Forward-Backward random process for the spectrum of 1D Anderson model (another proof and quantitative results)

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1. The critical one dimensional Anderson model.

2. Forward-Backward process for the construction of the eigenvectors.

3. Application I: A formula for the integral density of eigenvalues

4. Application II: with the limit theorem for product of random matrices.

5. Application III: A temperature profile
1. The critical one dimensional Anderson model.

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One dimensional critical Anderson model

The critical 1D Anderson model

On $[0, N]$

$$H^N = -\Delta + \frac{1}{\sqrt{N}} V_i^N$$

where $V_i^N$ are iid of same law as $V$ with $\mathbb{E}(V^2) = \sigma^2$
One dimensional critical Anderson model

\[ S(x) = \exp(B_x - \frac{1}{2}|x|) \]

We pick randomly an eigenvalue \( \mu^N \in \sigma(H^N) \) and write \( N|\psi_{\mu^N}(\lfloor Nt \rfloor)|^2 \) the corresponding rescaled eigenvector.

**Theorem [Rifkind, Virag]**

As \( N \to \infty \), we have the convergence in law

\[ (\mu^N, N|\psi_{\mu^N}(\lfloor Nt \rfloor)|^2) \to (\mu, S(\tau(\mu)[x - u])) \]

where \( \mu \) a random variable on \([-2, 2]\) with density \( \rho(\mu) = \frac{1}{\sqrt{1 - \frac{\mu^2}{4}}} \), \( u \) is uniform on \([0, 1]\) and \( S \) as defined above (all the three are independent).

And \( \tau(\mu) = \frac{\sigma^2}{2(1 - \frac{\mu^2}{4})} \).
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The set up (the non critical case).

On \([1, N] \cap \mathbb{Z}\)

\[ H = -\Delta + V_\omega \]

with \(V_\omega\) iid random variable with a continuous law.

The function \(u_n\) is an eigenvector of \(H\) with eigenvalue \(\lambda\) iff

\[
\begin{align*}
 u_{n+1} &= (v_n(\omega) - \lambda)u_n - u_{n-1} \\
 \Leftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} &= \begin{pmatrix} v_n(\omega) - \lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \\
 \Leftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} &= \prod_{k=1}^{n} T_\lambda(v_k(\omega)) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix},
\end{align*}
\]

Three natural random processes: the forward product of random matrices, the backward product of random matrices and the construction of the eigenvector.
\[ \log(|u_{n+1}|^2 + |u_n|^2) \approx \gamma(\lambda)n + \sigma(\lambda)B_n \] of the eigenvector
A forward-backward random process

Notation: \( z_n = u_{n+1} + iu_n = r_ne^{i\phi_n} \)
\( P_{f,1..k} \): the forward product of matrices up to \( k \)
\( P_{b,k+1,...,N} \): the backward product of matrices up to \( k \).
\( X = (\phi_0, \phi_1, \cdots, \phi_N) \).

Theorem [RD]: Law of the form of the eigenvector

For any test function \( G(\lambda, X) \), we have

\[
\mathbb{E} \left[ \sum_{\lambda \in \sigma(H), X = P h(\lambda)} G(\lambda, X) \right] = \int_{\mathbb{R}} d\lambda \sum_{k=1}^{N} \mathbb{E} P_{f,1..k} \otimes P_{b,k+1,...,N} \left[ G(\lambda, X) \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k^f) \right] \tag{1}
\]
A forward-backward random process

Notation: $z_n = u_{n+1} + i u_n = r_n e^{i \phi_n}$
$\mathcal{P}_f, 1..k :$ the forward product of matrices up to $k$
$\mathcal{P}_b, k+1, ..., N :$ the backward product of matrices up to $k$
$X = (\phi_0, \phi_1, \cdots, \phi_N)$.

Theorem [RD]: Law of the form of the eigenvector

For any test function $G(\lambda, X)$, we have

$$
\mathbb{E} \left[ \sum_{\lambda \in \sigma(H), \mathcal{P}_f = \mathcal{P} \mathcal{h}(\lambda)} G(\lambda, X) \right] = \int_{\mathbb{R}} d\lambda \sum_{k=1}^{N} \mathbb{E} [\mathcal{P}_f, 1..k \otimes \mathcal{P}_b, k+1, ..., N] \left[ G(\lambda, X) \delta_{\phi_f^k - \phi_b^k} \sin^2(\phi_f^k) \right]$$

(1)
Proof: calculations...

