Rigged configuration bijection for nonexceptional affine types

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Root systems

- A (symmetrizable) *Cartan matrix* $A = (A_{ij})_{i,j \in I}$ with index set $I = \{0, 1, \dots, n\}$:
  - $a_{ii} = 2$,
  - $a_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i \neq j$,
  - $a_{ij} = 0$ if and only if $A_{ij} = 0$,
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- The *fundamental weights* $\{\Lambda_i\}_{i \in I}$ are the dual vectors to the simple coroots with respect to $\langle \ , \ \rangle$.

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- Let $I_0 := I \setminus \{0\}$ be the index set for $\mathfrak{g}_0$.
- Let $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$, where the root/weight lattice of $\mathfrak{g}'$ is given by quotienting by the null root δ and is isomorphic to that of $\mathfrak{g}_0$.

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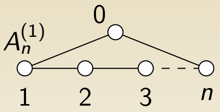

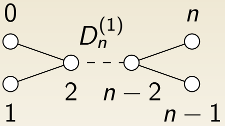

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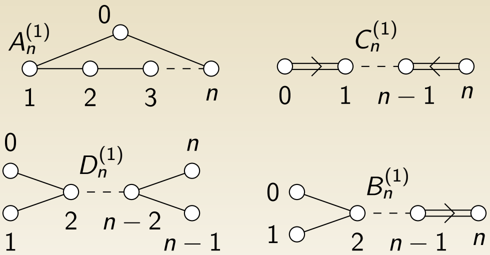

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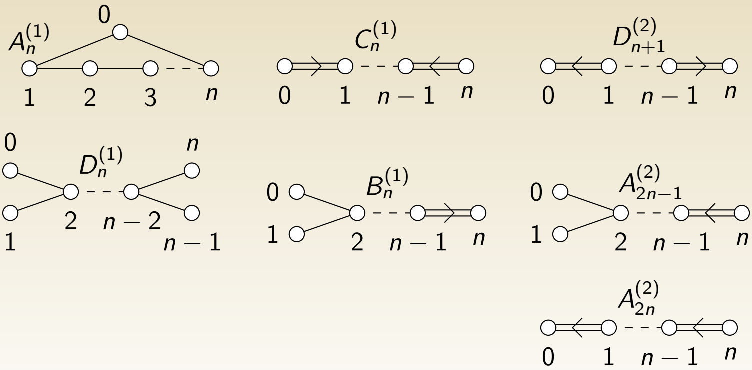
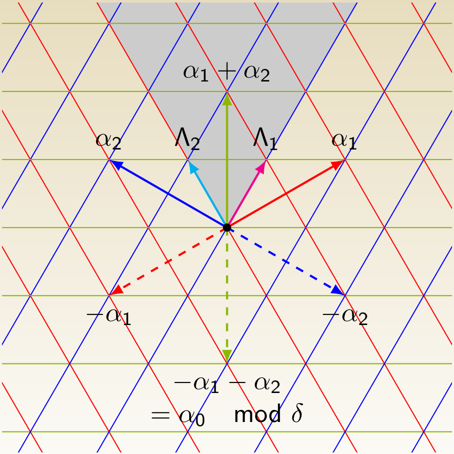
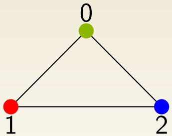


Type $A_2^{(1)}$



$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$



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- The crystal basis and the action of the quantum group (almost) can be encoded in an edge I-colored weighted directed graph.
- The finite-dimensional $U_q(\mathfrak{g}_0)$-crystals in types A_n, B_n, C_n, D_n, can be built by tensor products of the crystals of the vector representation and the spin representations for types B_n and D_n.

Crystals of the vector representation

$$A_n : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}$$

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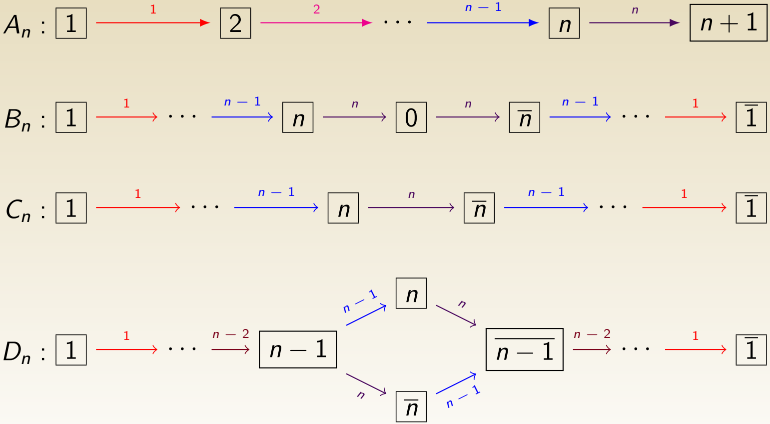

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Theorem (Lenart–Naito–Sagaki–Schilling–Shimozono, 2016-2017)

The graded (Demazure) characters of tensor products of KR modules $\bigotimes_r W^{r,1}$ are (nonsymmetric) Macdonald polynomials at $t = 0$ for type $X_n^{(1)}$.

