Stochastic duality, *t*KZ and degenerate Macdonald polynomials

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Motivation

- (Weak) Duality: many-body observables in one model may behave simple in dual model
- Connection between models with different particle content
- Algebraic construction of duality functions
- Application of Integrability and Macdonald polynomials to stochastic processes

Duality

$\mathbb{A}$ and $\mathbb{B}$ are vector spaces with basis vectors $|a\rangle$ and $|b\rangle$.

$$|\Psi\rangle := \sum_{a,b} \psi(a,b)|a\rangle \otimes |b\rangle.$$

Let $\mathbb{L} \in \mathrm{End}(\mathbb{A})$ and $\mathbb{M} \in \mathrm{End}(\mathbb{B})$ be linear operators (quantum Hamiltonian, Markov generator, ...) given explicitly by

$$\mathbb{L}|a\rangle = \sum_{a'} \ell(a',a)|a'\rangle, \qquad \mathbb{M}|b\rangle = \sum_{b'} m(b',b)|b'\rangle,$$

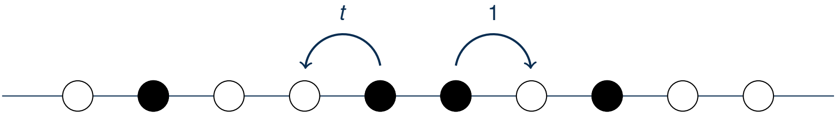
for certain matrix entries ℓ and m.

$\mathbb{M}$ and $\mathbb{L}$ are dual wrt ψ if

$$\mathbb{L}|\Psi\rangle = \mathbb{M}|\Psi\rangle$$

Asymmetric simple exclusion process (ASEP=XXZ)

Continuous time Markov chain of hopping particles on $\mathbb{Z}$:



Configurations $\mu = (\mu_1, \ldots, \mu_n) \qquad \mu_i \in \{0, 1\}$

$01 \mapsto 10$ with rate t
$10 \mapsto 01$ with rate 1

Example of ASEP self-duality

Let ν be an infinite composition with parts $\nu_i \in \{0,1\}$ with n nonzero parts

Let $\vec{x}(\mu) = (x_1 < \cdots < x_m)$ label the positions of ones in another composition $\mu \in \{0,1\}^m$.

Theorem (Schütz ('97), Borodin–Corwin–Sasamoto ('12))

The functions

$$\psi(\nu,\mu) = \prod_{x \in \vec{x}(\mu)} \left(\prod_{i<x} t^{\nu_i} \right) \nu_x$$

are duality functions between ASEPs with n and m particles.

This duality is employed by BCS to derive Tracy-Widom GUE current distributions in ASEP.

Matrix representation

Master equation for state vector $|\Psi\rangle$:

$$\frac{d}{dt}|\Psi\rangle = \mathbb{M}|\Psi\rangle, \qquad \mathbb{M} = \sum_{i\in\mathbb{Z}} \mathbb{M}_i$$

$\mathbb{M}_i$ acts on functions ψ of binary strings μ:

$$\mathbb{M}_i[\psi](\mu) = \sum_{\mu'} m_i(\mu,\mu')\psi(\mu'),$$

Conservation of probability: the matrix columns sum to zero.

$$\mathbb{M}_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & +t & 0 \\ 0 & +1 & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}, \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(local XXZ Hamiltonian)

Polynomial realisation of ASEP

$$\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}} \right) (s_i - 1),$$

where s_i acts on polynomials by the simple transposition $z_i \leftrightarrow z_{i+1}$.