\[ \theta_N \equiv \phi_N[2\pi]. \]

\[ \lambda \text{ is a eigenvalue } \iff \phi_N(\lambda, (V_\omega)) = \frac{\pi}{2} \]

\[ \mathbb{E} \left[ \sum_{\lambda \in \sigma(H), X = \mathcal{P} h(\lambda)} G(\lambda, X) \right] \]

\[ = \lim_{\varepsilon \to 0} \mathbb{E} \left[ \frac{1}{2\varepsilon} \sum_{n \in \mathbb{Z}} \int_{\pi n + \pi/2 - \varepsilon}^{\pi n + \pi/2 + \varepsilon} \sum_{\lambda : \theta_N(\lambda) = s} G(\lambda, X) \, ds \right]. \]

\[ = \lim_{\varepsilon \to 0} \mathbb{E} \left[ \int_{\mathbb{R}} G(\lambda, X) \frac{1}{2\varepsilon} 1_{\theta_N(\lambda) \in I_\varepsilon} \left| \frac{d\theta_N(\lambda)}{d\lambda} \right| \, d\lambda \right] \]

where \( I_\varepsilon = \frac{\pi}{2} + \bigcup_{n \in \mathbb{Z}} [\pi n - \varepsilon, \pi n + \varepsilon]. \)
\[ \frac{d\theta_N(\lambda)}{d\lambda} = \frac{d\phi_N(\lambda)}{d\lambda} \]

\[ = \frac{d}{d\lambda} \left[ \prod_{k=1}^{N} T(\nu_\omega(k) - \lambda) \phi_0 \right] \]

\[ = \sum_{k=1}^{N} \frac{d\phi_N}{d\phi_k} |_{\nu_\omega(N), \ldots, \nu_\omega(k+1)} \cdot \frac{d}{d\lambda} [T(\nu_\omega(k) - \lambda)](\phi_{k-1}) \]

And we have

\[ \frac{d}{d\lambda} [T(\nu_\omega(k) - \lambda)](\phi_{k-1}) = \sin^2 \phi_k. \]
\[
\sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int \ldots \int d\nu(v_1) \ldots d\nu(v_n) G(\lambda, X) \frac{d\phi_N}{d\phi_k} \cdot \sin^2(\phi_k) \right] \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in \mathbb{I}_{\epsilon}}
\]

We artificially add a variable \( \phi \).

\[
= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int \ldots \int d\nu(v_1) \ldots d\nu(v_k) \int_{\mathbb{S}^1} d\phi \delta_{\phi_k}(\phi) \int \ldots \int d\nu(v_{k+1}) \ldots d\nu(v_N) G(\lambda, X) \frac{d\phi_N}{d\phi} \cdot \sin^2(\phi_k) \right] \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in \mathbb{I}_{\epsilon}}
\]

\[
= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \mathbb{E} \mathcal{P}_{f,1\ldots,k} \otimes \mathcal{P}_{b,k+1,\ldots,N}^u \left[ G(\lambda, X) \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k) \frac{1}{2\epsilon} 1_{\phi_N \in \mathbb{I}_{\epsilon}/\pi \mathbb{Z}} \right] \right]
\]

with \( \mathcal{P}_{f,1\ldots,k} \otimes \mathcal{P}_{b,k+1,\ldots,N}^u \) the forward-backward process with \( \mu_b \) the uniform law on \( \mathbb{S}^1 \) and we can then conclude, by taking the limit

\[
\frac{1}{2\epsilon} 1_{\phi_N \in \mathbb{I}_{\epsilon}/2\pi \mathbb{Z}} d\phi_N \to \delta_0.
\]
\[
\sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int ... \int d\nu(v_1)...d\nu(v_n) G(\lambda, X) \frac{d\phi_N}{d\phi_k} \cdot \sin^2(\phi_k) \right] \frac{1}{2\varepsilon} 1_{\theta_N(\lambda) \in I_{\varepsilon}}
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We artificially add a variable \( \phi \).

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= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int ... \int d\nu(v_1)...d\nu(v_k) \int_{\mathbb{S}^1} d\phi \delta_{\phi_k}(\phi) \int ... \int d\nu(v_{k+1})...d\nu(v_N) G(\lambda, X) \frac{d\phi_N}{d\phi} \cdot \sin^2(\phi_k) \right] \frac{1}{2\varepsilon} 1_{\theta_N(\lambda) \in I_{\varepsilon}}
\]

\[
= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \mathbb{E} \mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,...,N}^{u} [G(\lambda, X) \delta_{\phi_k - \phi_b} \sin^2(\phi_k) \frac{1}{2\varepsilon} 1_{\phi_N \in I_{\varepsilon}/\pi\mathbb{Z}}] \right]
\]

with \( \mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,...,N}^{u} \) the forward-backward process with \( \mu_b \) the uniform law on \( \mathbb{S}^1 \) and we can then conclude, by taking the limit

\[
\frac{1}{2\varepsilon} 1_{\phi_N \in I_{\varepsilon}/2\pi\mathbb{Z}} d\phi_N \to \delta_0 .
\]
\[
\sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int \ldots \int dv(v_1) \ldots dv(v_n) G(\lambda, X) \frac{d\phi_N}{d\phi_k} \cdot \sin^2(\phi_k) \right] \frac{1}{2\epsilon} \mathbf{1}_{\phi_N(\lambda) \in \mathbb{I}_\epsilon}
\]