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Theorem (Naito–Sagaki, 2006–2008)

KR crystals $B^{r,1}$ can be constructed (uniformly) from extremal level-zero crystals of $U_q'(\mathfrak{g})$ and quotienting by "δ".

Kirillov–Reshetikhin crystals, exceptional types

Theorem (Kashiwara–Misra–Okado–Yamada, 2007)

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Theorem (Jones–Schilling, 2010)

For type $E_6^{(1)}$ and r a leaf node in type E_6, $B^{r,s}$ exists and was described by using a diagram automorphism.

Tensor products of KR crystals

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- KR crystals cannot be built up from other KR crystals by only using tensor products.

Example of $B^{1,1} \otimes B^{1,1}$ and $B^{1,2}$ in type $A_2^{(1)}$

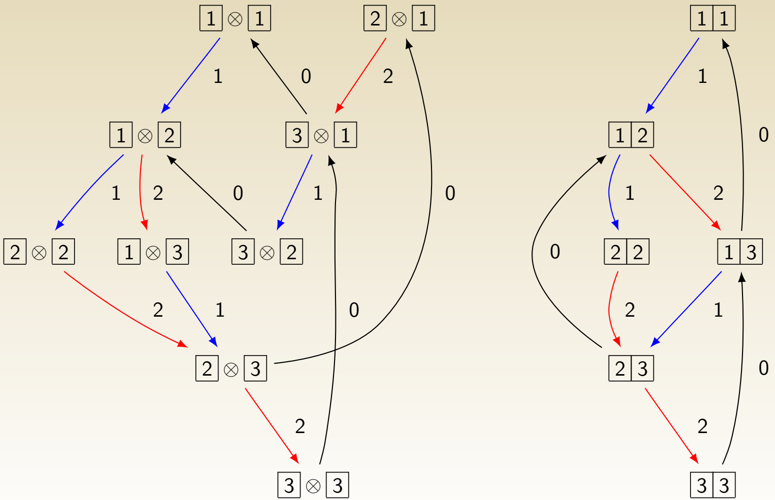

Diagram foldings

- A *diagram folding* is map $\phi \colon X_n^{(k)} \to \widehat{X}_n^{(k)}$ with *scaling factors* $\{\gamma_i \mid i \in I\}$ such that

$$\Lambda_i \mapsto \gamma_i \sum_{j \in \phi^{-1}(i)} \widehat{\Lambda}_j, \qquad \alpha_i \mapsto \gamma_i \sum_{j \in \phi^{-1}(i)} \widehat{\alpha}_j.$$

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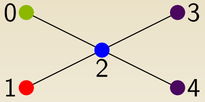
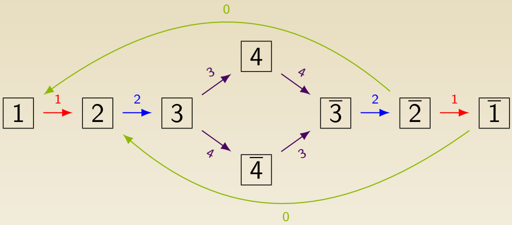
- A diagram folding induces a map on KR crystals $B^{r,s} \to \bigotimes_{a \in \phi^{-1}(r)} B^{a,\gamma_r s}$ by

$$f_i \mapsto \prod_{j \in \phi^{-1}(i)} \widehat{f}_j^{\gamma_i}.$$

Folding of type $D_4^{(1)}$ to $A_5^{(2)}$





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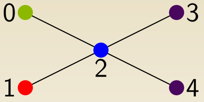

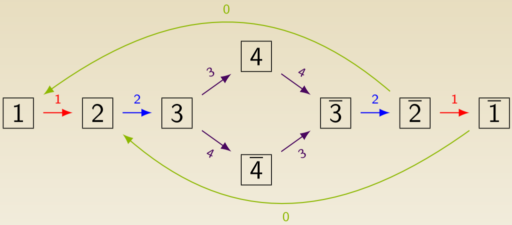

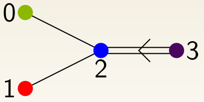

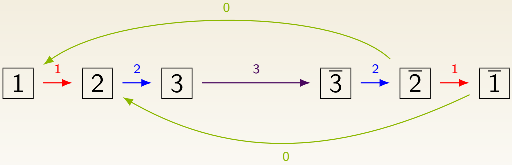

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 1$$

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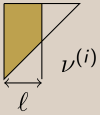
$$p_\ell^{(i)}(\nu, B) := \sum_{k \in \mathbb{Z}_{>0}} \min(k, \ell) \#\{B^{i,k} \in B\} - \sum_{j \in I_0} A_{ij}$$



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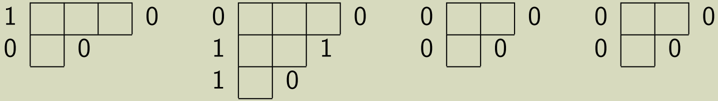

We write riggings on the right and vacancy numbers on the left.

