Loop models and K-theory

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Introduction

- Okounkov and collaborators (Maulik, Aganagic) ('12–'17), inspired by the work of Nekrasov+Shatashvili (∼'09) on 4d SUSY gauge theories, developed a framework for the relationship between quantum integrable systems (vertex models) and generalized cohomology theories (e.g., K-theory for trigonometric QIS).
- Some particular cases of this correspondence were discovered earlier (∼'05) by Di Francesco, Knutson and ZJ in the context of loop models.
- Loop models are interesting physically (nonlocal degrees of freedom) and combinatorially.
- One can then apply methods of algebraic geometry to these models. . . or vice versa.
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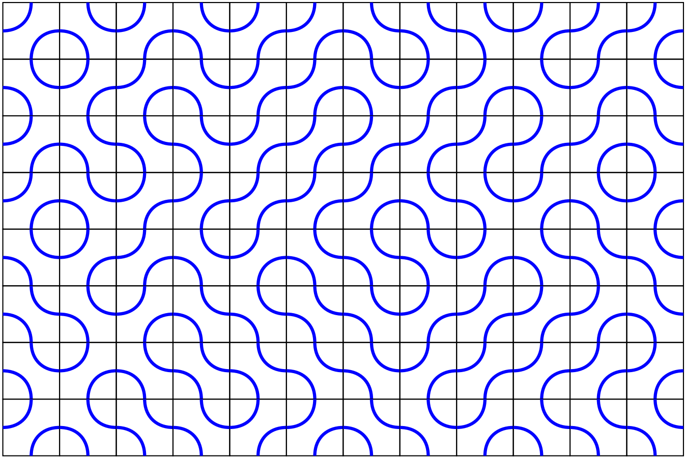
The Temperley–Lieb Loop model (equivalent to a model of *critical bond percolation*):



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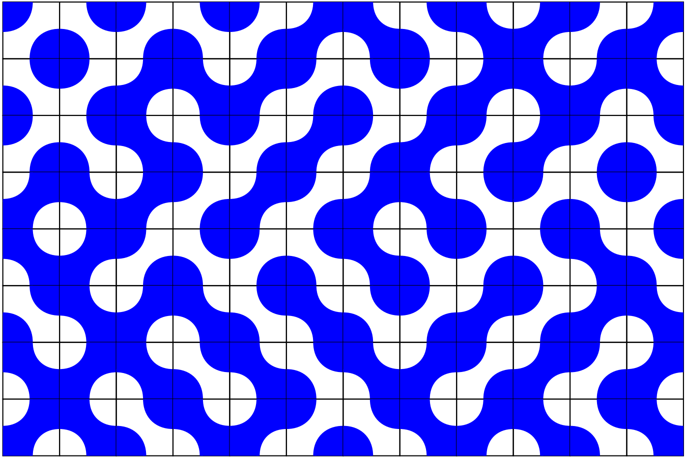

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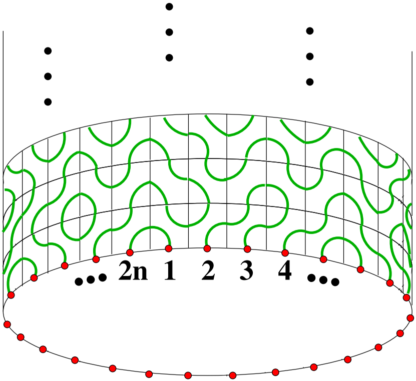
 with probability p,  with probability $1 - p$. $(0 < p < 1)$



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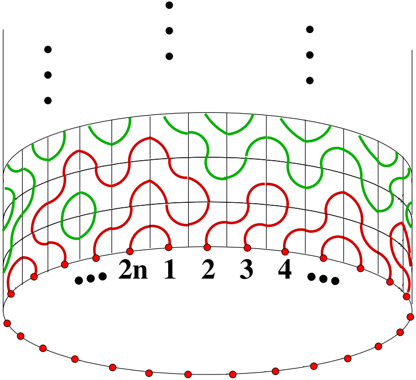


