

Two-point boundary correlation functions of dense loop models

Alexi Morin-Duchesne



Algebraic Methods in Mathematical Physics

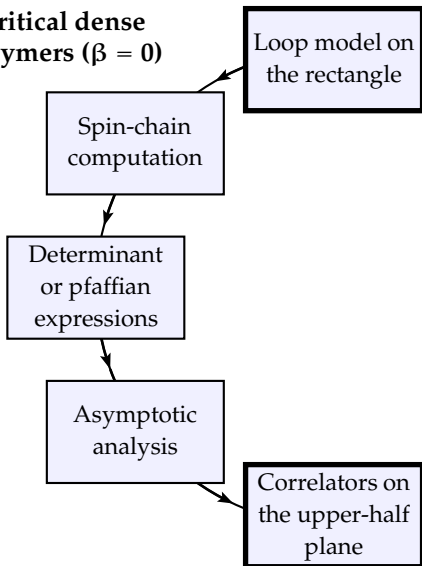
Montréal, 20/07/2018

Joint work with Jesper Jacobsen

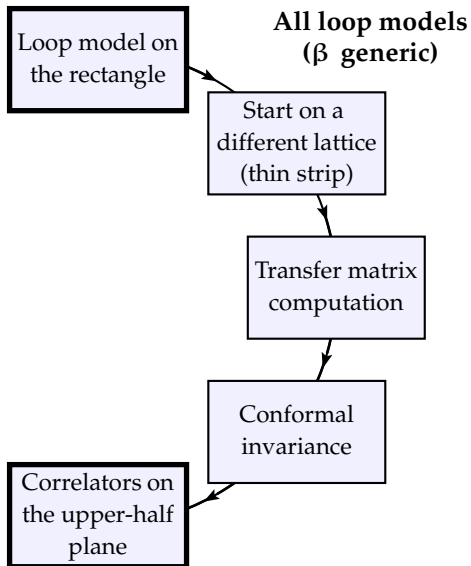
arXiv:1712.08657

Two approaches to correlation functions

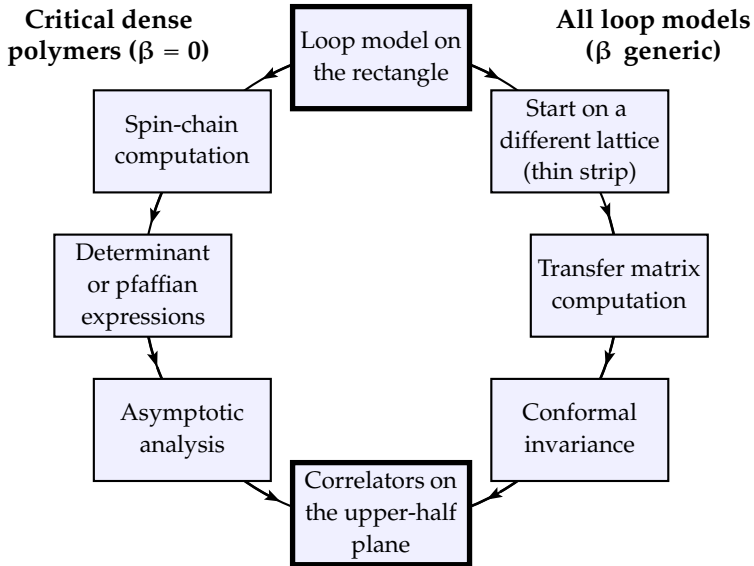
**Critical dense
polymers ($\beta = 0$)**



Two approaches to correlation functions



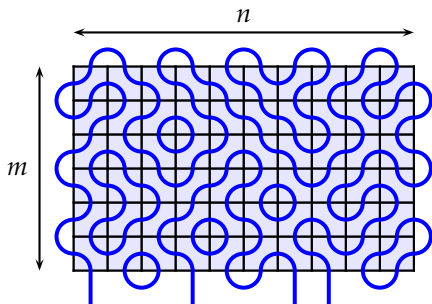
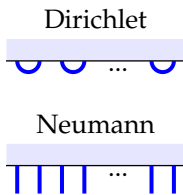
Two approaches to correlation functions



Dense loop model

- Loop model on the $m \times n$ rectangle

- Boundary conditions:



- Fugacity of the loops: $\begin{cases} \beta & \text{for bulk loops} \\ 1 & \text{for boundary loops} \end{cases}$

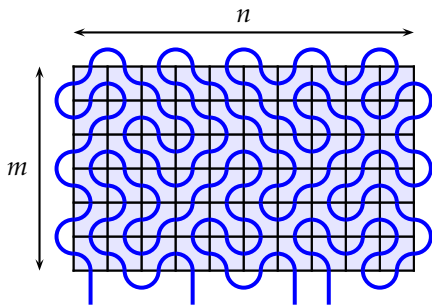
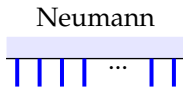
- Partition function: $Z = \sum_{\sigma} \beta^{n_{\sigma}}$

- Model of **critical dense polymers**: $\beta = 0$

Dense loop model

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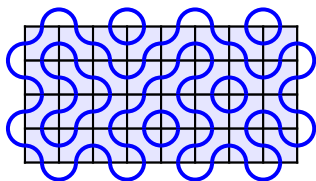
Boundary correlation functions

