Algebraic Heun operator and its applications

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Bispectral pairs

Formal definition

The pair of operators X, Y which satisfy $AW(3)$ algebra relations

$$[X, Y] = Z,$$

$$[Y, Z] = -2\nu YXY + A_2 Y^2 + A_1\{X, Y\} + BY + C_1 X + E_1,$$
$$[Z, X] = -2\nu XYX + A_1 X^2 + A_2\{X, Y\} + BX + C_2 Y + E_2,$$

$A_{1,2}, C_{1,2}, E_{1,2}, B, \nu$ are **structure constants** of the AW-algebra (Granovskii, Lutzenko, AZ (1992)).

The Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [Z, X], Y] = 0$$

holds for AW-algebra.

Casimir operator

$$Q = Z^2 - \nu \left(XY^2X + YX^2Y\right) + (2+\nu)\left(A_1 XYX + A_2 YXY\right) + \text{quadratic terms}$$

commutes with generators

$$[Q, X] = [Q, Y] = [Q, Z] = 0$$

Affine property:

change

$$X \to \alpha_1 X + \alpha_2, \; Y \to \beta_1 Y + \beta_2$$

leads to the same algebra with "shifted" parameters

Mutual tridiagonality

Assume that X, Y are finite-dimensional and that e_n is eigenbasis for X and d_n eigenbasis for Y:

$$Xe_n = \lambda_n e_n, \quad Yd_n = \mu_n d_n, \ n = 0, 1, 2, \ldots, N$$

Then

$$Xd_n = a_{n+1}d_{n+1} + b_n d_n + a_n d_{n-1}, \quad Ye_n = \xi_{n+1}e_{n+1} + \eta_n e_n + \xi_n e_{n-1}$$

Eigenbases expansions

$$e_s = \sum_{n=0}^{N} \sqrt{w_s}\varphi_n(\lambda_s)d_n, \ d_n = \sum_{s=0}^{N} \sqrt{w_s}\varphi_n(\lambda_s)e_s,$$

where $\varphi_n(x)$ are orthonormal polynomials defined by 3-term recurrence relation

$$a_{n+1}\varphi_{n+1}(x) + b_n\varphi_n(x) + a_n\varphi_{n-1}(x) = x\varphi_n(x), \quad \varphi_{-1} = 0, \ \varphi_0 = 1$$

These polynomials are orthogonal

$$\sum_{s=0}^{N} w_s\varphi_n(\lambda_s)\varphi_m(\lambda_s) = \delta_{nm}$$

Dual orthonormal polynomials $\chi_n(x)$

$$\xi_{n+1}\chi_{n+1}(x) + \eta_n\chi_n(x) + \xi_n\chi_{n-1}(x) = x\chi_n(x), \quad \chi_{-1} = 0, \ \chi_0 = 1$$

They are orthogonal

$$\sum_{s=0}^{N} \tilde{w}_s\chi_n(\mu_s)\chi_m(\mu_s) = \delta_{nm}$$

with respect to dual weights $\tilde{w}_s$

Dual expansion formulas

$$d_s = \sum_{n=0}^{N} \sqrt{\tilde{w}_s} \chi_n(\mu_s) e_n, \quad e_n = \sum_{s=0}^{N} \sqrt{\tilde{w}_s} \chi_n(\mu_s) d_s$$

Comparison leads to **Leonard duality**

$$\sqrt{w_s} \varphi_n(\lambda_s) = \sqrt{\tilde{w}_n} \chi_s(\mu_n)$$

Equivalently, this means that polynomials $\varphi_n(x)$ satisfy **2nd-degree difference equation**

$$A(s)\left(\varphi_n(\lambda_{s+1}) - \varphi_n(\lambda_s)\right) + C(s)\left(\varphi_n(\lambda_{s-1}) - \varphi_n(\lambda_s)\right) = \lambda_n \varphi_n(\lambda_s)$$

