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A discrete higher rank Racah algebra

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The classical Racah algebra

$$[K_1, K_2] = K_3$$
$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + dK_2 + e_1$$
$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + dK_1 + e_2$$

d, e_1 and e_2 structure constants

The standard realization

$$K_1 := x(x+\gamma+\delta+1)$$
$$K_2 := B(x)E_x - (B(x)+D(x))\mathbb{I} + D(x)E_x^{-1}$$

with the shift operator $E_x(x) = x+1$ and

$$B(x) := \frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}$$
$$D(x) := \frac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}$$

Racah polynomials

Definition

The Racah polynomials are defined as

$$r_n(\alpha,\beta,\gamma,\delta;x) := (\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n \times {}_4F_3\left[{-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \atop \alpha+1, \beta+\delta+1, \gamma+1}; 1\right]$$

K_2 has Racah polynomials as eigenvectors:

$$K_2 r_n(\alpha,\beta,\gamma,\delta;x) = n(n+\alpha+\beta+1) r_n(\alpha,\beta,\gamma,\delta;x)$$

Goal

Generalize the discrete model of Racah algebra to the multivariate case

Dunkl-operators

Definition

The Dunkl operators, corresponding to the abelian group $\mathbb{Z}_2^n$, are defined as follows:

$$T_i := \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i)$$

with real parameters $\mu_i > 0$ and reflection operators: $R_i f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, -x_i, \ldots, x_n)$.

Definition

The $\mathbb{Z}_n^2$ Laplace-Dunkl operator

$$\Delta = \sum_{i=1}^{n} T_i^2$$

A tensor algebra approach

- $\mathfrak{su}(1,1) = \mathrm{span}(x^2, T^2, \mathbb{E}+\gamma)$

  $[T^2, x^2] = 4(\mathbb{E}+\gamma), \quad [\mathbb{E}+\gamma, x^2] = 2x^2, \quad [\mathbb{E}+\gamma, T^2] = -2T^2$

- The Casimir

$$C := \frac{1}{4}\left((\mathbb{E}+\gamma)^2 - 2\,(\mathbb{E}+\gamma) - x^2 T^2\right)$$

- Consider the following algebra

$$\bigotimes_{i=1}^{n} \mathcal{U}(\mathfrak{su}(1,1))$$

- We have the operator

$$\Delta = \sum_{p=1}^{n} \underbrace{1 \otimes \ldots \otimes 1}_{j-1 \text{ times}} \otimes T^2 \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-j \text{ times}}$$

- The following operators commute with this operator:

$$C_{\{j\}} := \underbrace{1 \otimes \ldots \otimes 1}_{j-1 \text{ times}} \otimes C \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-j \text{ times}}$$

Definition

The comultiplication μ^* is an algebra morphism

$$\mu^* : \mathcal{U}(\mathfrak{su}(1,1)) \to \mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1))$$

acting as follows on the generators

$$\mu^*(x^2) = 1 \otimes x^2 + x^2 \otimes 1$$
$$\mu^*(T^2) = 1 \otimes T^2 + T^2 \otimes 1$$

The comultiplication is coassociative:

$$(1 \otimes \mu^*)\mu^* = (\mu^* \otimes 1)\mu^*$$

- We have the following operators

$$\mathbf{C}_1 := C, \qquad \mathbf{C}_k := (\underbrace{1 \otimes \ldots \otimes 1}_{k-2 \text{ times}} \otimes \mu^*)(\mathbf{C}_{k-1})$$

- Let $A \subset [n] := \{1, \ldots, n\}$. Using the τ map we define the generators

$$C_A := \left(\overrightarrow{\prod_{k \in [n] \setminus A}} \tau_k \right) (\mathbf{C}_{|A|})$$

$$\tau_k(A_1 \otimes \ldots \otimes A_l) := A_1 \otimes \ldots \otimes A_{k-1} \otimes 1 \otimes A_k \otimes \ldots \otimes A_l$$

- An example

$$C_{24} = \tau_3(\tau_1(\mathbf{C}_2)) \text{ with } \mathbf{C}_2 = \mu^*(C)$$

For ease of notation we write C_{24} instead of $C_{\{2,4\}}$

Using the comultiplication we find Casimirs that commute with Δ in

$$\mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1))$$

- The lower Casimirs:

$$C_1 := C \otimes 1 \otimes 1 \qquad C_2 := 1 \otimes C \otimes 1 \qquad C_3 := 1 \otimes 1 \otimes C$$

