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \\
& \Leftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \prod_{k=1}^{n} T_\lambda(v_k(\omega)) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix},
\end{aligned}$$

Three natural random processes : the forward product of random matrices, the backward product of random matrices and the construction of the eigenvector.

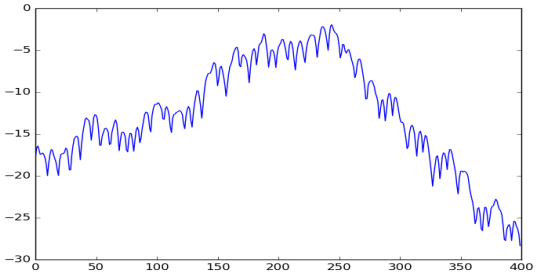
Random processes

$$\log(|u_{n+1}|^2 + |u_n|^2) \approx \gamma(\lambda)n + \sigma(\lambda)B_n$$





$\log(|u_{n+1}|^2 + |u_n|^2)$ of the eigenvector



A forward-backward random process

Notation : $z_n = u_{n+1} + iu_n = r_n e^{i\phi_n}$
$\mathscr{P}_{f,1..k}$: the forward product of matrices up to k
$\mathscr{P}_{b,k+1,\dots,N}$: the backward product of matrices up to k.
$X = (\phi_0, \phi_1, \cdots, \phi_N)$.

Theorem [RD]: Law of the form of the eigenvector

For any test function $G(\lambda, X)$, we have

$$\mathbb{E}\Big[\sum_{\lambda\in\sigma(H), X=\mathscr{P}h(\lambda)} G(\lambda, X)\Big]$$
$$= \int_{\mathbb{R}} d\lambda \sum_{k=1}^{N} \mathbb{E}_{\mathscr{P}_{f,1..k}\otimes\mathscr{P}_{b,k+1,\dots,N}}\Big[G(\lambda, X)\delta_{\phi_k^f - \phi_k^b}\sin^2(\phi_k^f)\Big] \quad (1)$$

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Proof : calculations...

$\theta_N \equiv \phi_N[2\pi]$.
λ is a eigenvalue $\Leftrightarrow \phi_N(\lambda,(V_\omega)) = \frac{\pi}{2}$

$$\mathbb{E}\Big[\sum_{\lambda\in\sigma(H),X=\mathscr{P}h(\lambda)} G(\lambda,X)\Big]$$
$$= \lim_{\varepsilon\to 0}\mathbb{E}\Big[\frac{1}{2\varepsilon}\sum_{n\in\mathbb{Z}}\int_{\pi n+\pi/2-\varepsilon}^{\pi n+\pi/2+\varepsilon}\sum_{\lambda:\theta_N(\lambda)=s} G(\lambda,X)ds\Big].$$
$$= \lim_{\varepsilon\to 0}\mathbb{E}\Big[\int_{\mathbb{R}} G(\lambda,X)\frac{1}{2\varepsilon}1_{\theta_N(\lambda)\in I_\varepsilon}|\frac{d\theta_N(\lambda)}{d\lambda}|d\lambda\Big]$$

where $I_\varepsilon = \frac{\pi}{2} + \cup_{n\in\mathbb{Z}}[\pi n-\varepsilon,\pi n+\varepsilon]$.

calculations...

$$\begin{aligned}\frac{d\theta_N(\lambda)}{d\lambda} &= \frac{d\phi_N(\lambda)}{d\lambda} \\ &= \frac{d}{d\lambda}\Big[\prod_{k=1}^{N} T(v_\omega(k)-\lambda)\phi_0\Big] \\ &= \sum_{k=1}^{N} \frac{d\phi_N}{d\phi_k}|_{v_\omega(N),\ldots,v_\omega(k+1)} \cdot \frac{d}{d\lambda}[T(v_\omega(k)-\lambda)](\phi_{k-1})\end{aligned}$$