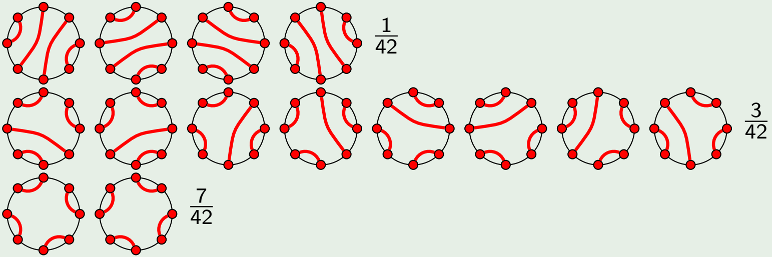
Case of the half-infinite cylinder geometry ("periodic boundary conditions")

Probability law of the connectivity of the external vertices?

The connectivity of the external vertices can be encoded into a link pattern = a planar pairing of $2n$ points on a circle.

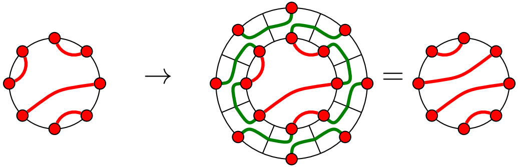
Example

In size $L = 2n = 8$,



Relation to Markov process on link patterns

Using a transfer matrix formalism, one can reformulate the computation of these probabilities in terms of a Markov process on link patterns (dependent on p):



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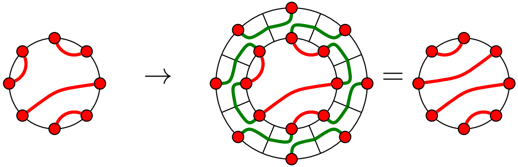

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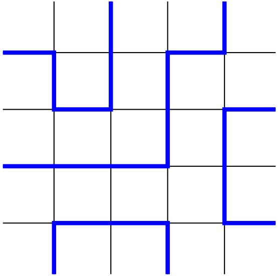
A Fully Packed Loop configuration (FPL) on a $n \times n$ square grid:



FPL configurations are in bijection with various other famous combinatorial objects, in particular Alternating Sign Matrices; their enumeration is given by

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429 \ldots$$

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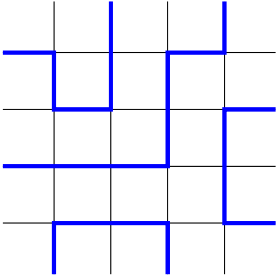

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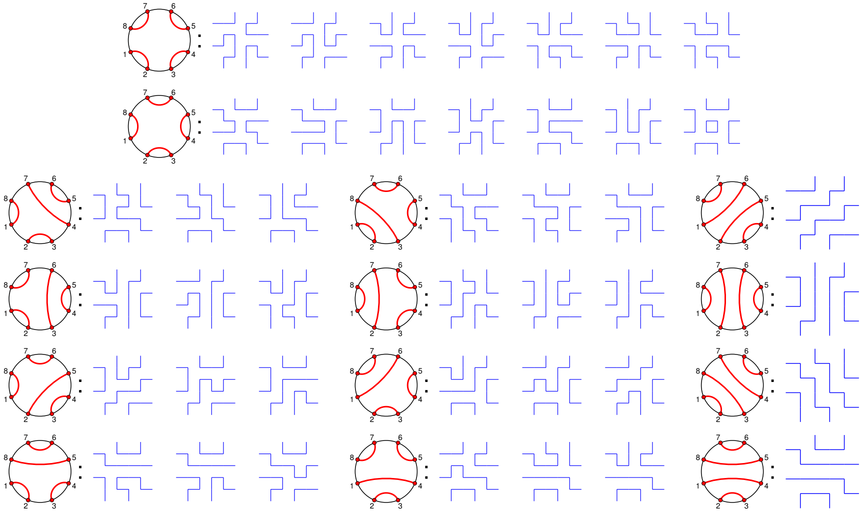
It is natural to group FPLs by connectivity of their endpoints: cf



Denote by $A(\pi)$ the number of FPLs with connectivity described by the link pattern π. Razumov and Stroganov observed (2001), and then Cantini and Sportiello proved (2010), that $A(\pi)$ is exactly the (unnormalized) probability of pattern π in the model of loops with the geometry of the cylinder.