Polynomial ASEP generator

$\mathbb{L}_i$ acts on *multlinear* monomials $|\nu\rangle = \prod_{k\in\mathbb{Z}} z_k^{\nu_k}$ as the ASEP generator.

$$\mathbb{L}_i(z_i z_{i+1}) = \mathbb{L}_i(1) = 0$$

$$\mathbb{L}_i(z_i) = z_{i+1} - tz_i,$$

$$\mathbb{L}_i(z_{i+1}) = tz_i - z_{i+1}.$$

Local duality and tKZ equation

The ASEP is locally self dual if there is a $|\Psi\rangle$ such that

$$\mathbb{L}_i|\Psi\rangle = \mathbb{M}_i|\Psi\rangle, \qquad \text{where } |\Psi\rangle = \sum_{\nu,\mu} \psi(\nu,\mu) \prod_{k\in\mathbb{Z}} z_k^{\nu_k} |\mu\rangle.$$

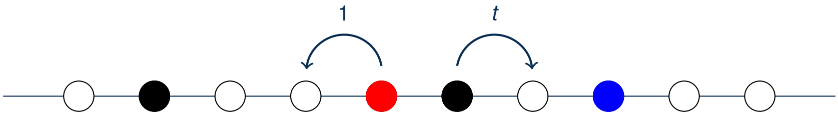
Connection with tKZ

ASEP duality can be rewritten as

$$s_i|\Psi\rangle = \check{R}(z_i/z_{i+1})|\Psi\rangle,$$

where $\check{R}$ is the (stochastic) $U_t(\widehat{sl}_2)$ quantum group R-matrix.

multi-species ASEP



Configurations $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \in \{0, \ldots, r\}$

$$\ldots \mu_i, \mu_{i+1} \ldots \mapsto \ldots \mu_{i+1}, \mu_i \ldots \begin{cases} \text{rate } 1 & \text{if } \mu_i < \mu_{i+1} \\ \text{rate } t & \text{if } \mu_i > \mu_{i+1} \end{cases}$$

Higher rank *t*KZ

$$s_i |\Psi\rangle = \check{R}(z_i/z_{i+1}) |\Psi\rangle,$$

where $\check{R}$ is the $U_t(\widehat{sl}_{r+1})$ quantum group R-matrix.

Admissible multi-species polynomial basis

$$|\Psi\rangle = \sum_{\mu,\nu} \psi(\nu,\mu) f_\nu(z)|\mu\rangle.$$

where $\{f_\nu\}$ is admissible if $\mathbb{L}_i$ acts on it as the mASEP generator

$\Rightarrow$

Solve *t*KZ with $\psi(\nu,\mu) = \delta_{\nu,\mu}$

Construct admissible polynomials using theory of (non-symmetric) Macdonald polynomials and Hecke algebra.

Hecke generators:

$$T_i = t - \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(1 - s_i), \qquad (T_i - t)(T_i + 1) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

tKZ solutions from Macdonald polynomials

A special admissible polynomial basis is given by

$$f_\delta = E_\delta, \quad \forall\, \delta = (\delta_1 \leq \cdots \leq \delta_n),$$

$$f_{s_i\mu} = T_i^{-1} f_\mu, \quad \text{when } \ \mu_i < \mu_{i+1},$$

where $E_\delta = E_\delta(z; q, t)$ is an non-symmetric Macdonald polynomial.

Additional cyclic relation:

$$f_{\mu_n,\mu_1,\ldots,\mu_{n-1}}(qz_n, z_1, \ldots, z_{n-1}; q, t) = q^{\mu_n} f_{\mu_1,\ldots,\mu_n}(z_1, \ldots, z_n; q, t),$$

introduces another parameter q.

Both E_μ and f_μ are monic polynomials in $\{z_i\}$ with coefficients in $\mathbb{Q}[q, t]$, i.e rational functions in q and t.

Main idea

- The polynomials $\{f_\mu(z;q,t)\}$ form an admissible basis
- The vector

$$|\Psi\rangle = \sum_\mu f_\mu(z;q,t)|\mu\rangle, \qquad f_\mu \in \mathbb{Q}(q,t)[z_1,\dots,z_n]$$

  is a solution of the tKZ equation.
- The polynomials f_μ may have poles at resonant points $q^k t^\ell = 1$
- After appropriately normalising

$$\lim_{q\to t^{-\ell}} (1-qt^\ell)^p\, f_\mu(z;q,t) = \sum_{\nu<\mu} \psi(\nu,\mu;t) f_\nu(z;t^{-\ell},t)$$

  for certain coefficients $\psi(\nu,\mu;t)$.
- These solutions produce non-trivial duality functions $\psi(\nu,\mu;t)$ in the mASEP.

