Topological characterization of phase transitions

Algebraic approach

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Outline

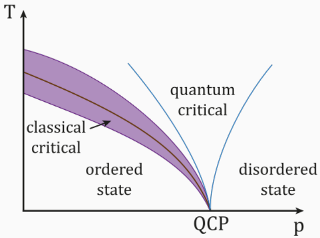
Introduction

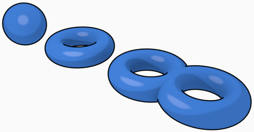
- **Classical phase transition**: Landau formalism
  Thermal fluctuation, critical point $\tau = \frac{T - T_c}{T_c}$.
  (Local) Order parameter χ.
  Critical exponent $\chi \propto (-\tau)^{\gamma}$.
- **Quantum phase transition**: Zero temperature, quantum fluctuation, vanishing of the energy gap near the critical point.
- **Topological quantum phase transition**: Robust phenomena, ground state degeneracy and non-local order parameter (Topological invariant).

General purpose

The main goal is to study phase transition phenomena from the point of view of topological invariants. More generally, we want to understand the relation between different kinds of phase transitions and topology.

Universality $\overset{?}{\longleftrightarrow}$ Topology





Main Example and Problem Statement

Classical Ising Model

The configuration space is a lattice $\mathbb{Z}^d$. The state space (phase space) for one site is $E = \{-1, +1\}$, then for the system $E^{\mathbb{Z}^d}$. So a local state is a function $s : \Lambda \to E$ with $\Lambda \in \mathbb{Z}^d$ and $|\Lambda| \leq \infty$. A local observable is a function $f : E^{\mathbb{Z}^d} \to \mathbb{R}$. The system dynamics (d=1) is given by the Hamiltonian function (energy observable):

$$H_\Lambda(s) = -J \sum_{\langle i,j \rangle_\Lambda} s_i s_j, \quad J > 0, \tag{1}$$

The partition function

$$Z_\beta = \sum_{\{s\}} e^{-\beta H_\Lambda(s)} = \sum_{\{s\}} \prod_i T(s_i, s_{i+1}), \qquad \beta = \left(k_B T\right)^{-1} \tag{2}$$

$$T(s_i, s_j) = e^{K s_i s_j}, \qquad K = \beta J \tag{3}$$

Build the transference matrix $T \in M_2(\mathbb{C})$ with entries $T_{ij} = T(s_i, s_j)$ by evaluating the possible values of $s_i = \pm 1$, in this case $Z_\beta = Tr(T^N)$, $N = |\Lambda|$.

The main idea is to relate quantum and classical criticality by the identification:

$$\langle f \rangle_\beta := tr(\hat{\rho}\hat{O}_{f,\beta}), \quad (4)$$

f is a classical observable, $\beta = (k_B T)^{-1}$, ρ is a self-adjoint, positive of norm 1 element in a C*-algebra A, and $\hat{O}_{f,\beta}$ is a self-adjoint element in A uniquely associated to f.

For the 2-d classical Ising model:

$$\langle f \rangle_\beta := Z_\beta^{-1} \sum_{\{s\}} f(s) e^{-\beta H_\Lambda(s)} = tr(\hat{\rho}\hat{f}) \quad (5)$$

where $Z_\beta = \sum_{\{s\}} e^{-\beta H_\Lambda(s)}$, $H_\Lambda(s) = -J \sum_{\langle i,j \rangle_\Lambda} s_i s_j$, $J > 0$, and $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}_q}$. The quantum Hamiltonian obtained by this operation is:

$$H_q = -\frac{1}{2\xi}\sigma_x. \quad (6)$$

Algebraic approach

For d=2 results in the 1-dimensional quantum Ising Model :

$$H_\lambda = -\sum_j \sigma_x^j \sigma_x^{j+1} - \lambda \sum_j \sigma_z^j. \tag{7}$$

Here $\sigma_x^j \sigma_x^{j+1} = \mathbb{1}_2 \otimes \cdots \mathbb{1}_2 \otimes \underbrace{\sigma_x}_{j} \otimes \sigma_x \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2$.