- Correlation function for two points x and y on the boundary:

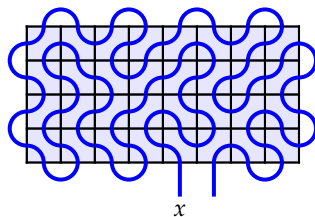
$$C(x, y) = \frac{Z(x, y)}{Z^0}$$

- Z^0 is the **reference partition function**
- Boundary conditions for Z^0 :

(β generic)



($\beta = 0$)



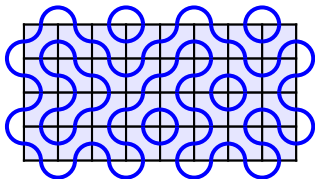
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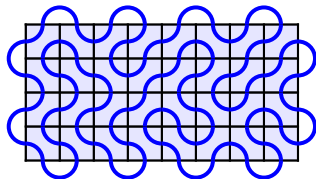
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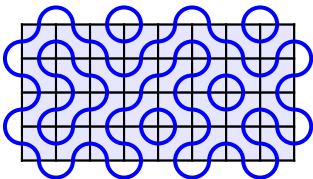
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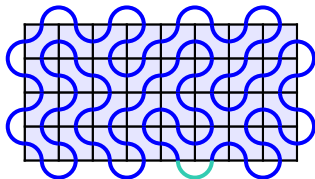
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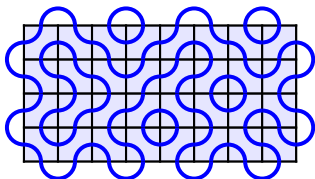
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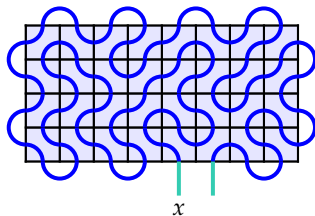
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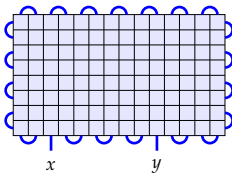
($\beta = 0$)



Six types of correlators: a, b and c

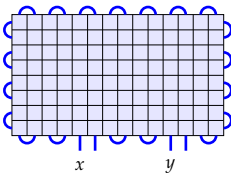
- Boundary conditions for $Z^{a,b,c}(x, y)$:

(a)



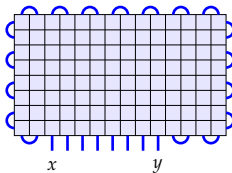
$$C^a(x, y) = \frac{Z^a(x, y)}{Z^0}$$

(b)



$$C^b(x, y) = \frac{Z^b(x, y)}{Z^0}$$

(c)

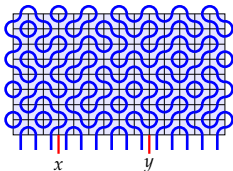


$$C^c(x, y) = \frac{Z^c(x, y)}{Z^0}$$

Six types of correlators: types d and e

- Boundary conditions and constraints for $Z^{d,e}(x, y)$:

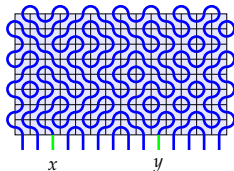
(d)



$$Z^d(x, y) = \sum_{\sigma} \beta^{n_{\beta}} \delta_{c_x, c_y}$$

$$C^d(x, y) = \frac{Z^d(x, y)}{Z^0}$$

(e)

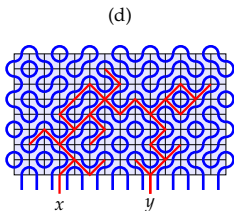


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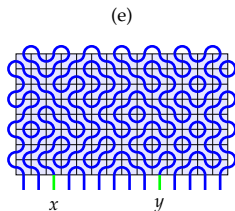
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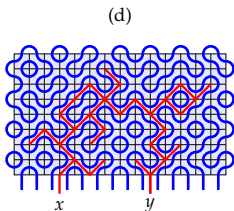


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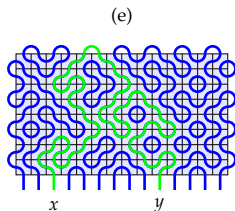
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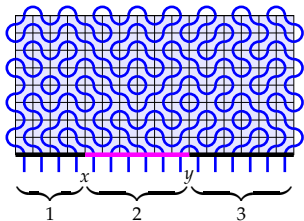


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Six types of correlators: type f

- Boundary divided in segments 1, 2 and 3
- n_{ij} : number of loops tying segments i and j



$$\begin{aligned}n_{12} &= 1 \\n_{23} &= 3 \\n_{13} &= 1\end{aligned}$$

- Partition function and correlation function:

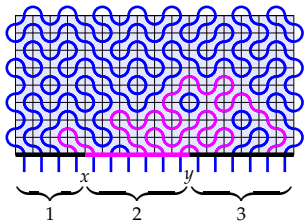
$$Z^f(x, y) = \sum_{\sigma} \beta^{n_{\beta}} \tau^{n_{12} + n_{23}} \quad C^f(x, y) = \frac{Z^f(x, y)}{Z^0}$$