Some ingredients

For given m and p , let

$$\mathrm{Coeff}_p[g, m] := \lim_{q \to t^{-m}} (1 - qt^m)^p g(z_1, \ldots, z_n),$$

if the limit exists.

Theorem (Unique sector)

*Given an anti-partition δ, then there exists a **unique** anti-partition ϵ such that*

$$f_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} f_\nu(z; q, t)$$

is well defined for all compositions $\nu \in \sigma(\epsilon)$, and such that

$$\mathrm{Coeff}_p[f_\mu, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) f_\nu(z; t^{-m}, t),$$

for all $\mu \in \sigma(\delta)$ and suitable coefficients $\psi(\nu, \mu; t)$.

Some ingredients

Theorem (Explicit expressions)

$$f_\mu(z_1,\dots,z_n;q,t) = \mathrm{Tr}\Big(A_{\mu_1}(z_1)\dots A_{\mu_n}(z_n)S_q\Big),$$

For example, if $\mu^- = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2})$, then

$$A_0(z) = 1 + z\phi, \quad A_1(z) = zk, \quad A_2(z) = z\phi^\dagger + z^2.$$

with t oscillator algebra generators

$$\phi\phi^\dagger = 1 - tk, \qquad \phi^\dagger\phi = 1 - k, \qquad tk\phi = \phi k, \qquad k\phi^\dagger = t\phi^\dagger k.$$

- this leads to a Matrix Product formula for Macdonald polynomials

First example

Theorem (Rank r to rank 1 duality)

Consider the anti-partition $\delta = (0^{n-m}, r^m)$*. Then* $\mathrm{Coeff}_1[f_\delta, m] \equiv \mathrm{Coeff}[f_\delta, m]$ *exists, and we have*

$$\mathrm{Coeff}[f_\mu, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) z^\nu, \quad \forall\, \mu \in \sigma(\delta),$$

where $\epsilon = (0^{n-rm}, 1^{rm})$*.*

The duality function $\psi(\nu, \mu)$ gives the original example

$$\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \left(\prod_{i<x} t^{\nu_i} \right) \nu_x$$

Second example

Theorem

Let

$$\delta = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}),$$

Then

$$\mathrm{Coeff}[f_\mu, p+m_1] = \sum_{\nu\in\sigma(\epsilon)} \psi(\nu,\mu) f_\nu(z; t^{-p-m_1}, t), \quad \forall\, \mu \in \sigma(\delta),$$

for appropriate coefficients $\psi(\nu,\mu)$*, where*

$$\epsilon = (0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p}).$$

The coefficients $\psi(\nu,\mu)$ can be explicitly calculated from the matrix product formula

Rank 2 duality

Fix two compositions

$$\mu \in \sigma(0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}), \qquad \nu \in \sigma(0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p}),$$

chosen such that the inequalities $\mu_k > \nu_k = 0$ and $\mu_k < \nu_k = 2$ do not occur for any $1 \leq k \leq n$.

Label the composition μ by two sets of positions: a set $\vec{x}(\mu) = (x_1 < \cdots < x_{m_1})$ which labels the positions of 1-particles, and a set $\vec{y}(\mu) = (y_1 < \cdots < y_{m_2})$ labelling the positions of 2-particles.

Define

$$\chi(\vec{x}, \vec{y}) := \#\{(x_i, y_j) \in (\vec{x}, \vec{y}) \mid x_i > y_j\}.$$

The duality is $\psi(\nu, \mu) = \Omega(\mu, \nu)$ with

$$\Omega(\mu, \nu) = -\chi(\vec{x}, \vec{y}) + \sum_{x \in \vec{x}(\mu)} \sum_{i<x} \mathbf{1}_{\nu_i \geq 1} + \sum_{y \in \vec{y}(\mu)} \sum_{i<y} \mathbf{1}_{\nu_i = 1} \mathbf{1}_{\nu_y = 1}.$$

Conclusion and Outlook

- Algebraic framework for constructing stochastic dualities in arbitrary rank mASEP
- Interplay between integrability and Macdonald polynomial theory
- Delicate analysis of resonant points
- Use higher rank dualities to compute mixed current distributions
- Polynomial representations of other stochastic models