In general a quantum dynamical system is described by the tuple $(A, \omega, \pi_\omega, H)$:
A is a C*-Algebra (unital).
An state $\omega : A \to \mathbb{C}$, s.t. $\omega(a^* a) \geq 0 \ \forall a \in A$, and $\omega(\mathbb{1}) = 1$. $\pi_\omega : A \to \mathcal{L}(\mathcal{H})$ a representation of A on a Hilbert space $\mathcal{H}$. H a self-adjoint operator over $\mathcal{H}$.

The initial problem is formulate in terms of the Pauli algebra $A = A_P$, but we are interested in the case $A = A_{CAR}(V)$ given by the relations:

$$\left\{a_J(u), a_J^*(v)\right\} = \langle u, v \rangle \mathbb{1} \tag{8}$$

This algebras are very related, as we will see below.

- An automorphism of A is an invertible linear map $\theta : A \to A$ satisfying

$$\theta(ab) = \theta(a)\theta(b), \qquad \theta(a^*) = \theta(a)^* \tag{9}$$

- A $\mathbb{Z}_2$-action on A is an automorphism $\theta : A \to A$ with $\theta^2 = \mathbb{1}$. An algebra A carrying a $\mathbb{Z}_2$-action decomposes as:

$$A = A_+ + A_-, \qquad A_\pm = \{a \in A \mid \theta(a) = \pm a\} \tag{10}$$

Example: Let $u = \mathcal{H} \to \mathcal{H}$ be a unitary operator $u^2 = \mathbb{1}$, $A = B(\mathcal{H})$, then

$$\theta(a) = uau^* \tag{11}$$

defines a $\mathbb{Z}_2$ action on A, so $A_\pm = \{a \in A \mid au \mp ua = 0\}$

Pauli and Fermionic algebras

- $I_L = [-L, L],\ L \in \mathbb{Z}_+$
- Pauli Algebra $A_L^P \simeq \bigotimes_{I_L} M_2$ generated by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 \equiv \mathbb{1} \tag{12}$$

- Fermionic Algebra $A_L^F \simeq A_{CAR}\big(l^2(I_L)\big)$ generated by

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0 = \{a_i^\dagger a_j^\dagger\} \tag{13}$$

- Let θ be a $\mathbb{Z}_2$-action such that for $i \in I_L$

$$\begin{aligned} \theta(\sigma_x^i) = -\sigma_x^i, \quad \theta(\sigma_y^i) = -\sigma_y^i, \quad \theta(\sigma_z^i) = \sigma_z^i, \qquad & A^P = A_+^P + A_-^P \\ \theta(a_i) = -a_i, \qquad & A^F = A_+^F + A_-^F \end{aligned} \tag{14}$$

Jordan-Wigner transformation

The JWT is an isomorphism $\alpha_L : A_L^P \to A_L^F$ ($L < \infty$)

$$\sigma_x^j = TS_j(a_j + a_j^\dagger), \quad \sigma_y^j = iTS_j(a_j - a_j^\dagger), \quad \sigma_z^j = \mathbb{1} - 2a_j^\dagger a_j \tag{15}$$

where $T = \prod_{k=-L}^{0} \sigma_z^k$ and $TS_j = \prod_{k=-L}^{j-1} \sigma_z^k$. Since the tail T depends on L, the diagram is not commutative

$$\begin{array}{ccc} A_L^P & \overrightarrow{\alpha_L} & A_L^F \\ \cap & & \cap \\ A_{L+1}^P & \overrightarrow{\alpha_{L+1}} & A_{L+1}^F \end{array} \tag{16}$$

A_{L+}^P is generated by σ_z^i and $\sigma_x^j \sigma_x^{j+1}$, since $T^2 = \mathbb{1}$, the restriction of α_L to the even subalgebra is not L-dependent, so $\lim_{L\to\infty} \alpha_L|_{A_{L+}^P}$ gives an isomorphism of A_+^P with A_+^F.