In the infinite-dimensional case $N = \infty$ there are several possibilities for duality property.
The "classical" duality means that orthogonal polynomials $P_n(x)$ satisfy both the recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x)$$

and 2nd-order differential equation

$$F_2(x) P_n''(x) + F_1(x) P_n'(x) + F_0 P_n(x) = \lambda_n P_n(x)$$

where $\deg(F_i) \le i$.
For generic (Askey-Wilson) duality the polynomials satisfy the difference equation

$$A(s)\left(P_n(x(s+1)) - P_n(x(s))\right) + C(s)\left(P_n(x(s-1)) - P_n(x(s))\right) = \lambda_n P_n(x(s))$$

where x_s is specific greed (AW-greed)

Main statement:

All dual orthogonal polynomials are described by AW-algebra

"Direct picture":

$$X = x(s), \quad Y = A(s)(T^+ - I) + C(s)(T^- - I)$$

$T^{\pm}$ are shift operators

$$T^{\pm} f(s) = f(s \pm 1)$$

"Dual picture"

$$Xe_n = e_{n-1} + b_n e_n + u_{n+1} e_{n+1}, \quad Ye_n = \lambda_n e_n$$

Example: Jacobi algebra

X is multiplication by x, Y is Gauss hypergeometric operator

$$X = x, \quad Y = x(1-x)\partial_x^2 + \big(\alpha + 1 - (\alpha+\beta+2)\, x\big)\partial_x$$

The dual picture

$$X e_n = e_{n-1} + b_n e_n + u_{n+1} e_{n+1}, \quad Y e_n = \lambda_n e_n$$

where

$$\lambda_n = -n(n+\alpha+\beta+1)$$

This corresponds to Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

$$Y P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(x)$$

and

$$X P_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n P_n^{(\alpha,\beta)}(x) + u_n P_n^{(\alpha,\beta)}(x)$$

Quadratic Jacobi algebra (Granovskii, Lutzenko, AZ, 1992)

$$[X,Y]=Z, \quad [Y,Z]=a_2\{X,Y\}+dY+c_2X+e_2, \quad [Z,X]=a_2X^2+dX$$

structures constants are

$$a_2=-2, \quad d=2, \quad c_2=(\alpha+\beta)(\alpha+\beta+2), \quad e_2=-(\alpha+1)(\alpha+\beta).$$

Casimir operator

$$Q=a_2\{X^2,Y\}+Z^2+(a_2^2+c_2)X^2+d\{X,Y\}+(da_2+2e_2)Y$$

is reduced to constant

$$Q\,f(x)=(\alpha^2-1)f(x)$$

q-algebra and q-polynomials

When $q \neq 0, \pm 1$ the parameter $\nu \neq 0$. New parameter q instead of ν

$$2\nu = q + q^{-1} - 2$$

Then AW-algebra can be presented in terms of two relations (Terwilliger, 2000)

$$X^2 Y + Y X^2 - (q + q^{-1}) XYX = A_1 Y + BX + C_1,$$

$$Y^2 X + X Y^2 - (q + q^{-1}) YXY = A_2 X + BY + C_2$$

Another form in terms of "linear q-algebra" (Granovskii, AZ 1993):

$$[X, Y]_q = Z, \ [Y, Z]_q = A_1 Y + BX + C_1, \ [Z, X]_q = A_2 Y + BX + C_2$$

where $[X, Y]_q = XY - qYX$ is "q-mutator"

Z_3 **form** (Wiegmann, Zabrodin, 1995, Terwilliger, 2004)

$$[X,Y]_q = a_3 Z + \omega_3, \ [Y,Z]_q = a_1 X + \omega_1, \ [Z,X]_q = a_2 Y + \omega_2$$

All q-polynomials from Askey scheme are described by representations of above algebra.

In case if $a_1 a_2 a_3 \neq 0$ it is possible to put

$$a_1 = a_2 = a_3 = 1$$

Then there are 4 independent parameters: $\omega_1, \omega_2, \omega_3$ and the value Q of the Casimir operator. They correspond to 4 parameters of Askey-Wilson polynomials.

Degenerations

When some parameters a_i are zero then the Aksey-Wilson polynomials degenerate to other polynomials from the Askey scheme.

For example when $a_1 = 0$ one has **big q-Jacobi** polynomials (or **q-Hahn** polynomials in finite-dimensional case).

The case $a_1 = \omega_1 = 0$ corresponds to **little q-Jacobi** polynomials etc.