- The intermediate Casimirs

$$C_{12} := \mu^*(C) \otimes 1 \qquad C_{23} := 1 \otimes \mu^*(C) \qquad C_{13} := \tau_2(\mu^*(C))$$

with $\tau_2(a \otimes b) := a \otimes 1 \otimes b$

- The total Casimir

$$C_{123} := (1 \otimes \mu^*)(\mu^*(C))$$

The centrally extended Racah algebra

$$
\begin{aligned}
C_{123} &= C_{12} + C_{23} + C_{13} - C_1 - C_2 - C_3 \\
[C_{12}, C_{23}] &=: 2F \\
[C_{23}, C_{13}] &= 2F \\
[C_{13}, C_{12}] &= 2F \\
[C_{12}, F] &= C_{23}C_{12} - C_{12}C_{13} + (C_2 - C_1)(C_3 - C_{123}) \\
[C_{23}, F] &= C_{13}C_{23} - C_{23}C_{12} + (C_3 - C_2)(C_1 - C_{123}) \\
[C_{13}, F] &= C_{12}C_{13} - C_{13}C_{23} + (C_1 - C_3)(C_2 - C_{123})
\end{aligned}
$$

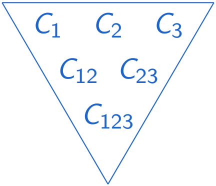
with C_1, C_2, C_3 and C_{123} central operators

The rank one Racah algebra

$$[K_1, K_2] = K_3$$
$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + dK_2 + e_1$$
$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + dK_1 + e_2$$

d, e_1 and e_2 structure constants

The centrally extended algebra



The higher rank Racah algebra

- The Racah algebra of rank $n-2$ is generated by the set $\{C_A | A \subset [n]\}$ and denoted by $R(n)$
- $\{C_{jk} | 1 \leq j < k \leq n\} \cup \{C_j | 1 \leq j \leq n\}$ is a generating set :

$$C_A = \sum_{\{i,j\} \subset A} C_{ij} - (|A| - 2) \sum_{i \in A} C_i$$

- So is the set $\{C_{[j \dots k]} | 1 \leq j \leq k \leq n\}$:

$$C_{ij} = C_{[i \dots j]} - C_{[i+1 \dots j]} - C_{[i \dots j-1]} + C_{[i+1 \dots j-1]} + C_i + C_j$$

labelling Abelian algebras

- By construction we have $[C_A, C_B] = 0$ if $A \subset B$, $B \subset A$ or $A \cap B = \emptyset$
- In particular C_i and $C_{[n]}$ are central
- labelling Abelian algebras

$$\mathscr{Y} = \langle C_{A_m} \rangle_{m=2\dots n-1} \text{ with } A_m \subset A_{m+1} \text{ and } |A_m| = m$$

- The rank $n-2$ equals the dimension of $\mathscr{Y}$

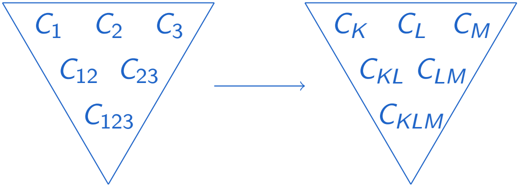
An important observation

- For every set $K \subset [n]$ using the comultiplication μ^* and τ

$$\mathcal{U}(\mathfrak{su}(1,1)) \to \bigotimes_{i=1}^{n} \mathcal{U}(\mathfrak{su}(1,1)) : C \to C_K$$

- For every triple of disjoint sets $K, L, M \subset [n]$

$$\mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1)) \to \bigotimes_{i=1}^{n} \mathcal{U}(\mathfrak{su}(1,1))$$



Bases of Dunkl-harmonics

- Consider two labelling Abelian algebras

$$\mathscr{Y}^{\psi} = \langle C_{[i]} \rangle_{i=2\ldots n-1} \text{ and } \mathscr{Y}^{\varphi} = \langle C_{[2\ldots i]} \rangle_{i=3\ldots n}$$