$$H_\lambda = -\sum_{i=1}^{N-1} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1}^\dagger - a_i a_{i+1}) + 2\lambda \sum_{i=1}^{N} a_i^\dagger a_i - \lambda N \tag{17}$$

Bogoliubov transformation

Define this new set of operators:

$$c_k = \sum_{i=1}^{N} \left(g_{ik} a_i + h_{ik} a_i^\dagger \right), \;\; c_k = \sum_{i=1}^{N} \left(\bar{g}_{ik} a_i^\dagger + \bar{h}_{ik} a_i \right) \tag{18}$$

Them satisfy the CAR algebra:

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = 0 = \{c_i^\dagger c_j^\dagger\} \tag{19}$$

And this requirement leads the following conditions:

$$gg^\dagger + hh^\dagger = \mathbb{1}, \;\; gh^t + hg^t = 0. \tag{20}$$

The Hamiltonian becomes diagonal in the new basis:

$$\sum_k \Lambda_k c_k^\dagger c_k + \mu \tag{21}$$

Then the ground state can be found:

$$c_k |\Omega(\lambda)\rangle = 0 \;\; \forall k \in \{1, \ldots, N\} \tag{22}$$

Local observables and ground state

- Hilbert space by site $\mathcal{H}_x = \mathbb{C}^2$
- $\mathcal{H}(\Lambda) = \bigotimes_{x\in\Lambda} \mathcal{H}_x$, $(\dim\mathcal{H}(\Lambda) = \left(\dim\mathcal{H}\right)^{|\Lambda|})$
- Local observables $A(\Lambda) = B(\mathcal{H}(\Lambda))$, $\Lambda \subset \Lambda' \Rightarrow A(\Lambda) \hookrightarrow A(\Lambda')$
- Heisenberg equation:
$$\frac{da(t)}{dt} = i[H_\Lambda, a(t)] \tag{23}$$
Setting $t = 0$, this defines a derivation
$$\begin{array}{rcccl} \delta_\Lambda & : & A(\Lambda) & \to & A(\Lambda) \\ & & a & \mapsto & \delta_\Lambda(a) = i[H_\Lambda, a] \end{array} \tag{24}$$
- For each $a \in A(\Lambda)$, $\delta(a) = i \lim_{\Lambda\uparrow\mathbb{Z}^d} [H_\Lambda, a]$ exists.
- A ground state is a state $\omega_0 : A \to \Lambda$ such that
$$-i\omega_0(a^*\delta(a)) \geq 0, \qquad \forall a \in A \tag{25}$$
- An even state on A^P or A^F is one which is invariant under θ.

GNS construction

- Let $\omega : A \to \mathbb{C}$ be a state on a C^*-algebra A. There exists a cyclic representation π_ω of A on a Hilbert space $\mathcal{H}_\omega$ with cyclic unit vector Ω_ω such that
$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle, \quad \forall a \in A \tag{26}$$
- The GNS representation $\pi_\omega(A)$ is irreducible iff ω is pure.
- A symmetry $\theta : A \to A$ is implementable in an Hilbert space $\mathcal{H}$ iff there is an unitary operator U such that:
$$U\pi(a)U^* = \pi(\theta a), \forall a \in A, \tag{27}$$
- Suppose A carries a $\mathbb{Z}_2$-action θ and consider a state $\omega : A \to \mathbb{C}$ that is $\mathbb{Z}_2$-invariant in the sense that $\omega\big(\theta(a)\big) = \omega(a)$ for all $a \in A$. We write this as $\theta^*\omega = \omega$, with $\theta^*\omega_\Omega := \omega_\Omega \circ \theta$. Then there is a unitary operator $u : \mathcal{H}_\omega \to \mathcal{H}_\omega$ satisfacing $u^2 = \mathbb{1}$, $u\Omega_\omega = \Omega_\omega$ and $u\pi_\omega(a)u^* = \pi_\omega\big(\theta(a)\big)$ for each $a \in A$.
- Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple of $\omega : A \to \mathbb{C}$, due to θ-invariance of ω
$$\mathcal{H}_\omega = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_\pm = \overline{\pi_\omega(A_\pm)\Omega_\omega} \tag{28}$$