- Related to the **valence bond entanglement entropy**: (Alet et al. '07)

$$\langle n_{12} + n_{23} \rangle = \frac{\sum_{\sigma} (n_{12} + n_{23}) \beta^{n_{\beta}}}{\sum_{\sigma} \beta^{n_{\beta}}} = \left. \frac{d(\ln C^f(x, y))}{d\tau} \right|_{\tau=1}$$

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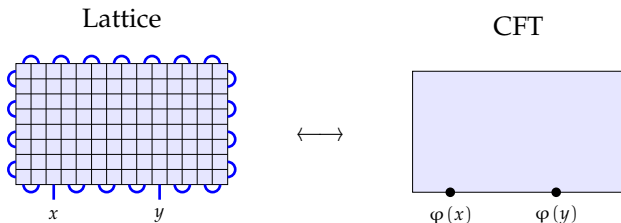
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Critical behavior



- Two-point function of **primary fields** on the upper-half plane \mathbb{H} :

$$C_{\mathbb{H}}(x, y) \simeq \langle \varphi(x) \varphi(y) \rangle_{\mathbb{H}} = \frac{K}{|x - y|^{2\Delta}}$$

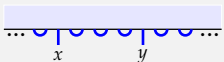
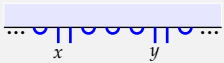
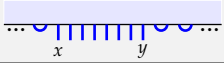
- Δ : conformal weight of the **boundary condition changing field**
- Partition function on the rectangle (Cardy-Peschel formula):

$$\ln Z = -mnf_b - (m + n)f_s - \ln(mn) \sum_{\text{corners}} \left(2\Delta - \frac{c}{16} \right) + \dots$$

where c is the **central charge**

Results for types a,b,c

■ Table of results (with $r = |x - y|$)

	Polymers ($\beta = 0$)	Generic β
(a) 	$C^a(x, y) = K r^{1/4}$	$C^a(x, y) = \frac{K}{r^{2\Delta_{1,2}}}$
(b) 	$C^b(x, y) = K_0 + K_1 \log r$	$C^b(x, y) = \frac{K}{r^{2\Delta_{1,3}}}$
(c) 	$C^c(x, y) = K r^{3/16}$	$C^c(x, y) = \frac{K}{r^{2\Delta_{0,1/2}}}$

- Loop weight: $\beta = 2 \cos \lambda \quad \lambda = \pi(1 - t) \quad t \in (0, 1)$
- Central charge and conformal weights:

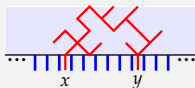
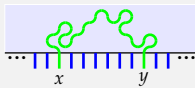
$$c = 13 - 6(t + t^{-1}) \quad \Delta_{r,s} = \frac{1-rs}{2} + \frac{r^2-1}{4t} + \frac{(s^2-1)t}{4}$$

- For polymers:

$$\lambda = \frac{\pi}{2} \quad t = \frac{1}{2} \quad c = -2 \quad \Delta_{r,s} = \frac{(2r-s)^2-1}{8}$$

Results for types d,e

■ Table of results (with $r = |x - y|$)

	Polymers ($\beta = 0$)	Generic β
(d) 	$C^d(x, y) = \frac{K}{r^{3/4}}$	$C^d(x, y) = \frac{K}{r^{2\Delta_{1,0}}}$
(e) 	$C^e(x, y) = \frac{K}{r^2}$	$C^e(x, y) = \frac{K}{r^{2\Delta_{1,-1}}}$

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■ Central charge and conformal weights:

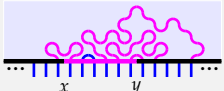
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■ For polymers:

$$\lambda = \frac{\pi}{2} \quad t = \frac{1}{2} \quad c = -2 \quad \Delta_{r,s} = \frac{(2r-s)^2-1}{8}$$

Results for types f

■ Table of results (with $r = |x - y|$)

	Polymers ($\beta = 0$)	Generic β
(f) 	$C^f(x, y) = \frac{K}{r^{\frac{2\theta}{\pi}(1 + \frac{2\theta}{\pi})}}$	$C^f(x, y) = \frac{K}{r^{2\Delta_{1 + \frac{2\theta}{\lambda}, 1 + \frac{2\theta}{\lambda}}}}$

■ Loop weights:

- bulk loops: $\beta = 2 \cos \lambda \quad \lambda = \pi(1 - t) \quad t \in (0, 1)$

- boundary loops: $\tau = \frac{\cos(\frac{\lambda}{2} + \theta)}{\cos(\frac{\lambda}{2})}$

■ Central charge and conformal weights:

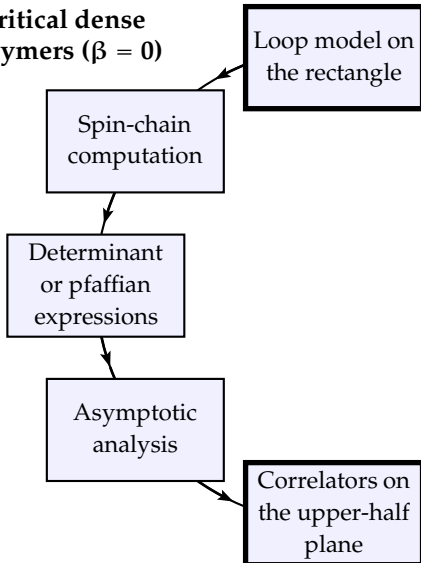
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■ For polymers:

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Approach 1: Exact derivations

**Critical dense
polymers ($\beta = 0$)**



Temperley-Lieb algebra

- The Temperley-Lieb algebra $TL_n(\beta) = \langle I, e_j \rangle$:

$$I = \begin{array}{|c|c|c|c|} \hline \text{||} & \text{||} & \dots & \text{||} \\ \hline 1 & 2 & \dots & n \\ \hline \end{array} \quad e_j = \begin{array}{|c|c|c|c|} \hline \text{||} & \dots & \text{||} & \text{||} \\ \hline 1 & j & j+1 & n \\ \hline \end{array} \quad j = 1, \dots, n-1$$

- Examples of Temperley-Lieb products for $n = 4$:

$$e_2 e_1 e_3 = \begin{array}{|c|c|c|c|} \hline \text{||} & \text{||} & \text{||} & \text{||} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \text{||} & \text{||} \\ \hline \end{array} = \beta \begin{array}{|c|c|} \hline \text{||} & \text{||} \\ \hline \end{array}$$

- Transfer tangle: (recall $\beta = 2 \cos \lambda$)

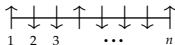
$$D(u) = \underbrace{\begin{array}{|c|c|c|c|} \hline u & u & \dots & u \\ \hline u & u & \dots & u \\ \hline \end{array}}_n = \frac{\sin(\lambda - u)}{\sin \frac{\lambda}{2}} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} + \frac{\sin u}{\sin \frac{\lambda}{2}} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array}$$

- Isotropic point: $u = \frac{\lambda}{2}$

$$\begin{array}{|c|} \hline \frac{\lambda}{2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{||} \\ \hline \end{array}$$

Spin-chain representation

- XXZ representation of $TL_n(\beta)$ for $\beta = q + q^{-1}$:



$$X_n(e_j) = \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{n-j-1}$$

- Spin-chain Hamiltonian and transfer matrix:

$$H = - \sum_{j=1}^{n-1} X_n(e_j) \quad D(u) = X_n(D(u))$$

$$= - \frac{1}{2} \left(\sum_{j=1}^{n-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \frac{q + q^{-1}}{2} (\sigma_j^z \sigma_{j+1}^z - \mathbb{I}) \right) - \frac{q - q^{-1}}{4} (\sigma_1^z - \sigma_n^z)$$

- Special values:

Polymers
($\beta = 0$)



XX spin-chain
($q = i$)

Correlators from the spin chain

- Partition functions on the rectangle: $\left(\begin{array}{|c|} \hline m \\ \hline \begin{array}{|c|} \hline x \quad y \\ \hline \end{array} \\ \hline n \\ \hline \end{array} \right)$

$$Z^{a,b,c}(x,y) = \langle v^{a,b,c} | D(\frac{\lambda}{2})^{m/2} | v^0 \rangle$$

with

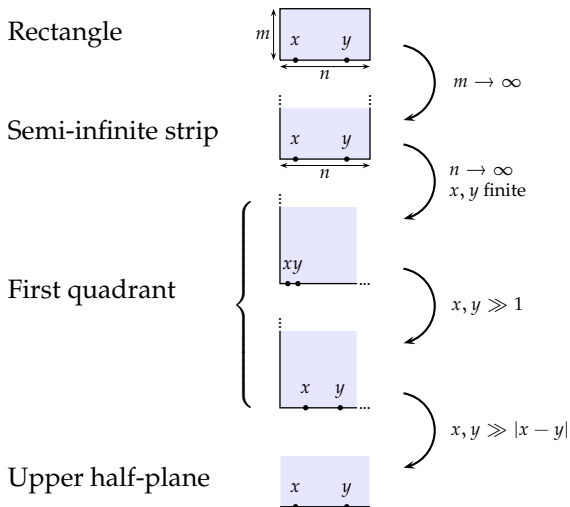
$$\begin{aligned} |v^0\rangle &= \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle & |v^a\rangle &= \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle \\ |v^b\rangle &= \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle & |v^c\rangle &= \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle \end{aligned}$$

- Correlation function on the upper-half plane: $\left(\begin{array}{|c|} \hline x \quad y \\ \hline \end{array} \right)$