History and facts

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Theorem (Salisbury-S., 2015-2018+)

Rigged configurations can be extended to model the crystal of the lower half of the quantum group $B(\infty)$ for Borcherds algebras, where they have an explicit description of the $*$-involution.

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- This gives the formula $X(B; t)$ as the generating function of the energy statistic on the classically highest weight elements in a tensor product of KR crystals B.
- The *fermionic formula* $M(B; t)$ is the generating function of the *cocharge* statistic on the classically highest weight B-rigged configurations.

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- The *fermionic formula* $M(B; t)$ is the generating function of the *cocharge* statistic on the classically highest weight B-rigged configurations.

Conjecture (Hatayama et al., 1999)

$$X(B; t) = M(B; t)$$

Inception and early 2000s

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- (Okado–Sakamoto–Schilling–S., 2017): General case for type $D_n^{(1)}$.
- (S., 2017): Type $D_4^{(3)}$ for factors $B^{1,s}$ and $B^{r,1}$.

Outline

1. Background
2. The $X = M$ Conjecture
3. **Proof and examples**
   - Main Theorem
   - Examples
   - Future directions

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Theorem (Okado–Schilling–S., 2017)

There exists a bijection Φ for all nonexceptional types from tensor products of KR crystals to rigged configurations that sends energy to cocharge.

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The proof is using diagram foldings and bijection in the general cases for types $A_n^{(1)}$ and $D_n^{(1)}$.

Remark

There exists direct (*i.e.*, not using diagram foldings) algorithm for bijections in all nonexceptional types.

Diagram folding on rigged configurations

Definition

Let $\gamma_i \nu^{(i)}$ denote scaling each row and rigging in $\nu^{(i)}$ by γ_i. Form *virtual rigged configuration* by $\widehat{\nu}^{(j)} = \gamma_i \nu^{(i)}$ for all $j \in \phi^{-1}(i)$.

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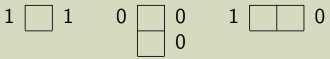
Example

$$\mathfrak{g} = B_3^{(1)}, B = B^{1,1} \otimes B^{3,1} \otimes B^{2,1},$$



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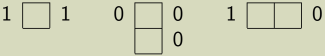

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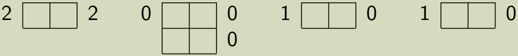

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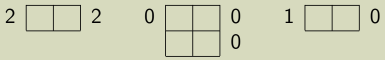

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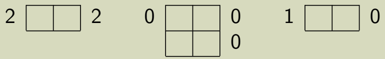

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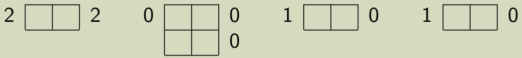

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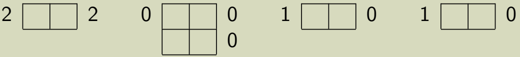

Bijection Φ for types $A_n^{(1)}$ and $D_n^{(1)}$

Definition ($B^{r,1}$ with r in the orbit of 0)

- Start at the classically highest weight element. Set $\ell = 0$.

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Same as above except start at $\boxed{r}$.

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Definition ($B^{r,s}$ type $D_n^{(1)}$ with $1 < r < n - 1$)

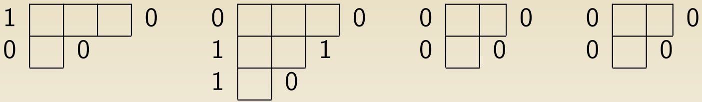
Same as above except start at $\boxed{r}$.

Definition ($B^{r,s}$ with $s > 1$)

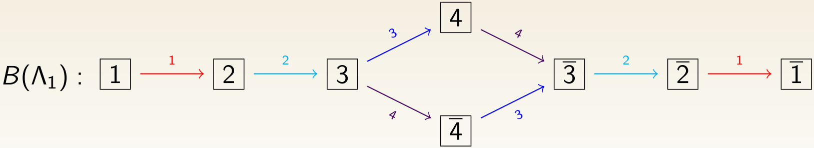
Split to $B^{r,1} \otimes B^{r,s-1}$ and do nothing on rigged configurations (changes vacancy numbers).