Araki approach

Remarkable result

Consider the fermionic algebra $A = Span\{a^\dagger, a, a^\dagger a, \mathbb{1}\} \cong M_2(\mathbb{C})$, in terms of Pauli matrices $\sigma_\pm = \sigma_x \pm i\sigma_y$, $a = \sigma_-$, $a^\dagger = \sigma_+$. The $\mathbb{Z}_2$ action can be implemented by the unitary σ_z, so $A_+ = Span\{a^\dagger a, \mathbb{1}\}$ and $A_- = Span\{a^\dagger, a\}$, then:

$$A_+ = \left\{ \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix},\ z_\pm \in \mathbb{C} \right\}; \quad A_- = \left\{ \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix},\ z_1, z_2 \in \mathbb{C} \right\} \tag{29}$$

- $\Omega = (1, 0)$, pure state $\omega(a) := \langle \Omega, a\Omega \rangle$, $\sigma_z \Omega = \Omega$.
- $\pi_\omega(A)$ is the defining representations of $M_2(\mathbb{C})$ on $\mathcal{H}_\omega = \mathbb{C}^2$, $\Omega_\omega = \Omega$.
- $\mathcal{H}_+ = \{(z, 0),\ z \in \mathbb{C}\}$ and $\mathcal{H}_- = \{(0, z),\ z \in \mathbb{C}\}$.
- Let $\pi_\pm$ be the restriction of $\pi_\omega(A_+)$ to $\mathcal{H}_\pm$

$$\pi_\pm \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix} = z_\pm \tag{30}$$

Inequivalent representations.

Remarkable result

Theorem

Suppose A carries a $\mathbb{Z}_2$-action θ as well as a $\mathbb{Z}_2$-invariant state $\omega : A \to \mathbb{C}$. suppose the representation $\pi_+(A_+)$ on $\mathcal{H}_+$ is irreducible. Then also the representation $\pi_-(A_+)$ on $\mathcal{H}_-$ is irreducible, and there are the following two possibilities for the representation $\pi_\omega(A)$ on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

- *$\pi_\omega(A)$ is irreducible (and hence ω is pure) iff $\pi_\pm(A_+)$ are inequivalent;*
- *$\pi_\omega(A)$ is reducible (and hence ω is mixed) iff $\pi_\pm(A_+)$ are equivalent.*

H. Araki and D.E. Evans, On a C*-algebra approach to phase transition in the two-dimensional Ising model, *Commun. Math. Phys.* 91, 489-503 (1983)

Self-dual formalism

Selfdual Algebra $A_{SD}(\mathcal{K}, \Gamma)$ (for diagonalization of quadratic Hamiltonians):

- $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$
- We have two conjugations: $T : \mathcal{H} \to \mathcal{H},\ T^* = T,\ T^2 = \mathbb{1}_{\mathcal{H}}$ and $\Gamma : \mathcal{K} \to \mathcal{K}$
- $B(h) = a^*(f) + a(Tg)$ (C^*-algebra isomorphism) with $h = (f, g) \in \mathcal{K}$
- $B^*(h) = B(h)^* = B(\Gamma h)$

Theorem
There is a bijective correspondence between basis projections $P : \mathcal{K} \to \mathcal{K}$ ($\Gamma P \Gamma = \mathbb{1}_{\mathcal{H}} - P$) and states ω_P on A^F that satisfy

$$\omega_P(B(h)^* B(h)) = \langle h \mid Ph \rangle, \quad \forall h \in \mathcal{K} \tag{31}$$

Such a state (quasi-free) ω_P is pure (so that the corresponding GNS representation π_P is irreducible).