And we have

$$\frac{d}{d\lambda}[T(V_\omega(k)-\lambda)](\phi_{k-1})\big] = \sin^2\phi_k.$$

calculations...

$$\sum_{k=1}^{N}\int_{\mathbb{R}} d\lambda\Big[\int\dots\int d\nu(v_1)\dots d\nu(v_n)G(\lambda,X)\frac{d\phi_N}{d\phi_k}\cdot\sin^2(\phi_k)\Big]\frac{1}{2\varepsilon}1_{\theta_N(\lambda)\in I_\varepsilon}$$

We artificially add a variable ϕ.

$$= \sum_{k=1}^{N}\int_{\mathbb{R}} d\lambda\Big[\int\dots\int d\nu(v_1)\dots d\nu(v_k)\int_{\mathbb{S}^1} d\phi\,\delta_{\phi_k}(\phi)$$

$$\int\dots\int d\nu(v_{k+1})\dots d\nu(v_N)G(\lambda,X)\frac{d\phi_N}{d\phi}\cdot\sin^2(\phi_k)\Big]\frac{1}{2\varepsilon}1_{\theta_N(\lambda)\in I_\varepsilon}$$

$$= \sum_{k=1}^{N}\int_{\mathbb{R}} d\lambda\,\big[\mathbb{E}_{\mathscr{P}_{f,1..k}\otimes\mathscr{P}^u_{b,k+1,\dots,N}}[G(\lambda,X)\delta_{\phi_k^f-\phi_k^b}\sin^2(\phi_k)\frac{1}{2\varepsilon}1_{\phi_N\in I_\varepsilon/\pi\mathbb{Z}}]$$

with $\mathscr{P}_{f,1..k}\otimes\mathscr{P}^u_{b,k+1,\dots,N}$ the forward-backward process with μ_b the uniform law on $\mathbb{S}^1$ and we can then conclude, by taking the limit $\frac{1}{2\varepsilon}1_{\phi_N\in I_\varepsilon/2\pi\mathbb{Z}}d\phi_N\to\delta_0$. .

calculations...

$$\sum_{k=1}^{N}\int_{\mathbb{R}} d\lambda \Big[\int \dots \int d\nu(v_1)\dots d\nu(v_n) G(\lambda, X)\frac{d\phi_N}{d\phi_k}\cdot \sin^2(\phi_k)\Big]\frac{1}{2\varepsilon}1_{\theta_N(\lambda)\in I_\varepsilon}$$

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Outline

Density of eigenvalues

$N(\lambda) =$ Expected number of eigenvalue smaller than λ.
$\mathscr{N}(\lambda)$ the integral density of states.

Corollary

With ρ_λ^k the law of ϕ_k (after k iteration of T_λ) we have

$$\frac{dN(\lambda)}{d\lambda} = \frac{1}{N}\sum_{k=1}^{N}\int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi)\rho_\lambda^k(\phi)\rho_\lambda^{N-k}\left(\frac{\pi}{2}-\phi\right)d\phi.$$

Corollary of the corollary

With ρ_λ the T_λ-invariant measure on $\mathbb{R}/2\pi\mathbb{Z}$, we have

$$\frac{d\mathscr{N}(\lambda)}{d\lambda} = \int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi)\rho_\lambda(\phi)\rho_\lambda\left(\frac{\pi}{2}-\phi\right)d\phi.$$

Outline

Limiting theorems for product of random matrices.

