Solutions of convex Bethe Ansatz equations and the zeros of (basic) hypergeometric orthogonal polynomials*

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Calogero-Moser Hamiltonian

$$H_{cm} = \frac{1}{2} \sum_{1 \leqslant j \leqslant n} (p_j^2 + x_j^2) + \sum_{1 \leqslant j < k \leqslant n} (x_j - x_k)^{-2}$$

Equations of motion

$$\ddot{x}_j = -x_j + 2 \sum_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} (x_j - x_k)^{-3} \quad (j = 1, \ldots, n)$$

Algebraic equations for the equilibrium configuration

$$x_j = 2 \sum_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} (x_j - x_k)^{-3} \quad (j = 1, \ldots, n) \qquad \text{(C)}$$

Theorem (Calogero '77)

The roots $x_1, \ldots, x_n$ of the Hermite polynomial

$$\boxed{H_n(x) := (-1)^n e^{x^2}\left(\frac{d^n}{dx^n} e^{-x^2}\right) = 2^n (x - x_1)\cdots(x - x_n)}$$

provide a solution to (C).

Idea of proof: via Stieltjes' 1885 'electrostatic interpretation'

- Diff. eq. for $H_n(x)$ implies that

$$x_j = 2 \sum_{\substack{1\leqslant k\leqslant n \\ k\neq j}} (x_j - x_k)^{-1} \quad (j = 1, \ldots, n) \tag{S}$$

  so the roots $x_1, \ldots, x_n$ minimize

$$V(x_1, \ldots, x_n) = \underbrace{\frac{1}{2} \sum_{1\leqslant j\leqslant n} x_j^2}_{\text{harmonic well}} - \underbrace{\sum_{1\leqslant j<k\leqslant n} \log|x_j - x_k|}_{\text{electrostatic potential}}$$

- Calogero's trick $\Rightarrow$ the (real-valued) solutions of (S) also satisfy (C).

□

2. What about your favorite integrable particle model?

Here's mine (vD '94):

$$\boxed{H_f = \sum_{1 \leqslant j \leqslant n} \left(\cosh(p_j) |V_j(\mathbf{x})| - \mathrm{Re}\left[V_j(\mathbf{x}) \right] \right)}$$

where

$$V_j(\mathbf{x}) = w(x_j) \prod_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} v(x_j + x_k) v(x_j - x_k)$$

with

$$v(x) = \frac{\sin(x + ig)}{\sin(x)}$$

$$w(x) = \frac{\sin(x + ig_a) \sin(x + ig_b) \cos(x + ig_c) \cos(x + ig_d)}{\sin^2(x) \cos^2(x)}$$

Note: This model belongs to the Ruijsenaars-Schneider family.

Observation (vD '05):

- H_f is nonnegative.
- The absolute lower bound $H_f = 0$ is attained iff

$$\boxed{p_j = 0 \quad \text{and} \quad \mathrm{Im}\big[V_j(\mathbf{x})\big] = 0} \qquad (j = 1, \ldots, n)$$

i.e.

$$\frac{\sin(x_j + ig_a)}{\sin(x_j - ig_a)} \frac{\sin(x_j + ig_b)}{\sin(x_j - ig_b)} \frac{\cos(x_j + ig_c)}{\cos(x_j - ig_c)} \frac{\cos(x_j + ig_d)}{\cos(x_j - ig_d)}$$
$$\times \prod_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} \frac{\sin(x_j + x_k + ig) \sin(x_j - x_k + ig)}{\sin(x_j + x_k - ig) \sin(x_j - x_k - ig)} = 1 \qquad \text{(BAE)}$$

$j = 1, \ldots, n.$

3. Bethe Ansatz equations

These are Bethe Ansatz equations studied by Ismail, Lin and Roan ('04):

Theorem (Ismail et al '04)

The system (BAE) is solved by the roots $x_1, \ldots, x_n$ of the nth degree Askey-Wilson polynomial.

Corollary (vD '05, Odake-Sasaki '05)

The ground-state equilibrium configuration of H_f is attained at the roots of the Askey-Wilson polynomial.

Note: RS $\leftrightarrow$ AW parameters

$$\boxed{q = e^{-2g},\ a = e^{-2g_a},\ b = e^{-2g_b},\ c = -e^{-2g_c},\ d = -e^{-2g_d}}$$

4. Askey-Wilson polynomials

(Askey-Wilson '85)

Explicit basic hypergeometric representation

$$p_n(\xi; a, b, c, d; q) := \frac{(ab, ac, ad; q)_n}{(2a)^n (abcdq^{n-1}; q)_n} \times {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\xi}, ae^{-i\xi} \\ ab, ac, ad \end{matrix} ; q, q \right]$$

$(\xi = 2x)$.

q-Difference equation

$$A(\xi)\big(p_n(\xi - i\log(q)) - p_n(\xi)\big) + A(-\xi)\big(p_n(\xi + i\log(q)) - p_n(\xi)\big) = E_n p_n(\xi)$$

with

$$A(\xi) = \frac{(1 - ae^{i\xi})(1 - be^{i\xi})(1 - ce^{i\xi})(1 - de^{i\xi})}{(1 - e^{2i\xi})(1 - qe^{2i\xi})},$$

$$E_n = q^{-n}(1 - q^n)(1 - abcdq^{n-1})$$

Orthogonality

- $p_n(\xi; a, b, c, d; q)$ is a polynomial of degree n in $\cos\xi$.
- The AW polynomials form an orthogonal basis with respect to the density

$$\Delta(\xi) = \left| \frac{(e^{2i\xi}; q)_\infty}{(ae^{i\xi}, be^{i\xi}, ce^{i\xi}, de^{i\xi}; q)_\infty} \right|^2 \quad 0 < \xi < \pi.$$