We artificially add a variable \(\phi\).

\[
= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \int \ldots \int dv(v_1) \ldots dv(v_k) \int_{\mathbb{S}^1} d\phi \delta_{\phi_k}(\phi) \int \ldots \int dv(v_{k+1}) \ldots dv(v_N) G(\lambda, X) \frac{d\phi_N}{d\phi} \cdot \sin^2(\phi_k) \right] \frac{1}{2\epsilon} \mathbf{1}_{\phi_N(\lambda) \in \mathbb{I}_\epsilon}
\]

\[
= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda \left[ \mathbb{E} \mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,...,N} \left[ G(\lambda, X) \delta_{\phi_k - \phi_b} \sin^2(\phi_k) \right] \frac{1}{2\epsilon} \mathbf{1}_{\phi_N \in \mathbb{I}_\epsilon / \pi \mathbb{Z}} \right]
\]

with \(\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,...,N}\) the forward-backward process with \(\mu_b\) the uniform law on \(\mathbb{S}^1\) and we can then conclude, by taking the limit
\[
\frac{1}{2\epsilon} \mathbf{1}_{\phi_N \in \mathbb{I}_\epsilon / 2\pi \mathbb{Z}} d\phi_N \to \delta_0.
\]
1. The critical one dimensional Anderson model.

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3. Application I: A formula for the integral density of eigenvalues

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5. Application III: A temperature profile
Density of eigenvalues

\[ N(\lambda) = \text{Expected number of eigenvalue smaller than } \lambda. \]
\[ N(\lambda) \text{ the integral density of states.} \]

**Corollary**

With \( \rho^k_\lambda \) the law of \( \phi_k \) (after \( k \) iteration of \( T_\lambda \)) we have

\[
\frac{dN(\lambda)}{d\lambda} = \frac{1}{N} \sum_{k=1}^{N} \int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi)\rho^k_\lambda(\phi)\rho^{N-k}_\lambda\left(\frac{\pi}{2} - \phi\right) d\phi.
\]

**Corollary of the corollary**

With \( \rho_\lambda \) the \( T_\lambda \)-invariant measure on \( \mathbb{R}/2\pi\mathbb{Z} \), we have

\[
\frac{dN(\lambda)}{d\lambda} = \int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi)\rho_\lambda(\phi)\rho_\lambda\left(\frac{\pi}{2} - \phi\right) d\phi.
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Let \((M_i)\) iid random variables with a “nice random law”.

Heuristic: the log of the norm behave as if it was the sum of iid random variables.

\[
\log(\|\prod_{i=1}^{n} M_i x\|) = \log(\|M_n \frac{\prod_{i=1}^{n-1} M_i x}{\|\prod_{i=1}^{n-1} M_i x\|}\|) + \log(\|\prod_{i=1}^{n-1} M_i x\|)
\]

\[
= \sum_{k=1}^{n} \log \left(\|M_k \frac{\prod_{i=1}^{k-1} M_i x}{\|\prod_{i=1}^{k-1} M_i x\|}\|\right)
\]

**Limit theorems for product of random matrices**

- \(\exists \gamma\) (Lyapunov exponent) such that \(\frac{1}{n} \log(\|\prod_{i=1}^{n} M_i x\|) \to \gamma\)
- \(\exists \sigma\) such that \(\frac{1}{\sqrt{n}} (\log(\|\prod_{i=1}^{n} M_i x\|) - \gamma n) \to G(0, \sigma^2)\)
- With \(S_t = \frac{1}{\sqrt{N}} (\log(\|\prod_{i=1}^{[Nt]} M_i x\|) - \gamma Nt) \to \sigma B_t\)
- \(\mathbb{P}(\log(\|\prod_{i=1}^{n} M_i x\|) - \gamma n > an) \sim e^{-nF(a)}\)
An equivalent to the Laplace transform

\[ L_N(\alpha) = \mathbb{E}(e^{\alpha\log(\|\prod_{i=1}^{N} M_i x\|)}) \]

An operator \( A(\alpha) : \mathcal{C}^\beta(S^1) \to \mathcal{C}^\beta(S^1) \)

\[ [A(0)f](x) = \mathbb{E}(f(\frac{Mx}{\|Mx\|})) \]

\[ [A(\alpha)f](x) = \mathbb{E}(e^{\alpha\log(\|Mx\|)}f(\frac{Mx}{\|Mx\|})) \]