Example of bijection in type $D_4^{(1)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{4,1} \otimes B^{2,2}$$



$$\Phi(\nu, J) =$$



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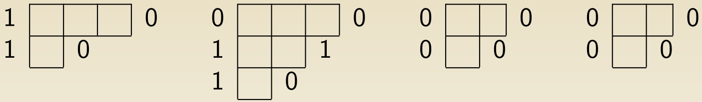

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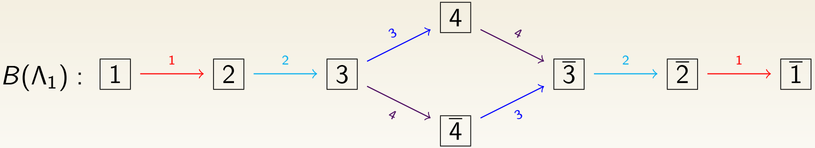

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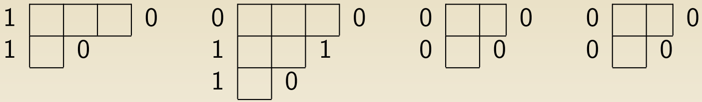

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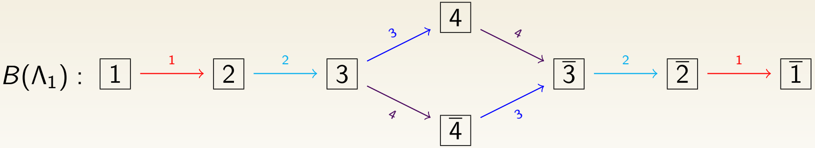

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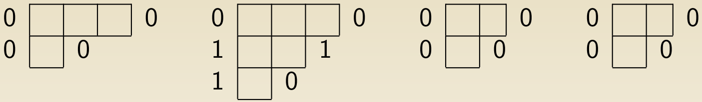

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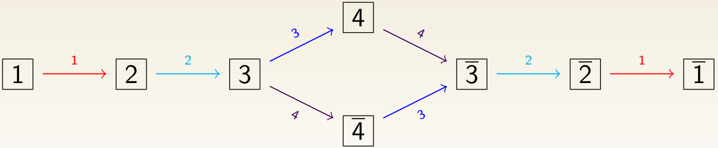

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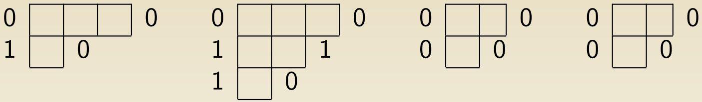

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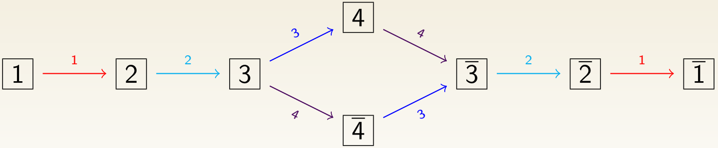

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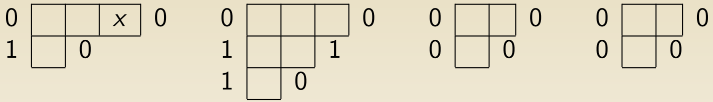

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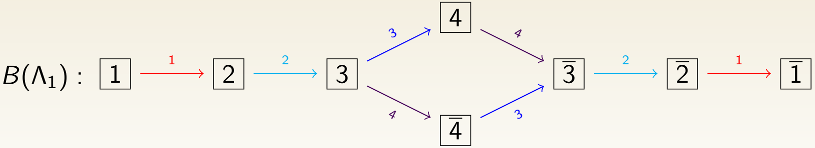

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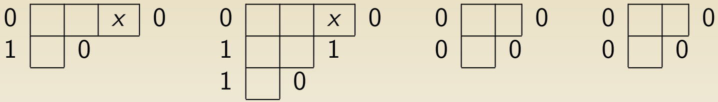

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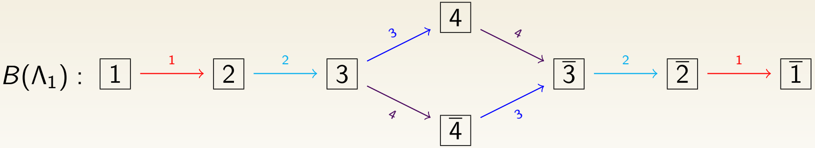

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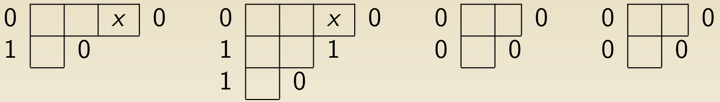

$$\Phi(\nu, J) = \boxed{1}\boxed{3}$$

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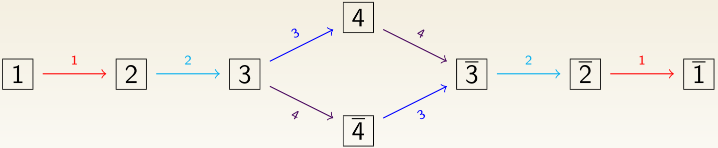

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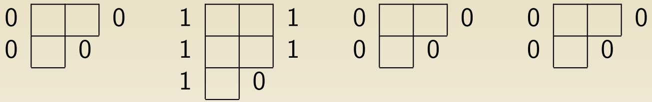