- In the Dunkl model bases diagonalizing $\mathscr{Y}^{\psi}$ and $\mathscr{Y}^{\varphi}$ can be constructed explicitly. Call them $\psi_{\vec{x}}$ and $\varphi_{\vec{k}}$
- The quantum numbers $\vec{x} = (x_1, \ldots x_{n-2})$ can be determined from:

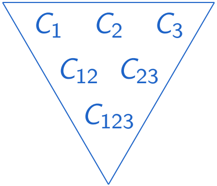
$$C_{[m]}\psi_{\vec{x}} = \kappa(x_{m-1}, \beta_{m-1})\psi_{\vec{x}}$$

with constants β_m that depend on the model used

Theorem

The overlap coefficients between these two bases are multivariate Racah polynomials

Proof for $R(3)$



$$\{\psi_x\} \curvearrowright \{\varphi_k\}$$

$$C_{12} \qquad C_{23}$$

The overlap coefficients are univariate Racah polynomials

- The connection coefficients $R_k(x)$ are given by

$$\sum_{x=0}^{N} R_k(x)\psi_x = \varphi_k$$

- One finds the explicit expression

$$R_k(x) := g^* r_k(\beta_1 - \beta_0 - 1, \beta_2 - \beta_1 - 1, -N - 1, \beta_1 + N; x)$$

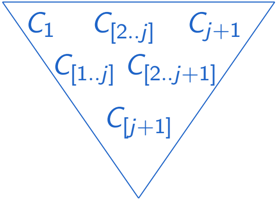
- From the eigenvalues of the operators

$$\begin{aligned}
C_1 &: \kappa(0, \beta_0) \\
C_2 &: \kappa(0, \beta_1 - \beta_0 - 1) \\
C_3 &: \kappa(0, \beta_2 - \beta_1 - 1) \\
C_{12} &: \kappa(x, \beta_1) \\
C_{23} &: \kappa(k, \beta_2 - \beta_0 - 1) \\
C_{123} &: \kappa(N, \beta_2).
\end{aligned}$$

Proof for $R(n)$

$\{\psi_{\vec{x}}\}$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\cdots$	$\mathcal{B}_{n-4}$	$\mathcal{B}_{n-3}$	$\{\varphi_{\vec{k}}\}$
C_{12}	C_{23}	C_{23}	C_{23}	$\cdots$	C_{23}	C_{23}	C_{23}
C_{123}	C_{123}	C_{234}	C_{234}	$\cdots$	C_{234}	C_{234}	C_{234}
C_{1234}	C_{1234}	C_{1234}	C_{2345}	$\cdots$	C_{2345}	C_{2345}	C_{2345}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$C_{[1..n-1]}$	$C_{[1..n-1]}$	$C_{[1..n-1]}$	$C_{[1..n-1]}$	$\cdots$	$C_{[1..n-2]}$	$C_{[2..n-1]}$	$C_{[2..n-1]}$
$C_{[n-1]}$	$C_{[n-1]}$	$C_{[n-1]}$	$C_{[n-1]}$	$\cdots$	$C_{[n-1]}$	$C_{[n-1]}$	$C_{[2..n]}$

- The elements differing, $C_{[1..j]}$ and $C_{[2..j+1]}$, generate a rank one Racah algebra



- This Racah algebra commutes with $\mathscr{Y}_{j-2} \cap \mathscr{Y}_{j-1}$ and therefore preserves its eigenspaces
- The connection coefficients at each step are univariate Racah polynomials leading to

$$\begin{aligned}\varphi_{\vec{k}} &= \sum_{\vec{x}} g_2^* R_{k_1}(x_1) R_{|\vec{k}|_2}(x_2) \dots R_{|\vec{k}|_{n-2}}(x_{n-2}) \psi_{\vec{x}} \\ &= \sum_{\vec{x}} g_2^* R_{\vec{k}}(\vec{x}) \psi_{\vec{x}}\end{aligned}$$

Multivariate Racah polynomials

Definition

The multivariate Racah polynomials are defined as

$$R_n(\vec{k};\vec{x};\vec{\beta};N) = \prod_{j=1}^{n-2} r_{k_j}(2|\vec{k}|_{j-1}+\beta_j-\beta_0-1, \beta_{j+1}-\beta_j-1, |\vec{k}|_{j-1}-x_{j+1}-1, |\vec{k}|_{j-1}+\beta_j+x_{j+1}; -|\vec{k}|_{j-1}+x_j)$$

with $\vec{k} := (k_1,\ldots,k_n)$, $|\vec{k}|_{j-1} = \sum_{p=1}^{j-1} k_p$, $\vec{x} = (x_1,\ldots,x_n)$, $\vec{\beta} := (\beta_0,\ldots,\beta_{n+1})$ and $N := x_{n+1}$.