In other words $|\Psi\rangle = \sum_\pi A(\pi)|\pi\rangle$ is the (unnormalized) steady state of the Markov process of loops:

$$T(p)|\Psi\rangle = |\Psi\rangle$$

Remark: there are (still conjectural!) variations: other types of b.c. on TL $\leftrightarrow$ different symmetry classes of ASM/FPL [Batchelor, de Gier & Nienhuis '01; Razumov-Stroganov '01; Pearce, de Gier & Rittenberg '01, ...]. The geometric approach advocated here should work equally well in those cases.

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Consider the probabilistic model (on the cylinder) with probabilities p_i depending on the column $i = 1, \dots, 2n$, which we parameterize as $p_i = \frac{t - q\, z_i}{z_i - q\, t}$, $1 - p_i = q^2 \frac{t - z_i}{z_i - q\, t}$, $q = e^{2\pi i/3}$.

This inhomogeneous model is still integrable, the z_i are the spectral parameters.

The corresponding steady state is $|\Psi(z_1, \dots, z_{2n})\rangle$.

$$T(z_1, \dots, z_{2n}|t)|\Psi(z_1, \dots, z_{2n})\rangle = |\Psi(z_1, \dots, z_{2n})\rangle$$

- Polynomiality.
  $|\Psi(z_1,\ldots,z_{2n})\rangle$ can be normalized in such a way that its components are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i.
- Factorization and symmetry.
  The components possess various linear factors and properties of symmetry by exchange of variables. (more later)
  In particular, their sum is a symmetric polynomial of all z_i.
- Recursion relations.
  Components of $|\Psi(z_1,\ldots,z_{2n})\rangle$ satisfy linear recursion relations; their sum is entirely determined by these:

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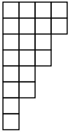
$$\sum_\pi \Psi_\pi(z_1,\ldots,z_{2n}) = IK_n(q;z_1,\ldots,z_{2n}) = Schur(\ $$  $$\ ;z_1,\ldots,z_{2n})$$

General q

So far the quantum parameter q was restricted to $e^{2i\pi/3}$. It is tempting to extend the Ψ_π to arbitrary q. This can be achieved by replacing the transfer matrix equation $T|\Psi\rangle = |\Psi\rangle$ to a q-difference equation: the qKZ equation introduced by Frenkel and Reshetikhin ('92).

In fact, $|\Psi\rangle$ will satisfy a slightly stronger version of the qKZ equation which I call the qKZ system.

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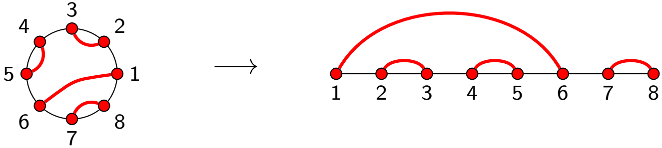

Polynomial solution of qKZ

The (level 1) qKZ system possesses a unique polynomial solution in the space of link patterns (of degree $n(n-1)$ in the zs), such that at $q = e^{\pm 2i\pi/3}$, it reduces to the (unnormalized) steady state of the Markov process introduced earlier.