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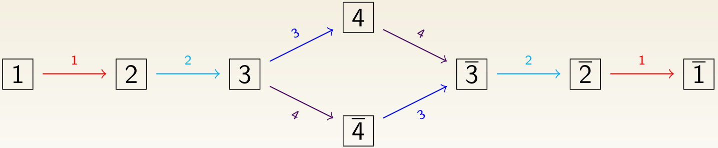

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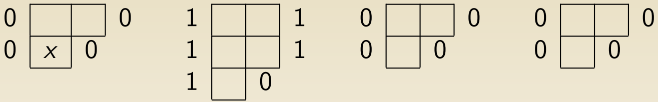

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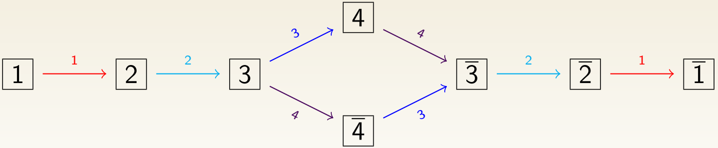

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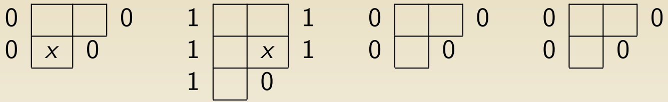

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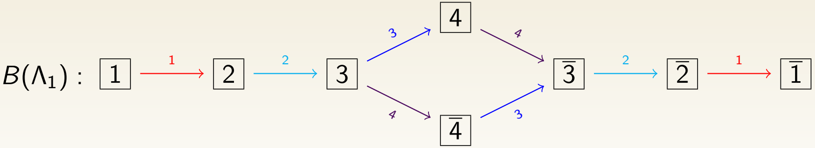

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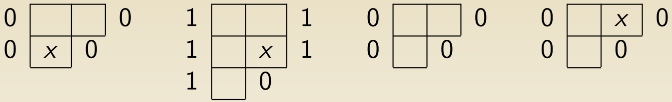

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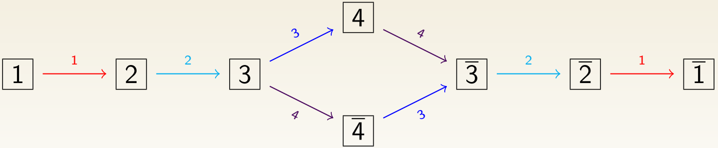

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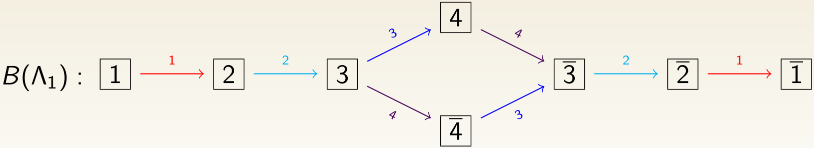

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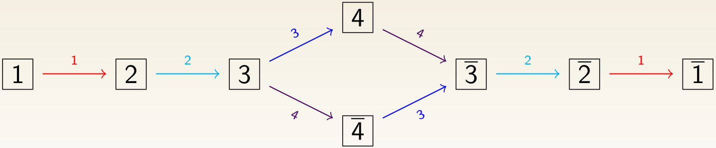

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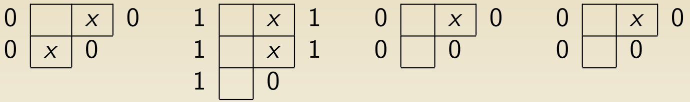
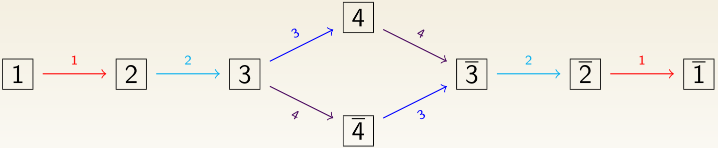


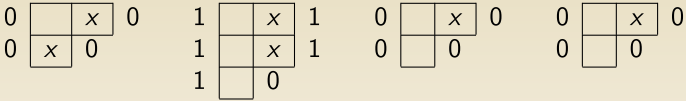
$$\Phi(\nu, J) = \boxed{1}\boxed{3}$$

$B(\Lambda_1)$:

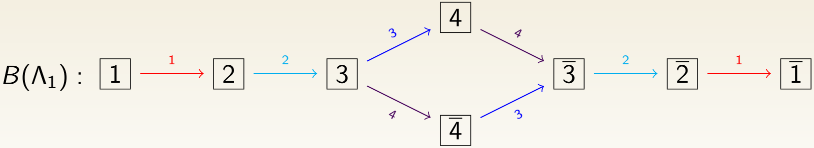


Example of bijection in type $D_4^{(1)}$

$$B^{1,1} \otimes B^{3,1} \otimes B^{4,1} \otimes B^{2,2}$$



$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}}$$



Example of bijection in type $D_4^{(1)}$

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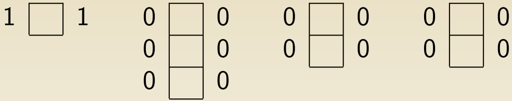
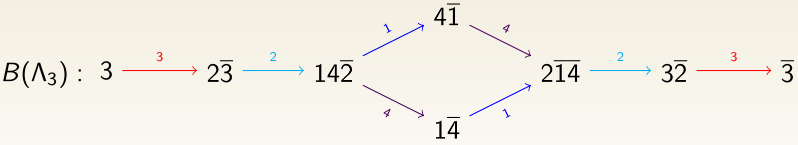


$$\Phi(\nu, J) = \begin{array}{|c|c|c|}\hline 1 & 3 & \bar{1} \\ \hline \end{array}$$



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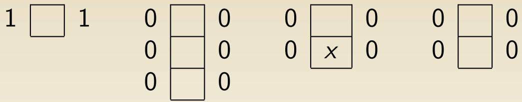
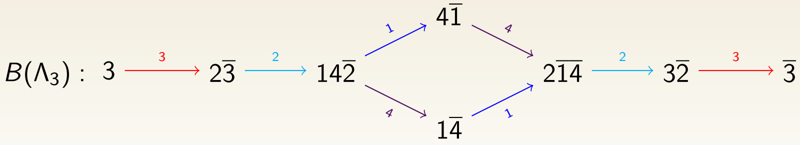


$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}}$$



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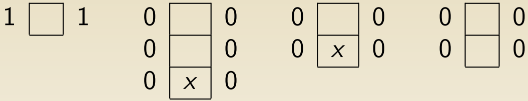

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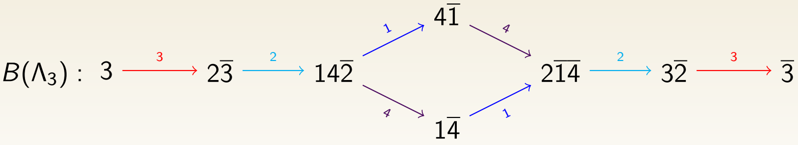

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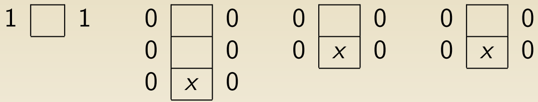
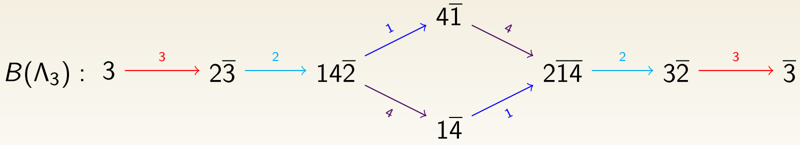


$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}}$$



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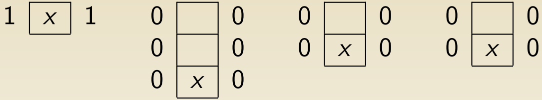
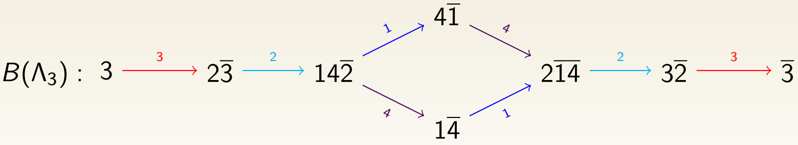


$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}}$$



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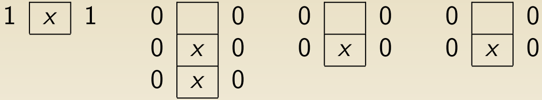
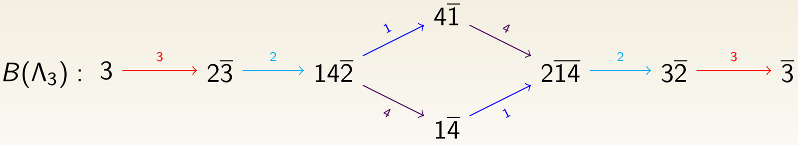


$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}}$$

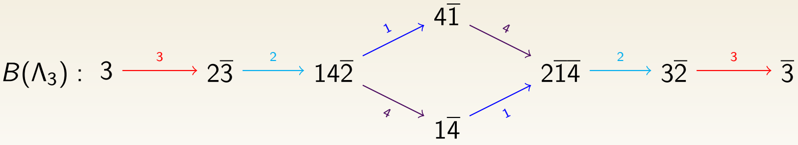


Example of bijection in type $D_4^{(1)}$

$$B^{3,1} \otimes B^{4,1} \otimes B^{2,2}$$

$$\begin{array}{ccc} 1 & \boxed{x} & 1 \end{array} \qquad \begin{array}{ccc} 0 & \boxed{} & 0 \\ 0 & \boxed{x} & 0 \\ 0 & \boxed{x} & 0 \end{array} \qquad \begin{array}{ccc} 0 & \boxed{x} & 0 \\ 0 & \boxed{x} & 0 \end{array} \qquad \begin{array}{ccc} 0 & \boxed{} & 0 \\ 0 & \boxed{x} & 0 \end{array}$$