The discrete model

- The Racah algebra acts as follows on the overlap coefficients:

$$C_A \left\langle \varphi_{\vec{k}} \,\middle|\, \psi_{\vec{x}} \right\rangle := \left\langle \varphi_{\vec{k}} \,\middle|\, C_A \,\middle|\, \psi_{\vec{x}} \right\rangle$$

- $C_{[m+1]} \in \mathscr{Y}^{\psi}$ then

$$\left\langle \varphi_{\vec{k}} \,\middle|\, C_{[m+1]} \,\middle|\, \psi_{\vec{x}} \right\rangle = \kappa(x_m, \beta_m) \left\langle \varphi_{\vec{k}} \,\middle|\, \psi_{\vec{x}} \right\rangle$$

- $C_{[m+1]}$ is multiplication by $\kappa(x_m, \beta_m)$
- Let $C_{[2\ldots m+2]} \in \mathscr{Y}^{\varphi}$ then

$$\left\langle \varphi_{\vec{k}} \,\middle|\, C_{[2\ldots m+2]} \,\middle|\, \psi_{\vec{x}} \right\rangle = \kappa(|\vec{k}|_m, \beta_{m+1} - \beta_0 - 1) \left\langle \varphi_{\vec{k}} \,\middle|\, \psi_{\vec{x}} \right\rangle$$

- $C_{[2\ldots m+2]}$ has the multivariate Racah polynomials as eigenvectors

Definition

The Racah operators $\mathcal{L}_m$ defined by J. Geronimo and P. Iliev are a set of commuting shift operators are given by

$$\mathcal{L}_m := \sum_{\vec{\nu}\in\{-1,0,1\}^m} f_{\vec{\nu}}\,(E_{\vec{\nu}} - \mathbb{I})$$

with $E_{\vec{\nu}}(g(\vec{x})) = g(\vec{x}+\vec{\nu})$ and $f_{\vec{\nu}}$ a rational function in $x_0, x_1 \ldots x_m, x_{m+1}$ and $\beta_1, \ldots, \beta_m$

- Its eigenvectors are multivariate Racah polynomials

$$\mathcal{L}_m R_{n-2}(\vec{k};\vec{x};\vec{\beta};N) = -|\vec{k}|_m(|\vec{k}|_m+\beta_{m+1}-\beta_0-1)R_{n-2}(\vec{k};\vec{x};\vec{\beta};N)$$

$$\begin{aligned}
&\left\langle \varphi_{\vec{k}} \,\middle|\, C_{[2\ldots m+2]} \,\middle|\, \psi_{\vec{x}} \right\rangle \\
&= \kappa(|\vec{k}|_m, \beta_{m+1} - \beta_0 - 1) \left\langle \varphi_{\vec{k}} \,\middle|\, \psi_{\vec{x}} \right\rangle \\
&= \kappa(|\vec{k}|_m, \beta_{m+1} - \beta_0 - 1) g^* R_{n-2}(\vec{k}; \vec{x}; \vec{\beta}; N) \\
&= g_* \big(-\mathcal{L}_m + \kappa(0, \beta_{m+1} - \beta_0 - 1) \big) g_*^{-1} g_* R_{n-2}(\vec{k}; \vec{x}; \vec{\beta}; N) \\
&= g_* \big(-\mathcal{L}_m + \kappa(0, \beta_{m+1} - \beta_0 - 1) \big) g_*^{-1} \left\langle \varphi_{\vec{k}} \,\middle|\, \psi_{\vec{x}} \right\rangle
\end{aligned}$$

- For $C_{[2\ldots m+2]}$ we conclude it acts as

$$g_* \big(-\mathcal{L}_m + \kappa(0, \beta_{m+1} - \beta_0 - 1) \big) g_*^{-1}$$