(vD '05, Odake-Sasaki '05, Bihun-Calogero '16)

Solve AW q-difference Eq. by Bethe Ansatz:

- Substitute Bethe Ansatz

$$p_n(\xi) = \big(\cos(\xi) - \cos(\xi_1)\big)\big(\cos(\xi) - \cos(\xi_2)\big) \cdots \big(\cos(\xi) - \cos(\xi_n)\big).$$

- Evaluate polynomial $T - Q$ identity at the Bethe roots

$$A(\xi_j) p_n\big(\xi_j - i\log(q)\big) + A(-\xi_j) p_n\big(\xi_j + i\log(q)\big) = 0 \quad (j = 1, \ldots, n)$$

- Making $A(\cdot)$ and $p_n(\cdot)$ explicit $\Longrightarrow$

$$\left(\frac{1 - ae^{i\xi_j}}{a - e^{i\xi_j}}\right)\left(\frac{1 - be^{i\xi_j}}{b - e^{i\xi_j}}\right)\left(\frac{1 - ce^{i\xi_j}}{c - e^{i\xi_j}}\right)\left(\frac{1 - de^{i\xi_j}}{d - e^{i\xi_j}}\right)$$

$$\times \prod_{1 \leqslant k \leqslant n,\, k \neq j} \frac{1 - qe^{i(\xi_j + \xi_k)}}{q - e^{i(\xi_j + \xi_x)}} \frac{1 - qe^{i(\xi_j - \xi_k)}}{q - e^{i(\xi_j - \xi_k)}} = 1$$

for $j = 1, \ldots, n$.

Note: These equations are nothing else but previous

(BAE)!!!

Conclusion: $\boxed{p_n(\xi) = p_n(\xi; a, b, c, d; q)} \Longrightarrow$

roots of AW polynomial solve (BAE).

(vD-Emsiz '18, following Yang-Yang '69)

Yang-Yang type Morse function:

$$V(\xi) = \sum_{1\leqslant j\leqslant n}\left((n+1-j)\xi_j + \int_0^{\xi_j} v_a(\theta)+v_b(\theta)+v_c(\theta)+v_d(\theta)d\theta\right) + \sum_{1\leqslant j<k\leqslant n}\left(\int_0^{\xi_j+\xi_k} v_q(\theta)d\theta + \int_0^{\xi_k-\xi_j} v_q(\theta)d\theta\right)$$

with

$$v_q(\theta) = \int_0^{\theta} \frac{(1-q^2)d\vartheta}{1-2q\cos(\vartheta)+q^2} = i\log\left(\frac{1-qe^{i\theta}}{e^{i\theta}-q}\right)$$

$(a, b, c, d, q \in (-1, 1))$.

This Morse function has a unique global minimum because

$$V(\boldsymbol{\xi}) \text{ is convex and } V(\boldsymbol{\xi}) \to +\infty \text{ as } |\boldsymbol{\xi}| \to \infty.$$

Upshot: System for the critical points

$$\begin{aligned} & \nu_a(\xi_j) + \nu_b(\xi_j) + \nu_c(\xi_j) + \nu_d(\xi_j) \\ & + \sum_{1 \leqslant k \leqslant n,\, k \neq j} \big(\nu_q(\xi_j + \xi_k) + \nu_q(\xi_j - \xi_k) \big) = 2\pi(n + 1 - j), \end{aligned} \tag{CP}$$

$(j = 1, \ldots, n)$ has a unique solution given by:

The global minimum $\boldsymbol{\xi}^{(n)}$ of $V(\boldsymbol{\xi})$

Note: (CP) = (BAE) up to exponentiation.

Proposition

The coordinates of the unique minimum

$$(\xi_1^{(n)}, \ldots, \xi_n^{(n)})$$

of the Morse function $V(\boldsymbol{\xi})$ provide the roots of the AW polynomial

$$p_n(\xi; a, b, c, d; q).$$

(vD-Emsiz '18)

Theorem (Bounds for AW roots)

The roots $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ of the AW polynomial $p_n(\xi; a, b, c, d; q)$ obey the inequalities

$$\frac{(n+1-j)\pi}{k_-^{(n)}(a,b,c,d;q)} \leqslant \xi_j^{(n)} \leqslant \frac{(n+1-j)\pi}{k_+^{(n)}(a,b,c,d;q)}$$

where

$$k_\pm^{(n)}(a,b,c,d;q) = (n-1)\left(\frac{1-|q|}{1+|q|}\right)^{\pm 1} + \frac{1}{2}\left(\left(\frac{1-|a|}{1+|a|}\right)^{\pm 1} + \left(\frac{1-|b|}{1+|b|}\right)^{\pm 1} + \left(\frac{1-|c|}{1+|c|}\right)^{\pm 1} + \left(\frac{1-|d|}{1+|d|}\right)^{\pm 1}\right).$$

Proof: Immediate from (CP) via mean value theorem using

$$\frac{1-|q|}{1+|q|} \leqslant \underbrace{\frac{1-q^2}{1-2q\cos(\vartheta)+q^2}}_{\nu_q'(\vartheta)} \leqslant \frac{1+|q|}{1-|q|}$$

(for $-1 < q < 1$). □

Example:

$n = 5$	$\xi_5^{(n)}$	$\xi_4^{(n)}$	$\xi_3^{(n)}$	$\xi_2^{(n)}$	$\xi_1^{(n)}$
Root	0.496	0.997	1.508	2.033	2.577
Lower bound	0.400	0.800	1.200	1.600	2.000
Upper bound	0.675	1.350	2.025	2.700	3.375

($n = 5$, $a = 0.300$, $b = -0.200$, $c = 0.150$, $d = 0.100$, and $q = 0.100$.)

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