The maximal degeneration $a_1 = a_2 = a_3 = 0$ leads to the algebra

$$[X, Y]_q = 1, \ [Y, Z]_q = 1, \ [Z, X]_q = 1$$

Each relation is the **q-oscillator**: $XY - qYX = 1$. This algebra is called equitable $sl_q(2)$ algebra (Terwilliger, 2005)

Bannai-Ito algebra

Put $q = -1$:

$$\{X, Y\} = a_3 Z + \omega_3,\ \{Y, Z\} = a_1 X + \omega_1,\ \{Z, X\} = a_2 Y + \omega_2$$

Describes **Bannai-Ito polynomials** (=limit $q \to -1$ of Askey-Wilson polynomials)

Degenerations: $a_1 = 0$ corresponds to **big -1 Jacobi** polynomials (L.Vinet, AZ, 2010);

$a_1 = \omega_1 = 0$ corresponds to **little -1 Jacobi** polynomials.

Applications

1. **Structure relations of polynomials form Askey scheme**
2. **Explanation of duality property**
3. **6j- and 3j-symbols**
4. **Application to completely integrable systems**
5. **DAHA and nonsymmetric AW-polynomials**

Algebraic Heun pencil

Let X, Y - bipsectral pair. Consider the operator

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 \mathcal{I}$$

Properties

(i) M is generic **bilinear** operator with respect to X and Y.

(ii) M possesses a **bispectrality** property

Assume that X and Y are finite-dimensional. Then

$$Xe_n = \lambda_n e_n, \quad Yd_n = \mu_n d_n, \; n = 0, 1, 2, \ldots, N$$

and

$$Xd_n = a_{n+1}d_{n+1} + b_n d_n + a_n d_{n-1}, \quad Ye_n = \xi_{n+1}e_{n+1} + \eta_n e_n + \xi_n e_{n-1}$$

It is clear that M is 3-diagonal with respect to **both** bases e_n and d_n:

$$Me_n = \{e_{n-1}, e_n, e_{n+1}\}, \quad Md_n = \{d_{n-1}, d_n, d_{n+1}\}$$

Nomura and Terwilliger (2007) proved that the most general operator with this (bispectrality) property coincides with M.

Consider another example: **Jacobi algebra**

$$X = x, \quad Y = x(1-x)\partial_x^2 + (\nu_1 x + \nu_2)\partial_x$$

The Jacobi polynomials are eigenfunctions of the operator Y

$$Y P_n^{(\omega_1,\omega_2)}(x) = \lambda_n P_n^{(\omega_1,\omega_2)}(x)$$

The operator X is 3-diagonal:

$$X P_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x)$$

The operator M is 2nd-order differential operator wich is tridiagonal with respect to Jacobi polynomials

$$M P_n(x) = \xi_{n+1} Pn + 1(x) + \eta_n P_n(x) + \zeta_n P_{n-1}(x)$$

where

$$\xi_n = \tau_1 \lambda_{n-1} + \tau_2 \lambda_n + \tau_3,$$

$$\zeta_n = \tau_1 \lambda_n + \tau_2 \lambda_{n-1} + \tau_3,$$

$$\eta_n = (\tau_1 + \tau_2) b_n + \tau_3 b_n + \tau_4 \lambda_n + \tau_0$$

Main statements (A.Grünbaum, L.Vinet, AZ, 2016)

Proposition 1. The most general 2nd-order differential operator which is 3-diagonal on Jacobi polynomials coincides with M.

Proposition 2. The operator M coincides with generic Heun operator.

Heun equation and Heun operator

Heun equation

$$\psi''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d}\right)\psi'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-d)}\,\psi(x) = 0$$

Heun operator

$$H = x(x-1)(x-d)\partial_x^2 + x(x-1)(x-d)\left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d}\right)\partial_x + \alpha\beta x - q$$