- The set $\{C_{[p\dots q]} | 1 \leq p \leq q \leq n\}$ generates the algebra
- To construct the remaining operators $C_{[p\dots q]}$, consider the map σ

$$\sigma : \mathrm{Alg}[x_0, \dots, x_s; \beta_0, \dots \beta_s; E_1, \dots, E_s]$$
$$\downarrow$$
$$\mathrm{Alg}[x_1, \dots, x_{s+1}; \beta_1, \dots \beta_{s+1}; E_2, \dots, E_{s+1}]$$

- Its action is

$$\sigma(x_i) = x_{i+1}, \quad , \sigma(\beta_i) = \beta_{i+1}, \quad \sigma(E_{x_i}) = E_{x_{i+1}}$$

Theorem

Define the following operators:

$$C_{[m+1]} = \kappa(x_m, \beta_m)$$
$$C_{[2..m+2]} = -\mathcal{L}_m + \kappa(0, \beta_{m+1} - \beta_0 - 1)$$
$$C_{[p..q]} = \sigma^{p-2}(C_{[2...q-p+2]})$$

The algebra defined by these operators generates a discrete realization of $R(n)$

Example

- According to the theorem $C_{34} = \sigma(C_{23})$ and

$$C_{23} = -\mathcal{L}_1(x_0, x_1, x_2, \beta_0, \beta_1, \beta_2, E_{x_1}) + \frac{(\beta_2 - \beta_0)(\beta_2 - \beta_0 - 2)}{4}$$

- Therefore C_{34} must be

$$C_{34} = -\mathcal{L}_1(x_1, x_2, x_3, \beta_1, \beta_2, \beta_3, E_{x_2}) + \frac{(\beta_3 - \beta_1)(\beta_3 - \beta_1 - 2)}{4}$$

- We give a sketch of the proof for C_{34}

Sketch of the proof

Let $|\phi_{\vec{m}}\rangle$ be a normalized basis such that C_{34} acts diagonal on this basis

$$\begin{aligned}
C_{34}\left\langle\varphi_{\vec{k}}\,\middle|\,\psi_{\vec{x}}\right\rangle &= \sum_{\vec{m}}\left\langle\varphi_{\vec{k}}\,\middle|\,\phi_{\vec{m}}\right\rangle\left\langle\phi_{\vec{m}}\,\middle|\,C_{34}\,\middle|\,\psi_{\vec{x}}\right\rangle \\
&= \sum_{\vec{m}}\left\langle\varphi_{\vec{k}}\,\middle|\,\phi_{\vec{m}}\right\rangle\nu(\vec{m})\left\langle\phi_{\vec{m}}\,\middle|\,\psi_{\vec{x}}\right\rangle \\
&= \sum_{\vec{m}}\left\langle\varphi_{\vec{k}}\,\middle|\,\phi_{\vec{m}}\right\rangle T^{\phi}_{\vec{x}}\left\langle\phi_{\vec{m}}\,\middle|\,\psi_{\vec{x}}\right\rangle \\
&= T^{\phi}_{\vec{x}}\sum_{\vec{m}}\left\langle\varphi_{\vec{k}}\,\middle|\,\phi_{\vec{m}}\right\rangle\left\langle\phi_{\vec{m}}\,\middle|\,\psi_{\vec{x}}\right\rangle \\
&= T^{\phi}_{\vec{x}}\left\langle\varphi_{\vec{k}}\,\middle|\,\psi_{\vec{x}}\right\rangle
\end{aligned}$$

Assuming $T^{\phi}_{\vec{x}}$ an operator acting on the grid $\vec{x}$ but not dependent on $\vec{m}$ with eigenvectors $\langle\phi_{\vec{m}}\,|\,\psi_{\vec{x}}\rangle$ and eigenvalues $\nu(\vec{m})$