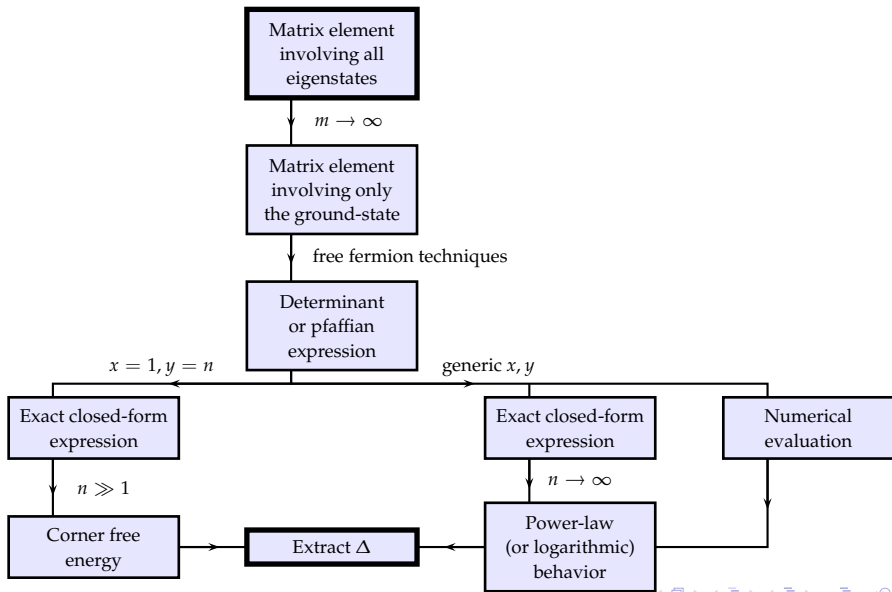
$$\frac{Z^{a,b,c}(x,y)}{Z^0} \xrightarrow{?} \frac{K}{|x-y|^{2\Delta^{a,b,c}}}$$

From the rectangle to the upper-half plane

- Sequence of limits in the lattice calculation:



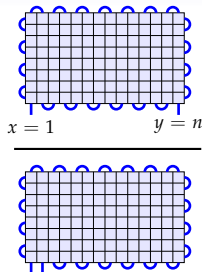
Strategy to compute Δ for $\beta = 0$



Correlators of type a

- Corner free energy analysis:

$$\ln C^a(1, n) = \ln \left(\frac{Z^a(1, n)}{Z^0} \right) = \frac{1}{2} \ln n - \frac{1}{2} \ln 2$$



- Expected from CFT:

$$\ln Z/Z' = -n(f_s - f'_s) - 2 \ln n \sum_{\text{corners}} (\Delta - \Delta') + \dots$$

- Result of the exact calculation for x, y generic:

$$C^a(x, y) = K(y - x)^{1/4} \quad K = \overset{\text{ Barnes' } G(z) \text{ function}}{G^2\left(\frac{3}{2}\right)} 2^{1/4}$$

- Conformal weights: $\Delta^a = -\frac{1}{8} \quad \Delta^0 = 0$

Correlators of type b

- Corner free energy analysis:

$$C^b(1, n) = \frac{4}{\pi} \ln n + 1 + \frac{4}{\pi} (\overset{\text{Euler-Mascheroni constant}}{\gamma} + 2 \ln 2 - \ln \pi - 1) + \dots$$

$$\ln C^b(1, n) = \ln(\ln n) + \ln(4/\pi) + \dots$$

- Result of the exact calculation for x, y generic:

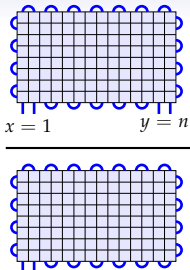
$$C^b(x, y) = K_0 \ln(y - x) + K_1 \quad K_0 = \frac{2}{\pi} \quad K_1 = 1 + \frac{2}{\pi} (\gamma + \ln 2)$$

■ **Logarithmic field of conformal weight:** $\Delta^b = 0$

- CFT: two-point function of a log. field in a rank 2 Jordan cell:

$$\langle \omega(z_0) \omega(z_1) \rangle = \frac{K_0 \ln |z_0 - z_1| + K_1}{|z_0 - z_1|^{2\Delta}}$$

- The constant K_0 is **universal** (Vasseur, Jacobsen '14)



Correlators of type c

- Corner free energy analysis:

$$\ln C^c(1, n) = \ln \left(\frac{Z^c(1, n)}{Z^0} \right) = n \frac{G}{\pi} + \frac{3}{8} \ln n + \dots$$

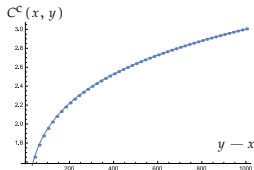
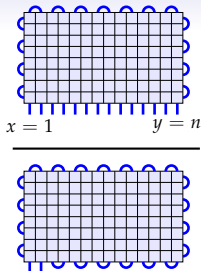
Catalan's constant

- Expected from CFT:

$$\ln Z/Z' = -n(f_s - f'_s) - 2 \ln n \sum_{\text{corners}} (\Delta - \Delta') + \dots$$

- Results for generic x, y :

- Pfaffian formula for $C^c(x, y)$
- No closed form obtained
- Numerical evaluation: $\Delta^c \simeq -0.09405$



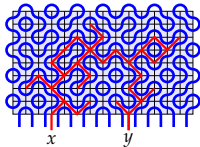
- **Conformal weight:** $\Delta^c = -\frac{3}{32} = -0.09375$

Correlators of type d, e and f

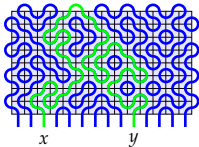
■ Pfaffian formulas for $C^{d,e,f}(x, y)$