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\bar{1}}$$



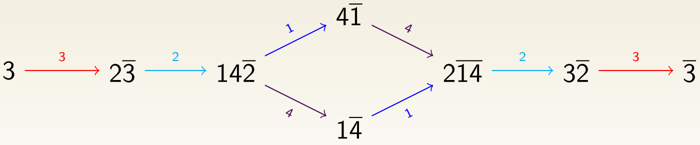
Example of bijection in type $D_4^{(1)}$

$$B^{3,1} \otimes B^{4,1} \otimes B^{2,2}$$

$$
1\ \begin{array}{|c|}\hline x \\ \hline\end{array}\ 1
\qquad
\begin{array}{c} 0 \\ 0 \\ 0 \end{array}
\begin{array}{|c|}\hline \\ \hline x \\ \hline x \\ \hline\end{array}
\begin{array}{c} 0 \\ 0 \\ 0 \end{array}
\qquad
\begin{array}{c} 0 \\ 0 \end{array}
\begin{array}{|c|}\hline x \\ \hline x \\ \hline\end{array}
\begin{array}{c} 0 \\ 0 \end{array}
\qquad
\begin{array}{c} 0 \\ 0 \end{array}
\begin{array}{|c|}\hline \\ \hline x \\ \hline\end{array}
\begin{array}{c} 0 \\ 0 \end{array}
$$

$$\Phi(\nu, J) = \begin{array}{|c|c|c|}\hline 1 & 3 & \overline{1} \\ \hline\end{array} \otimes \overline{3}$$

$B(\Lambda_3)$:

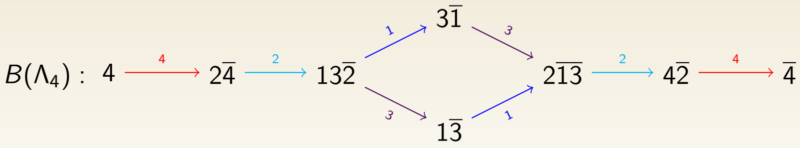


Example of bijection in type $D_4^{(1)}$

$$B^{4,1} \otimes B^{2,2}$$

$\emptyset$ $\quad$ 0 $\square$ 0 $\quad$ $\emptyset$ $\quad$ 0 $\square$ 0

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3}$$

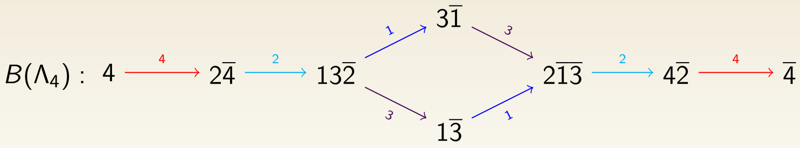


Example of bijection in type $D_4^{(1)}$

$$B^{4,1} \otimes B^{2,2}$$

$\emptyset$ 0 $\square$ 0 $\emptyset$ 0 $\boxed{x}$ 0

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3}$$

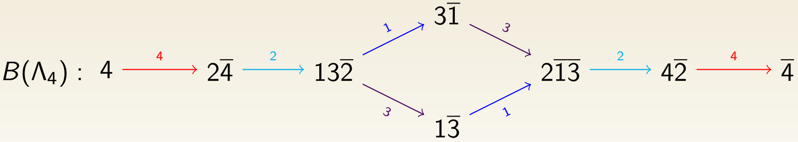


Example of bijection in type $D_4^{(1)}$

$$B^{4,1} \otimes B^{2,2}$$

$$\emptyset \qquad 0 \, \boxed{x} \, 0 \qquad \emptyset \qquad 0 \, \boxed{x} \, 0$$

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3}$$

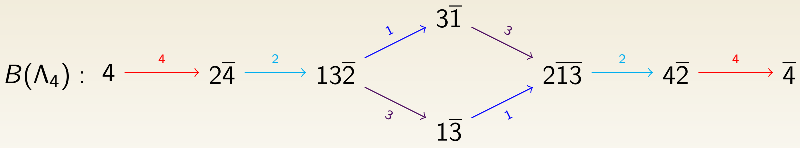


Example of bijection in type $D_4^{(1)}$

$$B^{4,1} \otimes B^{2,2}$$

$$\emptyset \quad 0 \; \boxed{x} \; 0 \qquad \emptyset \quad 0 \; \boxed{x} \; 0$$