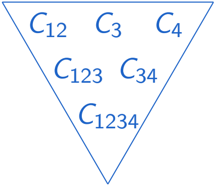
Sketch of the proof

- If we find a basis $|\phi_{\vec{m}}\rangle$ such that $\langle \phi_{\vec{m}} \,|\, \psi_{\vec{x}} \rangle$ are multivariate Racah polynomials then $T^{\phi}_{\vec{x}}$ is a Racah operator
- Consider the following Abelian algebra $\mathscr{Y}^{\phi} = \langle C_{12}, C_{34} \rangle$ and related basis $\{\phi_{\vec{m}}\}$
- It differs by one element from the basis $\mathscr{Y}^{\psi} = \langle C_{12}, C_{123} \rangle$ related to the basis $\{\psi_{\vec{x}}\}$:

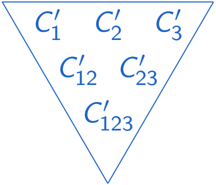
$$
\begin{array}{ccc}
\{\psi_{\vec{x}}\} & \curvearrowright & \{\phi_{\vec{m}}\} \\
C_{12} & & C_{12} \\
\boxed{C_{123}} & & \boxed{C_{34}}
\end{array}
$$

Sketch of the proof

- Those two elements are in the rank one Racah algebra $R^{12,3,4}(3)$ preserving the eigenspaces of $\mathscr{Y}_\psi \cap \mathscr{Y}_\phi = \langle C_{12} \rangle$



- C_{34} plays the role of C'_{23} in the standard rank one Racah algebra



Sketch of the proof

- By comparing the spectra of the operators in both algebras one finds the following set of equations

$$\begin{aligned}
C_{12} \cong C_1' :& \quad \kappa(x_1, \beta_1) = \kappa(0, \beta_0') \\
C_3 \cong C_2' :& \quad \kappa(0, \beta_2 - \beta_1 - 1) = \kappa(0, \beta_1' - \beta_0' - 1) \\
C_4 \cong C_3' :& \quad \kappa(0, \beta_3 - \beta_2 - 1) = \kappa(0, \beta_2' - \beta_1' - 1) \\
C_{123} \cong C_{12}' :& \quad \kappa(x_2, \beta_2) = \kappa(x_1', \beta_1') \\
C_{34} \cong C_{23}' :& \quad \kappa(m_2, \beta_3 - \beta_1 - 1) = \kappa(k_1', \beta_2' - \beta_0' - 1) \\
C_{1234} \cong C_{123}' :& \quad \kappa(x_3, \beta_3) = \kappa(x_2', \beta_2')
\end{aligned}$$

- Solving this set of equation, one obtains

$$\begin{array}{lll}
\beta_0' = \beta_1 + 2x_1 & \beta_1' = \beta_2 + 2x_1 & \beta_2' = \beta_3 + 2x_1 \\
x_1' = x_2 - x_1 & x_2' = x_3 - x_1 & k_1' = m_2.
\end{array}$$

- The connection coefficients are given by

$$\langle \phi_{\vec{m}} | \psi_{\vec{x}} \rangle = g'_* R_1(\vec{k}'; \vec{x}'; \vec{\beta}'; N) \delta_{m_1, x_1}$$

- C_{34} is the operator

$$C_{34} = g'_* \big(-\mathcal{L}_1(0, x'_1, x'_2, \beta'_0, \beta'_1, \beta'_2, E_{x'_1}) + \kappa(0, \beta'_2 - \beta'_0 - 1) \big) g'^{-1}_*$$

- Perfoming the substitutions one obtains

$$C_{34} = g'_* \big(-\mathcal{L}_1(x_1, x_2, x_3, \beta_1, \beta_2, \beta_3, E_{x_2}) + \kappa(0, \beta_3 - \beta_1 - 1) \big) g'^{-1}_*$$

- One can conjugate the algebra so that the gauge coefficients of all operators disappear
- C_{34} becomes

$$C_{34} = -\mathcal{L}_1(x_1, x_2, x_3, \beta_1, \beta_2, \beta_3, E_{x_2}) + \kappa(0, \beta_3 - \beta_1 - 1)$$

- This is exactly $\sigma(C_{23})$ and concludes the proof

Theorem

Define the following operators:

$$C_{[m+1]} = \kappa(x_m, \beta_m)$$
$$C_{[2..m+2]} = -\mathcal{L}_m + \kappa(0, \beta_{m+1} - \beta_0 - 1)$$
$$C_{[p..q]} = \sigma^{p-2}(C_{[2...q-p+2]})$$

The algebra defined by these operators generates a discrete realization of $R(n)$