(recall $\tau = \frac{\cos(\frac{\lambda}{2} + \theta)}{\cos(\frac{\lambda}{2})}$)

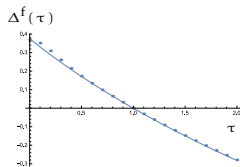
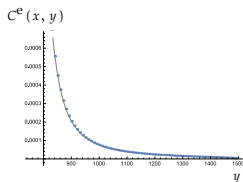
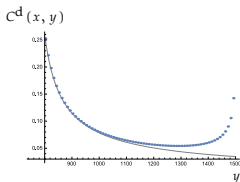
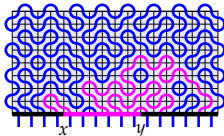
(d)



(e)



(f)



■ Numerical evaluation:

$\Delta^d \simeq 0.37247$

$\Delta^e \simeq 0.994845$

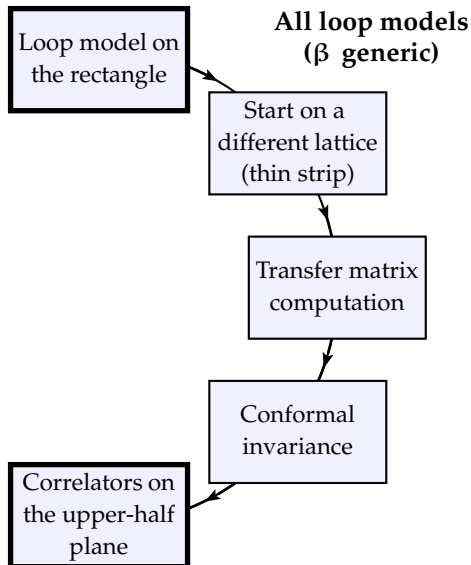
■ Conformal weights:

$\Delta^d = \frac{3}{8}$

$\Delta^e = 1$

$\Delta^f = \frac{\theta}{\pi} (1 + \frac{2\theta}{\pi})$

Approach 2: CFT derivations

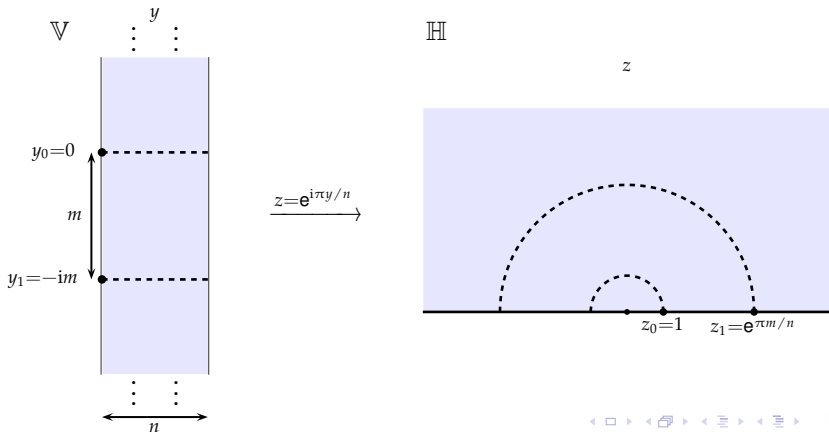


Covariance of two-point functions

- Transformation law for two-point functions of primary fields:

$$\langle \phi(y_0)\phi(y_1) \rangle_{\mathbb{V}} = \left| \frac{dy}{dz} \right|_{y=y_0}^{-\Delta} \left| \frac{dy}{dz} \right|_{y=y_1}^{-\Delta} \langle \phi(z_0)\phi(z_1) \rangle_{\mathbb{H}}$$

- The domains \mathbb{V} and \mathbb{H} :



Finite-size corrections

- Reminder: the transfer matrix is defined as

$$D(u) = X_n(D(u)) = X_n \left(\begin{array}{|c|c|c|c|} \hline u & u & u & u \\ \hline u & u & u & u \\ \hline \end{array} \right)$$

- Behavior of the eigenvalues of the low-lying states of $D(\frac{\lambda}{2})$:

$$\Lambda_i = \exp \left(-nf_b - f_s - \underbrace{\frac{2\pi}{n} \left(\Delta(i) - \frac{c}{24} \right)}_{\text{finite-size corrections}} + \dots \right)$$

- f_b : bulk free energy
 - f_s : surface free energy
- } Do not depend on i
- The finite-size corrections depend on the **central charge** and a **conformal weight**

Correlators of type a

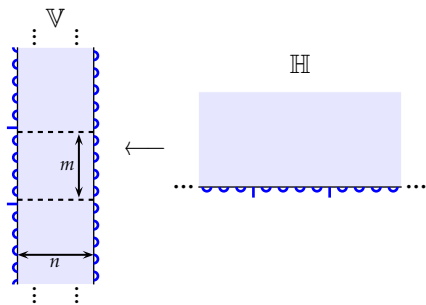
- Compute $C_{\mathbb{V}}^a(y_0, y_1)$ in two ways:

1) using **conformal invariance**:

$$C_{\mathbb{H}}^a(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^a}}$$

$$C_{\mathbb{V}}^a(y_0, y_1) = \frac{K' e^{\frac{\pi m}{n} \Delta^a}}{\left(e^{\frac{\pi m}{n}} - 1\right)^{2\Delta^a}}$$

$$\xrightarrow{m \gg n} K' e^{-\frac{\pi m}{n} \Delta^a}$$



Correlators of type a

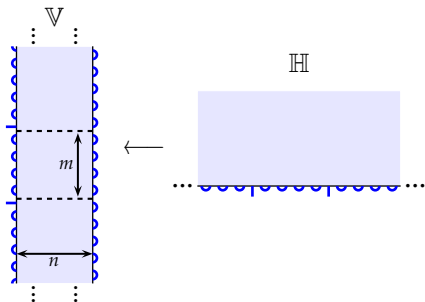
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$$\xrightarrow{m \gg n} K' e^{-\frac{\pi m}{n} \Delta^a}$$



- 2) using the **transfer matrix**:

$$C_{\mathbb{V}}^a(y_0, y_1) \xrightarrow{m \gg n} \tilde{K} \left(\frac{\Lambda_1}{\Lambda_0} \right)^{m/2} = \tilde{K} e^{-\frac{\pi m}{n} (\Delta_{1,2} - \overbrace{\Delta_{1,1}}^{=0})} = \tilde{K} e^{-\frac{\pi m}{n} \Delta_{1,2}}$$

■ **Conformal weight:** (Saleur, Bauer '89) $\Delta^a = \Delta_{1,2}$

- For $\beta = 0$: $\Delta_{1,2} = -\frac{1}{8}$

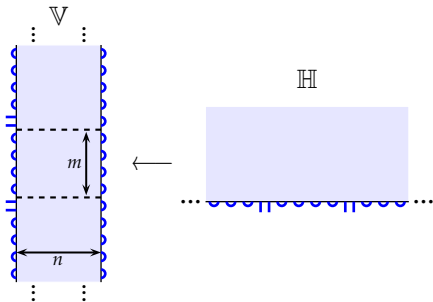
Correlators of type b

- Compute $C_{\mathbb{V}}^b(y_0, y_1)$ in two ways:

1) using **conformal invariance**:

$$C_{\mathbb{H}}^b(z_0, z_1) = \frac{K_0}{|z_0 - z_1|^{2\Delta^{b,0}}} + \frac{K_1}{|z_0 - z_1|^{2\Delta^{b,1}}}$$

$$C_{\mathbb{V}}^b(y_0, y_1) \xrightarrow{m \gg n} K'_0 e^{-\frac{\pi m}{n} \Delta^{b,0}} + K'_1 e^{-\frac{\pi m}{n} \Delta^{b,1}}$$



2) using the **transfer matrix**:

$$C_{\mathbb{V}}^b(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}_0 \left(\frac{\Lambda_0}{\Lambda_0} \right)^{m/2} + \tilde{K}_1 \left(\frac{\Lambda_2}{\Lambda_0} \right)^{m/2} = \tilde{K}_0 e^{-\frac{\pi m}{n} \Delta_{1,1}} + \tilde{K}_1 e^{-\frac{\pi m}{n} \Delta_{1,3}}$$

■ **Conformal weights:** (Saleur, Bauer '89) $\Delta^{b,0} = \Delta_{1,1}$ $\Delta^{b,1} = \Delta_{1,3}$

- This boundary condition change is **not a primary field**

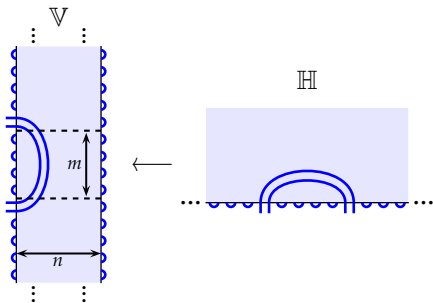
Correlators of type b

- Compute $C_{\mathbb{V}}^b(y_0, y_1)$ in two ways:

1) using **conformal invariance**:

$$C_{\mathbb{H}}^b(z_0, z_1) = \frac{K_0}{|z_0 - z_1|^{2\Delta^{b,0}}} + \frac{K_1}{|z_0 - z_1|^{2\Delta^{b,1}}}$$

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- This boundary condition change is **not a primary field**

Correlators of type b for $\beta = 0$

■ For $\beta = 0$: $\Delta_{1,1} = \Delta_{1,3} = 0$

■ Rank 2 Jordan cell in the transfer matrix: $D(\frac{\lambda}{2}) \simeq \begin{pmatrix} \Lambda_0 & 1 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$

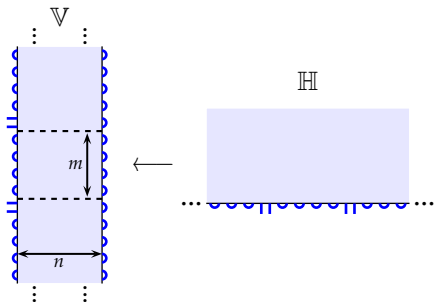
■ Compute $C_{\mathbb{V}}^b(y_0, y_1)$ using the **transfer matrix**:

$$C_{\mathbb{V}}^b(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}_0 m + \tilde{K}_1 + \dots$$