Correspondence of parameters

$$\gamma = \nu_2,\ \delta = -\nu_1 - \nu_2,\ \varepsilon = 2\tau_2,\ d = -\tau_4,$$

$$\alpha\beta = -\tau_3 - \nu_1\tau_2,\ q = -\lambda - \tau_2\nu_2$$

Remark: necessarily $\tau_1 + \tau_2 \neq 0$, otherwise the operator M becomes degenerate (confluent) Heun operator. Hence one can put

$$\tau_1 + \tau_2 = 1.$$

Racah-Heun algebra

$$[Y, M] = Z,$$
$$[M, Z] = \alpha_1\{Y, M\} + \alpha_2 M^2 + \gamma_1 Y + \delta M + \kappa Y^2 + \epsilon_1,$$
$$[Z, Y] = \alpha_2\{Y, M\} + \alpha_1 Y^2 + \gamma_2 M + \delta Y + \epsilon_2.$$

Resembles Racah algebra. The only difference - the term κY^2, where

$$\kappa = 6\tau_4(\tau_4 + 1)$$

When $\tau_4 = 0$ or $\tau_4 = -1$ this algebra becomes the ordinary Racah algebra. Otherwise this is **non-self-dual** algebra. It describes the Heun operator M.

Tridiagonalization of Jacobi polynomials and Racah-Wilson polynomials

Consider special case $\tau_4 = 0$.

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_0 \mathcal{I}$$

Again one can put $\tau_1 + \tau_2 = 1$.
The operator M can be reduced to hypergeometric

$$M = x^2(1-x)\,\partial_x^2 + x\,(\alpha + 1 + 2\,\tau_2 - (\alpha+\beta+2+2\tau_2)x)\,\partial_x$$

$$-(\tau_2(\alpha+\beta+2) - \tau_3)\,x + (\alpha+1)\tau_2 + \tau_0.$$

Duality between L and M

$$L = \tau_1 X^{-1} M + \tau_2 M X^{-1} - (\tau_0 + 2\tau_1\tau_2) X^{-1} + 2\tau_1\tau_2 - \tau_3$$

where

$$X^{-1}\,f(x) = x^{-1}\,f(x)$$

Eigenfunctions

$$M\,\psi_n(x) = \Lambda_n \psi_n(x)$$

where

$$\psi_n(x) = x^{\nu_2 - 1} P_n^{(\widetilde{\alpha}, \widetilde{\beta})}(1/x),$$

$$\Lambda_n = n(n-1) + a_2 n + d_2$$

Expansion

$$\psi(x;\Lambda) = \sum_{k=0}^{\infty} G_k(\Lambda) P_k^{(\alpha,\beta)}(x)$$

Put

$$G_k(\Lambda) = G_0(\Lambda)\,\Xi_k\,Q_k(\Lambda)$$

Then $Q_n(\Lambda)$ are orthogonal polynomials satisfying

$$E_n^{(1)} Q_{n+1}(\Lambda) + E_n^{(2)} Q_n(\Lambda) + E_n^{(3)} Q_{n-1}(\Lambda) = \Lambda\, Q_n(\Lambda)$$

where coefficients $E_n^{(i)}$ coincide with those for Racah-Wilson polynomials.

This leads to

Proposition. **Racah-Wilson polynomials can be obtained by tridiagonalization of Jacobi polynomials** (Genest, Ismail, Vinet, AZ, 2016)

This yields integral representation of Racah-Wilson polynomials (Koornwinder, 1985)

$$Q_n(\Lambda) = \frac{1}{h_n G_0(\Lambda) \Xi_n} \int_0^1 \psi(x;\Lambda) P_k^{(\alpha,\beta)}(x) x^{\alpha} (1-x)^{\beta} dx.$$

Finite-dimensional reduction

Condition $\nu_2 = N + 1$ means that the operator M preserves space of polynomials of degree $\leq N$. This leads to finite-dimensional reduction. Then eigenfunctions of M are polynomials

$$\psi_n(x) = x^N \, P_n^{(-\alpha-\beta-\nu_1-N-1,\beta)}(1/x)$$

Expansion

$$\psi_n(x) = \sum_{k=0}^{N} R_{nk} P_k^{(\alpha,\beta)}(x)$$

leads to Racah polynomials R_{nk}.

Heun-Hahn operator

2nd order difference operator

$$W = A(x)T^{+} + B(x)T^{-} + C(x)\mathcal{I}$$

where $T^{\pm} f(x) = f(x \pm 1)$.