Example ($L = 2n = 4$)

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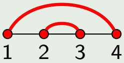
Ψ 

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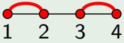
Ψ 

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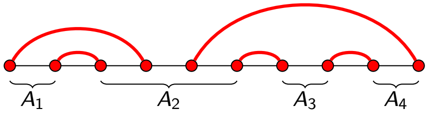
Factorization and symmetry

Given a link pattern π, one can separate vertices into maximal groups of neighbors that are not paired with each other:



Then any solution of the qKZ system satisfies

$$\Psi_\pi = \prod_k \prod_{\substack{i,j\in A_k \\ i<j}} (q\, z_j - q^{-1} z_i)\ \Phi_\pi$$

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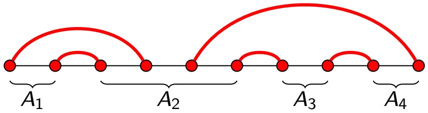

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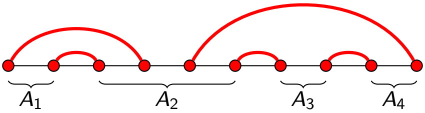

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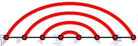
(This immediately implies uniqueness of Ψ, since the qKZ system allows to build the entries inductively starting from )

Generalities

- It is natural to try to interpret these polynomials $\Psi_\pi(z_1, \ldots, z_{2n}; q)$ as (numerators of) Hilbert series (characters) of certain modules (of commutative algebras!)
- We failed to achieve this 15 years ago with my collaborators. . . because it involves rather sophisticated constructions in equivariant K-theory!
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Orbital varieties

In general, orbital varieties are irreducible components of the intersection of a nilpotent orbit closure with a Borel subalgebra. Here we consider the orbit $\{M^2 = 0,\ M \in \mathfrak{gl}(L)\}$ so

$$\mathcal{O} = \{M^2 = 0,\ M\ L \times L \text{ upper triangular complex matrix}\}$$

Its irreducible components, the orbital varieties, are indexed by link patterns:

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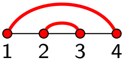
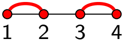
Example

At $L = 2n = 4$, two components:

$\mathcal{O}$ 

$$= \left\{ M = \begin{pmatrix} 0 & m_{13} & m_{14} \\ & m_{23} & m_{24} \\ & & 0 \\ & & \end{pmatrix} \right\}$$

$\mathcal{O}$ 

$$= \left\{ M = \begin{pmatrix} m_{12} & m_{13} & m_{14} \\ & 0 & m_{24} \\ & & m_{34} \\ & & \end{pmatrix} : \ m_{12} m_{24} + m_{13} m_{34} = 0 \right\}$$

Orbital varieties cont'd

In general, the orbital varieties $\mathcal{O}_\pi$ can be described explicitly:

$$\mathcal{O}_\pi = \Big\{ M \text{ upper triangular} : M^2 = 0 \text{ and } \operatorname{rank} M^{i\urcorner}_{\;j} \le \#\{\text{arcs inside } [i,j]\}, \quad i,j = 1,\dots,N \Big\}$$

where $M^{i\urcorner}_{\;j}$ is the sub-matrix of M below and to the left of (i,j).

The torus action

It is natural to make the torus $T = \mathbb{C}^{L+1}$ act on $\mathcal{O}$ by scaling and conjugation by diagonal matrices. Equivalently, the entries of the matrices are transformed as

$$M_{ij} \mapsto q^2 z_i z_j^{-1} M_{ij}, \qquad t = (q, z_1, \ldots, z_L) \in T$$

This action extends to the coordinate ring $\mathbb{C}[\mathcal{O}_\pi]$ (i.e., ring of polynomial functions on $\mathcal{O}_\pi$), so we can define the character

$$\chi_\pi = \mathrm{tr}_{\mathbb{C}[\mathcal{O}_\pi]} t = \sum_{k,n_1,\ldots,n_L} q^k z_1^{n_1} \ldots z_L^{n_L} \dim \mathbb{C}[\mathcal{O}_\pi]_{k,n_1,\ldots,n_L}$$

It is well-known that χ_π is a rational function, and that

$$K_\pi = \prod_{1 \le i < j \le L} (1 - q^2 z_i z_j^{-1})\, \chi_\pi$$

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The modules

Up to $L = 4$, these K_π are equal (up to a monomial) to the Ψ_π.

At $L = 6$, still true except for component !!!

We need to define appropriate modules over $\mathbb{C}[\mathcal{O}_\pi]$.

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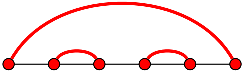
At $L = 6$, still true except for component  !!!