■ Using **conformal invariance**:

$$C_{\mathbb{H}}^b(z_0, z_1) = \frac{K_0 \ln |z_0 - z_1| + K_1}{|z_0 - z_1|^{2\Delta^b}}$$

$$C_{\mathbb{V}}^b(z_0, z_1) = e^{-\frac{\pi m}{n} \Delta^b} (K'_0 m + \tilde{K}'_1)$$



■ **Logarithmic field with $\Delta^b = 0$**

Correlators of types c, d, e, f

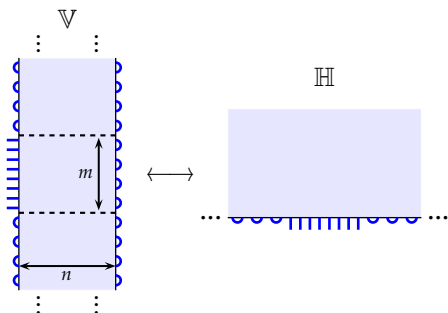
- Same strategy for types c, d, e, f:
two ways to compute $C_{\mathbb{V}}(y_0, y_1)$:

1) using **conformal invariance**

2) using the spectra of the
transfer matrix of the
**One- and Two-Boundary
Temperley-Lieb algebras**

(Jacobsen, Saleur '08)

(Dubail, Jacobsen, Saleur '09)



Conformal weights:

$$\Delta^c = \Delta_{0, \frac{1}{2}}$$

$$\Delta^d = \Delta_{1, 0}$$

$$\Delta^e = \Delta_{1, -1}$$

$$\Delta^f = \Delta_{\frac{2\theta}{\lambda} + 1, \frac{2\theta}{\lambda} + 1}$$

- For $\beta = 0$:

$$\Delta^c = -\frac{3}{32}$$

$$\Delta^d = \frac{3}{8}$$

$$\Delta^e = 1$$

$$\Delta^f = \frac{\theta}{\pi} \left(1 + \frac{2\theta}{\pi} \right)$$

Correlators of types c, d, e, f

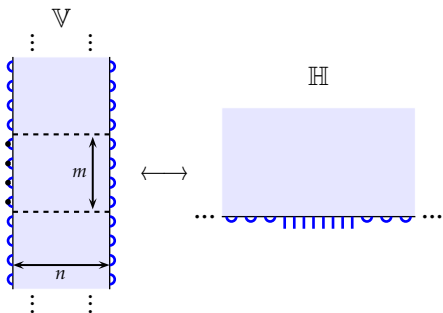
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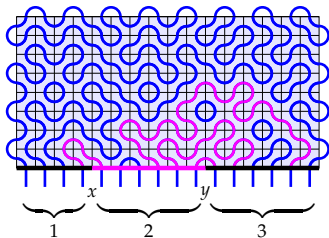
$$\Delta^f = \frac{\theta}{\pi} \left(1 + \frac{2\theta}{\pi} \right)$$

Valence bond entanglement entropy

- Correlator of type f on the upper-half plane:

$$C_{\mathbb{H}}^f(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^f}}$$

$$\Delta^f = \Delta_{\frac{2\theta}{\lambda}+1, \frac{2\theta}{\lambda}+1}$$



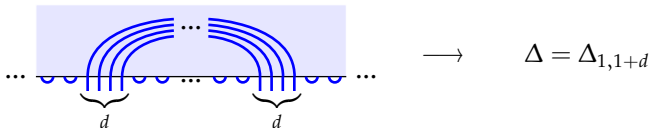
$$\begin{aligned} n_{12} &= 1 \\ n_{23} &= 3 \\ n_{13} &= 1 \end{aligned}$$

- Reminder: $\beta = 2 \cos \lambda$ $\tau = \frac{\cos(\frac{\lambda}{2} + \theta)}{\cos(\frac{\lambda}{2})}$
- Valence bond entanglement entropy: (Jacobsen, Saleur '08)

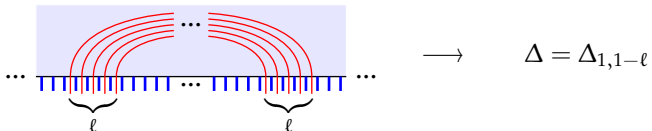
$$\begin{aligned} \langle n_{12} + n_{23} \rangle &= \frac{\sum_{\sigma} (n_{12} + n_{23}) \beta^{n_{\sigma}}}{\sum_{\sigma} \beta^{n_{\sigma}}} = \left. \frac{d(\ln C_{\mathbb{H}}^f(x, y))}{d\tau} \right|_{\tau=1} \\ &= \frac{2(\lambda/\pi)}{\pi(1 - \lambda/\pi)} \frac{\cos(\lambda/2)}{\sin(\lambda/2)} \ln(y - x) \end{aligned}$$

More two-point correlators

- Correlator of d defects in a Dirichlet boundary:



- Correlator of ℓ clusters in a Neumann boundary:



Outlook

Overview:

- perfect agreement between exact results and CFT predictions
- corner free energy for the logarithmic field: $\sim \ln(\ln n)$

Possible future avenues: extend the method to

- exact derivations for other values of β
- periodic boundary conditions
- correlators in the bulk
- three-point functions

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Thank you!