The main condition:

$$W\pi_n(x) = \pi_{n+1}(x)$$

What is the most general operator W on the grid $x = 0, 1, 2, \ldots, N$ with this property?

Answer:

$$A(x) = (x - N)(\kappa x^2 + \mu_1 x + \mu_0), \quad B(x) = x(\kappa x^2 + \nu_1 x + \nu_0),$$

$$C(x) = -A_1(x) - A_2(x) + r_1 x + r_0$$

Main statement

Operator W coincides with algebraic Heun-Hahn operator

$$W = M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 \mathcal{I}$$

with Y difference operator for **Hahn** polynomials

Consequence

Operator W is 3-diagonal in Pochhammer basis

$$W(-x)_n = \{(-x)_{n+1}, (-x)_n, (-x)_{n-1}\}$$

Heun-Racah algebra

Pair of operators Y, W satisfy commutation relations

$$[Y,[Y,W]] = g_1 Y^2 + g_2\{Y,W\} + g_3 Y + g_4 W + g_5 \mathcal{I}$$

and

$$[W,[W,Y]] = e_1 Y^2 + e_2 Y^3 + g_2 W^2 + g_1\{Y,W\} + g_3 W + g_6 Y + g_7$$

Mostly Racah algebra. But - two extra-terms.

$$e_2 = 2(\tau_1+\tau_2)^2$$

$$e_1 = 6\tau_4^2 + 3(\tau_1+\tau_2)(\tau_3 + (2N+\beta-\alpha)\tau_4) -$$
$$(\tau_1^2+\tau_2^2)(3N(\alpha+1)-2) - 2(3N(\alpha+1)-5)\tau_1\tau_2$$

What happens when $e_1 = e_2 = 0$? Racah algebra. There are two possible cases: W_1 and W_2

$$W_1 = \frac{[X,Y]}{2} + \gamma X - \frac{Y}{2} + \varepsilon \mathcal{I}$$

$$W_2 = -\frac{[X,Y]}{2} - \gamma X - \frac{Y}{2} - \varepsilon \mathcal{I}$$

Moreover

$$Y + W_1 + W_2 = 0$$

This means that Y, W_1, W_2 form **equitable Racah algebra**

Operators W_1, W_2 becomes **first-order** operators

$$W_1 = A_1(x)T^+ + C_1(x), \quad W_2 = A_2(x)T^- + C_2(x)$$

Quasi-exactly solvable problems

Let M is a tridiagonal operator (infinite-dimensional, in general)

$$M e_n = A_{n+1} e_{n+1} + B_n e_n + C_n e_{n-1}, \quad n = 0, 1, 2, \dots$$

When M commutes with the projection operator π_J?

$$\pi_J e_n = \begin{cases} e_n, & \text{if} \quad n \le J \\ 0, & \text{if} \quad n > J \end{cases}$$

Operator π_J projects the infinite-dimensional space $\{e_0, e_1, \dots\}$ onto finite-dimensional subspace $e_0, e_1, \dots, e_J$.
Commutation condition $[M, \pi_J] = 0$ means restriction of the operator M onto finite-dimensional subspace.

Necessary and sufficient conditions

$$A_{J+1} = C_{J+1} = 0$$

Application to algebraic Heun operator

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 \mathcal{I}$$

$$Xe_n = \lambda_n e_n, \quad Ye_n = \xi_{n+1} e_{n+1} + \eta_n e_n + \xi_n e_{n-1}$$

The recurrence coefficients of M are

$$A_n = (\tau_1 \lambda_n + \tau_2 \lambda_{n-1} + \tau_4)\xi_n,$$

$$C_n = (\tau_1 \lambda_{n-1} + \tau_2 \lambda_n + \tau_4)\xi_n,$$

$$B_n = (\tau_1 + \tau_2)\eta_n \lambda_n + \tau_3 \lambda_n + \tau_4 \eta_n + \tau_0$$

Commutativity conditions

$$\tau_1 \lambda_{J+1} + \tau_2 \lambda_J + \tau_4 = 0, \quad \tau_1 \lambda_J + \tau_2 \lambda_{J+1} + \tau_4 = 0$$

Solution of these condition

$$\tau_1 = \tau_2, \quad \tau_1 \left(\lambda_J + \lambda_{J+1}\right) + \tau_4 = 0.$$

It is always possible to find τ_1, τ_4 to fulfill this condition.