Let (M_i) iid random variables with a "nice random law".
Heuristic : the log of the norm behave as if it was the sum of iid random variables.

$$\log(\|\prod_{i=1}^{n} M_i x\|) = \log(\|M_n \frac{\prod_{i=1}^{n-1} M_i x}{\|\prod_{i=1}^{n-1} M_i x\|}\|) + \log(\|\prod_{i=1}^{n-1} M_i x\|)$$
$$= \sum_{k=1}^{n} \log\left(\|M_k(\frac{\prod_{i=1}^{k-1} M_i x}{\|\prod_{i=1}^{k-1} M_i x\|})\|\right)$$

Limit theorems for product of random matrices

- $\exists\gamma$ (Lyapunov exponant) such that $\frac{1}{n}\log(\|\prod_{i=1}^{n} M_i x\|) \to \gamma$
- $\exists\sigma$ such that $\frac{1}{\sqrt{n}}(\log(\|\prod_{i=1}^{n} M_i x\|) - \gamma n) \rightharpoonup G(0,\sigma^2)$
- With $S_t = \frac{1}{\sqrt{N}}(\log(\|\prod_{i=1}^{\lfloor Nt \rfloor} M_i x\| - \gamma N t) \rightharpoonup \sigma B_t$
- $\mathbb{P}(\log(\|\prod_{i=1}^{n} M_i x\|) - \gamma n > an) \sim e^{-nF(a)}$.

Product of random matrices, idea of the proof.

An equivalent to the Laplace transform

$$L_N(\alpha) = \mathbb{E}\big(e^{\alpha \log(\|\Pi_{i=1}^N M_i x\|))}\big)$$

An operator $A(\alpha) : \mathscr{C}^\beta(S^1) \to \mathscr{C}^\beta(S^1)$

$$[A(0)f](x) = \mathbb{E}(f(\frac{Mx}{\|Mx\|}))$$

$$[A(\alpha)f](x) = \mathbb{E}(e^{\alpha \log(\|Mx\|)} f(\frac{Mx}{\|Mx\|}))$$

$A(0)$ has a unique largest eigenvalue $(=1)$. We call $g(\alpha)$ the largest eigenvalue.

$$L_N(\alpha) = [A(\alpha)^N 1](x) \sim g(\alpha)^N$$

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Product of random matrices, idea of the proof.

- With α fixed, this gives the large deviation estimate.

$$\mathbb{P}(\log(\|\prod_{i=1}^{N} M_i x\|) > \nu N) \leq \frac{\mathbb{E}(e^{\alpha \log(\|\prod_{i=1}^{N} M_i x\|))})}{e^{\alpha \nu N}} \sim \left(\frac{g(\alpha)}{e^{\alpha \nu}}\right)^N$$

- With $\alpha = \frac{is}{\sqrt{N}}$. because $g(\alpha)$ is analytic around 0 :

$$\left(\frac{g(\alpha)}{e^{\gamma \alpha}}\right)^N = (1 - \sigma^2 \frac{s^2}{N} + O(\frac{s^3}{N^3}))^N \to e^{-\sigma^2 s^2}$$

which gives the central limit theorem.

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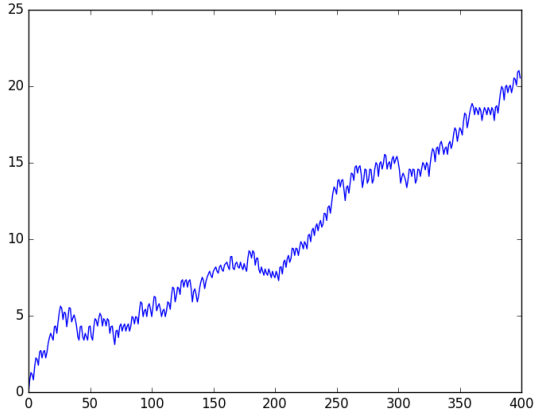
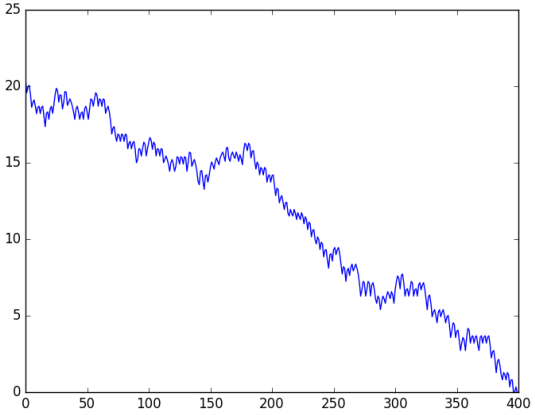
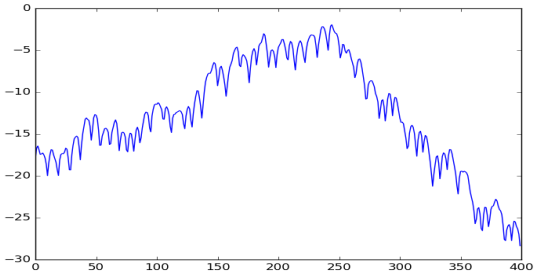
The random processes