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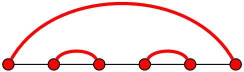
At $L = 6$, still true except for component  !!!

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Some combinatorial data

Given a link pattern π, call two arcs α and β neighboring if they can be connected without crossing another arc and if neither contains the other.

Given such α, β, denote $e_{\alpha,\beta}\pi$ the new link pattern obtained by reconnecting the endpoints of α, β in the other noncrossing way:

Also given an arc α inside a link pattern π, denote depth(α) the number of arcs crossed going from α to upwards infinity.

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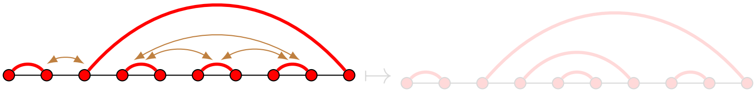

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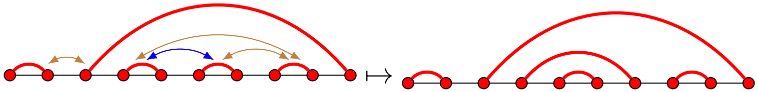

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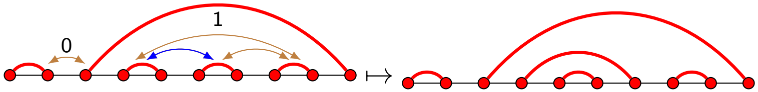

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The main theorem

Given a link pattern π, and neighboring arcs α and β, $\mathcal{O}_\pi \cap \mathcal{O}_{e_{\alpha,\beta}\pi}$ is a divisor inside $\mathcal{O}_\pi$.

We can therefore define the module $\mathcal{M}_\pi$ of rational functions on $\mathcal{O}_\pi$ with poles of order at most depth(α) = depth(β) on $\mathcal{O}_\pi \cap \mathcal{O}_{e_{\alpha,\beta}\pi}$, α, β running over neighboring pairs of π.

The torus T naturally acts on $\mathcal{M}_\pi$, so we can define its Hilbert series χ'_π and $K'_\pi = \prod_{1\le i<j\le L}(1 - q^2 z_i z_j^{-1})\, \chi'_\pi$. Then

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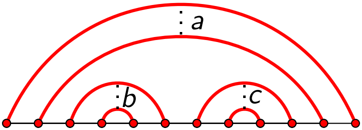
The (a, b, c) case

Now consider link patterns of the form



Then $\Psi_{(a,b,c)} = |PP(a, b, c)|$ where $PP(a, b, c)$ is the set of lozenge tilings of a $a \times b \times c$ hexagon, or plane partitions of $c \times b$ with maximal part a. [conjectured by Zuber for FPLs; proven by DF, Z-J, Zuber, '03]

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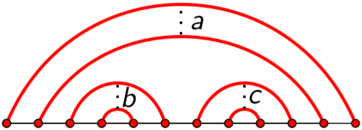

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(a, b, c) and Gröbner degeneration

Theorem (Knutson, Z-J, '16)

There is a Gröbner degeneration of $\mathcal{M}_\pi$ (where $\pi = (a, b, c)$) into a direct sum of coordinate rings of toric varieties naturally indexed by plane partitions.

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PDF's conjecture

The same strategy works for other series of examples. In fact, we recover this way more than the Razumov–Stroganov correspondence; we get a proof (for various series of examples) of

Conjecture (Di Francesco, '06)

For every link pattern π, Ψ_π can be decomposed as a sum of products of the form

$$\Psi_\pi = \sum_{f \in FPL_\pi} \prod_{a=1}^{n(n-1)} \left(q^{\alpha_{f,a}} z_{j_{f,a}} - q^{-\alpha_{f,a}} z_{i_{f,a}} \right)$$

where $\alpha_{f,a} \in \{1,2\}$, and the indexing set FPL_π is the set of FPLs with connectivity π.

This implies positivity of coefficients of $\Psi_{\pi|homogeneous}(-q-q^{-1})$.