The operator

$$\tilde{M} = \pi_J M \pi_J = M \pi_J = \pi_J M$$

is the restriction of the operator M to finite-dimensional subspace.

Simple example: Jacobi algebra. In this case M is the ordinary Heun operator and $\tilde{M}$ corresponds to **quasi-exactly solvable** problems (Turbiner, 1988). This leads to potentials which admits exact solution of Scródinger equation for several levels $n = 0, 1, \ldots, J$.

Time and band limiting

What happens if one tries to restrict simultaneously both bispectral operators X and Y?

Consider finite-dimensional case: $n = 0, 1, \dots, N$. Two eigenbases e_n and d_n and two projectors π_1 and π_2

$$\pi_1 e_n = \begin{cases} e_n, & \text{if} \quad n \le J_1 \\ 0, & \text{if} \quad n > J_1 \end{cases}$$

$$\pi_2 d_n = \begin{cases} d_n, & \text{if} \quad n \le J_2 \\ 0, & \text{if} \quad n > J_2 \end{cases}$$

They satisfy

$$\pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2.$$

If $J_1 = N$ then $\pi_1 = \mathcal{I}$. Similarly, if $J_2 = N$ then $\pi_2 = \mathcal{I}$.

Combination of projectors yields two **restriction operators**:

$$V_1 = \pi_1\pi_2\pi_1 = K_1K_2, \quad V_2 = \pi_2\pi_1\pi_2 = K_2K_1,$$

where

$$K_1 = \pi_1\pi_2, \quad K_2 = \pi_2\pi_1$$

Operators V_1 and V_2 are symmetric and hence are diagonalizable and have the same eigenvalues (may be degenerate).

When $J_1 = J_2 = N$ both V_1 and V_2 are identity operators $V_1 = V_2 = I$. When only $J_2 = N$ and J_1 arbitrary then $V_1 = V_2 = J_1$ and the operator V_1 has $J_1 + 1$ eigenvalues equal to 1 and $N - J_1$ zero eigenvalues.

But what are eigenvectors and eigenvalues of operators V_1, V_2 for arbitrary J_1, J_2?

The operators V_1, V_2 are **highly nonlocal**. For example:

$$V_{ik}^{(1)} = \sum_{s=0}^{J_2} \tilde{w}_s \chi_i(\mu_s) \chi_k(\mu_s)$$

Hence the problem of funding eigenvectors and eigenvalues is **very complicated**.

Main idea - to find a **3-diagonal matrix** M which commutes with both V_1 and V_2

$$[M, V_1] = [M, V_2] = 0.$$

Because eigenvalue problem for 3-diagonal matrices is much easier to solve.

This idea back to seminal papers by Slepian, Landau, Pollak in 50-ths on time-band limiting. Developed by A.Grúnbaum and his colleagues in 80-ths

Try algebraic Heun operator

$$M = \tau_1\{X, Y\} + \tau_3 X + \tau_4 Y + \tau_0$$

such that M commutes with both projectors

$$[M, \pi_1] = [M, \pi_2] = 0$$

Then M will also commute with V_1, V_2.

This leads to restrictions on coefficients τ_3, τ_4

$$\tau_1 \left(\lambda_{J_1} + \lambda_{J_1+1}\right) + \tau_4 = 0, \quad \tau_1 \left(\mu_{J_2} + \mu_{J_2+1}\right) + \tau_3 = 0$$

It is **always** possible to find coefficients τ_3, τ_4.

The **only exception** - Bannai-Ito spectrum:

$$\lambda_n = (-1)^n \left(n\alpha + \beta\right)$$

BTL for anti-Krawtchouk polynomials

For BI polynomials the above method does not work - there are no nontrivial 3-diagonal commuting operator M commuting with V_1, V_2.

Instead, one can try **5-diagonal** operator M.

Consider special example: anti-Krawtchouk polynomials (V.Genest, L.Vinet, AZ, 2014).