\( A(0) \) has a unique largest eigenvalue \((=1)\). We call \( g(\alpha) \) the largest eigenvalue.

\[ L_N(\alpha) = [A(\alpha)^N 1](x) \sim g(\alpha)^N \]
An equivalent to the Laplace transform

\[ L_N(\alpha) = \mathbb{E}(e^{\alpha \log(\|\prod_{i=1}^N M_i x\|)}) \]

An operator \( A(\alpha) : \mathbb{C}^\beta(S^1) \rightarrow \mathbb{C}^\beta(S^1) \)

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\( A(0) \) has a unique largest eigenvalue (\( = 1 \)). We call \( g(\alpha) \) the largest eigenvalue.

\[ L_N(\alpha) = [A(\alpha)^N1](x) \sim g(\alpha)^N \]
With $\alpha$ fixed, this gives the large deviation estimate.

$$\mathbb{P}(\log(\| \prod_{i=1}^{N} M_i x \|) > \nu N) \leq \frac{\mathbb{E}(e^{\alpha \log(\| \prod_{i=1}^{N} M_i x \|)})}{e^{\alpha \nu N}} \sim \left( \frac{g(\alpha)}{e^{\alpha \nu}} \right)^N$$

With $\alpha = \frac{is}{\sqrt{N}}$, because $g(\alpha)$ is analytic around 0:

$$\left( \frac{g(\alpha)}{e^{\gamma \alpha}} \right)^N = (1 - \sigma^2 \frac{s^2}{N} + O(\frac{s^3}{N^3}))^N \to e^{-\sigma^2 s^2}$$

which gives the central limit theorem.
With $\alpha$ fixed, this gives the large deviation estimate:

$$\Pr(\log(\| \prod_{i=1}^{N} M_{i}x \|) > \nu N) \leq \frac{\mathbb{E}(e^{\alpha \log(\| \prod_{i=1}^{N} M_{i}x \|)})}{e^{\alpha \nu N}} \sim \left( \frac{g(\alpha)}{e^{\alpha \nu}} \right)^N$$

With $\alpha = \frac{is}{\sqrt{N}}$. because $g(\alpha)$ is analytic around 0:

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which gives the central limit theorem.
The random processes

\[ \log(|u_{n+1}|^2 + |u_n|^2) \approx \gamma(\lambda) n + \sigma(\lambda) B_n \]

\[ \log(|u_{n+1}|^2 + |u_n|^2) \text{ of the eigenvector} \]
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The model (Huveneer, De Roeck)

A disordered harmonic chain coupled at its ends to two thermal baths.
A temperature profile model

Temperature associated to each eigenvector

\[ T(\psi_\lambda) = \frac{|\psi_\lambda(1)|^2 T_1 + |\psi_\lambda(N)|^2 T_N}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2} \]

At each point

\[ T(x) = \sum_\lambda |\psi_\lambda(x)|^2 T(\psi_\lambda) \]

For \( N \) large and because of the exponential decay \( T(\psi_\lambda) = T_1 \) (resp. \( T_N \)) if \( |\psi_\lambda(1)| > |\psi_\lambda(N)| \) (resp. \( |\psi_\lambda(1)| < |\psi_\lambda(N)| \))

Limiting temperature profile

We have

\[
\lim_{N \to \infty} \mathbb{E}\left[ T(\lfloor \sqrt{N} x + \frac{N}{2} \rfloor) \right] = T_1 + (T_N - T_1) \int_{\mathbb{R}} \mathbb{P}\left( \mathcal{N}(0,1) \leq \frac{2\gamma(\lambda)}{\sigma(\lambda)} x \right) dN(\lambda)
\]

where \( dN(\lambda) \) is the integrated density of states, \( \gamma(\lambda) \) the Lyapunov exponent and \( \sigma(\lambda) \) the limit variance.
A temperature profile model

Temperature associated to each eigenvector

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At each point

\[ T(x) = \sum_{\lambda} |\psi_\lambda(x)|^2 T(\psi_\lambda) \]

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Limiting temperature profile

We have

\[ \lim_{N \to \infty} \mathbb{E}[T(\lfloor \sqrt{N}x + \frac{N}{2} \rfloor)] = T_1 + (T_N - T_1) \int_{\mathbb{R}} P\left( \mathcal{N}(0,1) \leq \frac{2\gamma(\lambda)}{\sigma(\lambda)} x \right) dN(\lambda) \]

where \( dN(\lambda) \) is the integrated density of states, \( \gamma(\lambda) \) the Lyapunov exponent and \( \sigma(\lambda) \) the limit variance.
Thank you for your attention.