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3} \otimes 13\overline{2}$$



Example of bijection in type $D_4^{(1)}$

$$B^{2,2}$$

$$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset$$

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3} \otimes 13\overline{2}$$

Example of bijection in type $D_4^{(1)}$

$$B^{2,2}$$

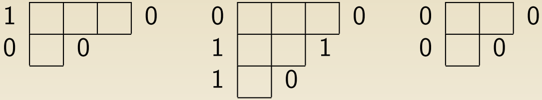
$$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset$$

$$\Phi(\nu, J) = \boxed{1}\boxed{3}\boxed{\bar{1}} \otimes \bar{3} \otimes 13\bar{2} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

Example of bijection in type $A_5^{(2)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{2,2}$$

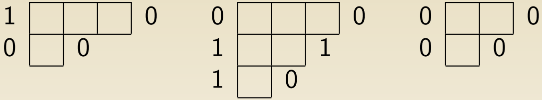
$(\nu, J) =$



Example of bijection in type $A_5^{(2)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{2,2} \mapsto \widehat{B}^{1,3} \otimes \widehat{B}^{1,3} \otimes \widehat{B}^{1,4} \otimes \widehat{B}^{2,2}$$

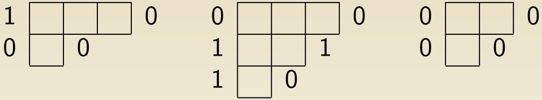
$(\nu, J) =$



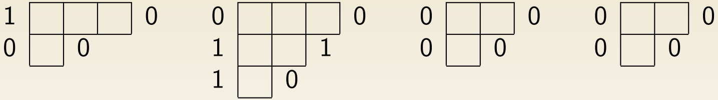
Example of bijection in type $A_5^{(2)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{2,2} \mapsto \widehat{B}^{1,3} \otimes \widehat{B}^{1,3} \otimes \widehat{B}^{1,4} \otimes \widehat{B}^{2,2}$$

$(\nu, J) =$

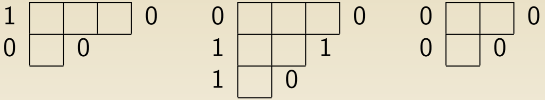


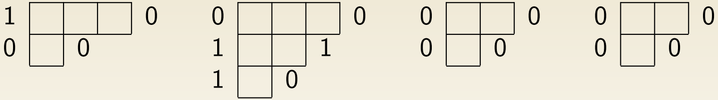
$(\widehat{\nu}, \widehat{J}) =$



Example of bijection in type $A_5^{(2)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{2,2} \mapsto \widehat{B}^{1,3} \otimes \widehat{B}^{1,3} \otimes \widehat{B}^{1,4} \otimes \widehat{B}^{2,2}$$

$(\nu, J) =$


$(\widehat{\nu}, \widehat{J}) =$


$$\Phi(\widehat{\nu}, \widehat{J}) = \boxed{1}\boxed{3}\boxed{\overline{1}} \otimes \overline{3} \otimes 13\overline{2} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

Example of bijection in type $A_5^{(2)}$

$$B^{1,3} \otimes B^{3,1} \otimes B^{2,2} \mapsto \widehat{B}^{1,3} \otimes \widehat{B}^{1,3} \otimes \widehat{B}^{1,4} \otimes \widehat{B}^{2,2}$$

$$(\nu, J) = \begin{array}{c|l|c} 1 & \square\square\square & 0 \\ 0 & \square & 0 \end{array} \qquad \begin{array}{c|l|c} 0 & \square\square\square & 0 \\ 1 & \square\square & 1 \\ 1 & \square & 0 \end{array} \qquad \begin{array}{c|l|c} 0 & \square\square & 0 \\ 0 & \square & 0 \end{array}$$

$$(\widehat{\nu}, \widehat{J}) = \begin{array}{c|l|c} 1 & \square\square\square & 0 \\ 0 & \square & 0 \end{array} \qquad \begin{array}{c|l|c} 0 & \square\square\square & 0 \\ 1 & \square\square & 1 \\ 1 & \square & 0 \end{array} \qquad \begin{array}{c|l|c} 0 & \square\square & 0 \\ 0 & \square & 0 \end{array} \qquad \begin{array}{c|l|c} 0 & \square\square & 0 \\ 0 & \square & 0 \end{array}$$

$$\Phi(\widehat{\nu}, \widehat{J}) = \boxed{1 \mid 3 \mid \overline{1}} \otimes \overline{3} \otimes 13\overline{2} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

$$\Phi(\nu, J) = \boxed{1 \mid 3 \mid \overline{1}} \otimes \begin{array}{|c|} \hline 1 \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

Extensions to exceptional types

Theorem (S., 2017)

There exists a bijection Φ *for factors* $B^{r,1}$ *in all types. Moreover, the* $X = M$ *conjecture holds.*

Extensions to exceptional types

Theorem (S., 2017)

There exists a bijection Φ for factors $B^{r,1}$ in all types. Moreover, the $X = M$ conjecture holds.

The algorithm/proof unifies all of the previous cases for $B^{r,1}$ except for type $B_n^{(1)}$.

Thank you!