$$\log(|u_{n+1}|^2 + |u_n|^2) \approx \gamma(\lambda)n + \sigma(\lambda)B_n$$





$\log(|u_{n+1}|^2 + |u_n|^2)$ of the eigenvector

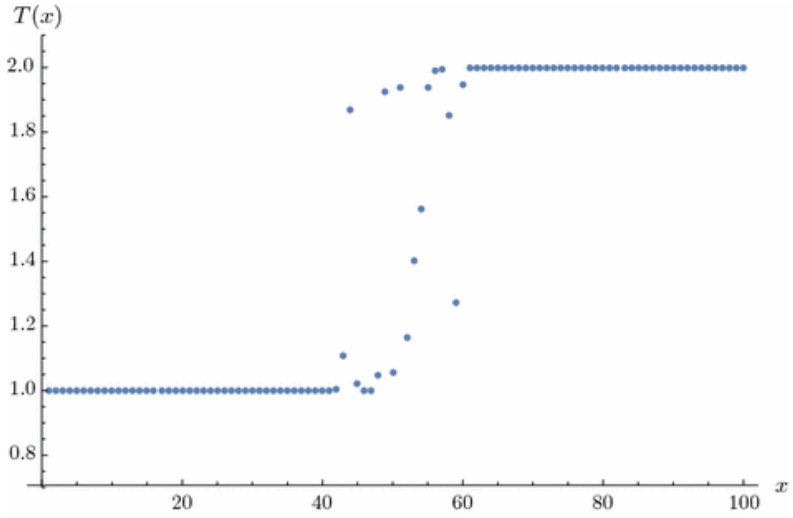


Outline

A temperature profile.

The model(Huveneer, De Roeck)

A disordered harmonic chain coupled at its ends to two thermal baths.



A temperature profile model

Temperature associated to each eigenvector

$$T(\psi_\lambda) = \frac{|\psi_\lambda(1)|^2 T_1 + |\psi_\lambda(N)|^2 T_N}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2}$$

At each point

$$T(x) = \sum_\lambda |\psi_\lambda(x)|^2 T(\psi_\lambda)$$

For N large and because of the exponential decay $T(\psi_\lambda) = T_1$ (resp. T_N) if $|\psi_\lambda(1)| > |\psi_\lambda(N)|$ (resp. $|\psi_\lambda(1)| < |\psi_\lambda(N)|$)

Limiting temperature profile

We have

$$\lim_{N\to\infty} \mathbb{E}\big[T(\lfloor \sqrt{N}x + \frac{N}{2} \rfloor)\big] = T_1 + (T_N - T_1) \int_{\mathbb{R}} \mathbb{P}\Big(\mathcal{N}(0,1) \leq \frac{2\gamma(\lambda)}{\sigma(\lambda)} x\Big) dN(\lambda)$$

where $dN(\lambda)$ is the integrated density of states, $\gamma(\lambda)$ the Lyapunov exponent and $\sigma(\lambda)$ the limit variance.

A temperature profile model

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The End

Thank you for your attention.