Programme sur les probabilités en théorie des nombres
21 mai au 1er juin 2018

Programme on Probability in Number Theory
May 21 - June 1, 2018

Certain three-way prime number races

Greg Martin [*]

gerg@math.ubc.ca

Let $\pi(x; q, a)$ denote the number of primes not exceeding x that are congruent to $a \pmod{q}$. Given a positive integer q and distinct reduced residue classes $a_1, a_2, \dots, a_r \pmod{q}$, a prime number race addresses the question of how often the inequalities $\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r)$ hold. Rubinstein and Sarnak showed in 1994 that the set of such positive real numbers x has a well-defined (logarithmic) density $\delta(q; a_1, a_2, \dots, a_r)$. Their result, as well as all the results discussed herein, were conditional on the assumption that the nontrivial zeros of Dirichlet L-functions (mod q) all lie on the critical line (that is, their real parts all equal 1/2) and that their imaginary parts are linearly independent over the rational numbers.

For a given $r \geq 2$, Rubinstein and Sarnak proved that these densities $\delta(q; a_1, a_2, \dots, a_r)$ tend uniformly to $1/r!$ as q tends to infinity. In 2013, Fiorilli and the speaker established an asymptotic formula for $\delta(q; a_1, a_2)$ as q tends to infinity in terms of certain "variances." In the same year, Lamzouri established a similar asymptotic formula for all $r \geq 3$; the latter asymptotic formula involves certain "covariances" in addition to the variances required in the $r = 2$ case. Later work of Harper and Lamzouri also relies upon an analysis of the covariances of the relevant r-dimensional distributions when studying prime number races with many contestants.

In this talk, we describe work in progress with Jiawei Lin in which we investigate certain three-way prime number races, namely those for which $a_1^2 \equiv a_2^2 \equiv a_3^2 \pmod{q}$. For these races, we implement an alternate approach that produces an asymptotic formula for $\delta(q; a_1, a_2, a_3)$ in terms of certain (sub)variances, without ever having to examine any covariances. We do so by exploiting the special structure of these races to apply tools from univariate probability (such as a quantitative one-dimensional central limit theorem) in each coordinate separately, simplifying the analysis.

[*]Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC V6J 1Z2, CANADA