Anti-spin algebra

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2,$$

Irreducible representation with diagonal L_2

$$L_2 e_n = x_n e_n, \quad n = 0, 1, \dots, N$$

Spectrum of BI type

$$x_n = (-1)^n (n + 1/2)$$

The operator L_1 is 3-diagonal

$$L_1 e_n = a_{n+1} e_{n+1} + b_n e_n + a_n e_{n-1}$$

with

$$a_n^2 = \frac{(N+1)^2 - n^2}{4}, \quad n = 1, 2, \dots, N$$

and

$$b_0 = (-1)^N (N+1)/2, \; b_n = 0, \; n = 1, 2, \dots, N$$

Dual basis

$$L_1 d_n = x_n d_n, \quad L_2 d_n = a_{n+1} d_{n+1} + b_n d_n + a_n d_{n-1}$$

Casimir operator

$$Q = L_1^2 + L_2^2 + L_3^2$$

takes the value

$$Q = (N + 1/2)(N + 3/2)$$

Search commuting operator M in terms of quartic and cubic expressions

$$M = \{L_1^2, L_2^2\} + \alpha_1\{L_1^2, L_2\} + \alpha_2\{L_2^2, L_1\} +$$

$$\alpha_3 L_1^2 + \alpha_4 L_2^2 + \alpha_5 L_1 + \alpha_6 L_2$$

M is **pentadiagonal**

$$M e_n = G_n e_{n-2} + F_n e_{n-1} + H_n e_n + F_{n+1} e_{n+1} + G_{n+2} e_{n+2}$$

Projector

$$\pi_{N_1} e_n = \begin{cases} e_n, & n \le N_1 \\ 0, & n > N_1 \end{cases} .$$

Commutativity $[M, \pi_{N_1}] = 0$ holds iff

$$G_{N_1+1} = G_{N_1+2} = F_{N_1+1} = 0$$

Similalrly, "dual" projector

$$\pi_{N_2} d_n = \begin{cases} d_n, & n \le N_2 \\ 0, & n > N_2 \end{cases} .$$

Commutativity $[M, \pi_{N_2}] = 0$ conditions

$$\tilde{G}_{N_2+1} = \tilde{G}_{N_2+2} = \tilde{F}_{N_2+1} = 0$$

6 equations for 6 unknowns. Solution:

$$\alpha_1 = (-1)^{N_1},\ \alpha_2 = (-1)^{N_2},\ \alpha_3 = -1 - \kappa_1,$$

$$\alpha_4 = -1 - \kappa_2,\ \alpha_5 = (-1)^{N_2}\kappa_1,\ \alpha_6 = (-1)^{N_1}\kappa_2,$$

where

$$\kappa_1 = 2{N_1}^2 + 4N_1 + 5/2, \quad \kappa_2 = 2{N_2}^2 + 4N_2 + 5/2.$$

Perspectives and open problems

- **Classification of algebraic Heun operators**. Hahn (1971) and Takemura (2017) considered **q-version** of Heun operator. It is observed that it coincides with algebraic Heun operator for **little q-Jacobi** polynomials.

- **Elliptic Heun operator**. Krichever and Zabrodin (1995) constructed **elliptic Lamé** equation on the base of Sklyanin algebra. How to construct corresponding Heun pencil?

- **Multivariate analogs of AHO**. Ruijsenaars and Van Diejen considered multivariate exactly solvable models related with generalized Heun operators. What is their algebraic description?

- **Heun pencil and biorthogonal functions**. The AHO is a **linear pencil**

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0$$

Consider the equation

$$M\psi = 0$$

If eigenvalue $\lambda = -\tau_0$ then we have the **ordinary** eigenvalue problem. Other choices lead to **generalized eigenvalue problem**

$$M_1\psi = \lambda\, M_2\psi$$

where M_1, M_2 - 3-diagonal operators. This leads to theory of **biorthogonal rational functions** (AZ, 1999). What are the most general BRF arising in this problem?

- **Applications to integrable systems**. It is known that after separation of variables for some integrable systems (i.e. Kepler problem) in elliptic coordinates the Heun operators appear as integrals. How to understand this phenomenon from algebraic point of view?

- **Classical analogs of AHO**. It is known that AW-algebra has its classical version with Poisson brackets instead of commutators (A.Korovnichenko, AZ, 2000). What is classical version of AHO?

Thank you for your attention!