H. Araki., On Quasifree States of CAR and Bogoliubov Automorphisms, *Publ. RIMS Kyoto Univ.* 6, 385-442 (1970)

Phase transition of Ising Model and representations

$$H_\lambda = -\sum_j \sigma_x^j \sigma_x^{j+1} - \lambda \sum_j \sigma_z^j, \tag{32}$$

$\lambda \ll 1$	$\lambda \gg 1$
$\lvert\rightarrow\rightarrow\cdots\rightarrow\rangle$ or $\lvert\leftarrow\leftarrow\cdots\leftarrow\rangle$	$\lvert\uparrow\uparrow\cdots\uparrow\rangle$

Let $W_- : \mathcal{K} \to \mathcal{K}$ be the $\mathbb{Z}_2$-action on $\mathcal{K}$ defining the $\mathbb{Z}_2$-action θ_- on A^F and let E_+ be the projection onto the positive energy space for $H_{\lambda(SD)}$ in $\mathcal{K}$, then

$$\begin{aligned} \pi_{\omega_0^F} &= \pi_{E_+} \\ \pi_{\theta_-^* \omega_0^F} &= \pi_{W_- E_+ W_-} \end{aligned} \tag{33}$$

Theorem (Araki-Matsui)
The unique $\mathbb{Z}_2$-invariant ground state ω_0 of the Hamiltonian of the Ising model is pure (and hence forms the unique ground state) iff both of the following hold

1. $E_+ - W_- E_+ W_- \in B_2(\mathcal{K})$;
2. $dim(E_+\mathcal{K} \cap (\mathbb{1} - W_- E_+ W_-)\mathcal{K})$ *is even*

- $\mathbb{Z}_2$-index between two basis projections E_1, E_2

$$\sigma(E_1, E_2) = (-1)^{\dim E_1 \cap (1 - E_2)} \tag{34}$$

- For the Ising model

$$\sigma\big(E_+, (\mathbb{1} - W_- E_+ W_-)\big) = \begin{cases} +1, & |\lambda| \geq 1 \quad \omega_0 \text{ is pure} \\ -1, & |\lambda| < 1 \quad \omega_0 = \frac{1}{2}(\omega_0^+ + \omega_0^-) \end{cases} \tag{35}$$

where $\omega_0^\pm$ are pure and transform under the $\mathbb{Z}_2$-action θ as $\omega_0^\pm \circ \theta = \omega_0^\mp$

H. Araki and T. Matsui, Ground states of the XY-model, *Commun. Math. Phys.* 101, 213-245 (1985)

Shale-Stinespring Theorem

Context

V: real vector space of even $2m$ or infinite dimension.
g: bilinear, symmetry and non-degenerated form over V.
J: orthogonal complex structure compatible with g, i.e. a real linear operator in V with the properties:

$$g(u,v) = g(Ju,Jv) \ \forall \ u,v \ \in V, \quad J^2 = -\mathbb{1}_V, \tag{36}$$

V_J: complexification of V by defining $\mathrm{i}v := Jv$. We can define an inner product:

$$\langle u,v\rangle_J = g(u,v) + \mathrm{i}g(Ju,v), \tag{37}$$

Turns $(V_J, \langle\cdot,\cdot\rangle_J)$ in a Hilbert space. Fock space is the exterior algebra of V_J, $\mathcal{F}_J(V) = \overline{\bigwedge^\bullet V_J}$. $\{u_i\}_{i=1}^k \in V_J$ with $k < dimV_J$, and s.t. $u_i \wedge u_j - u_j \wedge u_i = 0$, then $u_1 \wedge \cdots \wedge u_k \in \mathcal{F}_J(V)$. Define the operators:

$$\begin{aligned} a_J^\dagger(v)(u_1 \wedge \cdots \wedge u_k) &:= v \wedge u_1 \wedge \cdots \wedge u_k, \\ a_J(v)(u_1 \wedge \cdots \wedge u_k) &:= \textstyle\sum_{i=1}^k (-1)^{i-1} \langle v, u_i\rangle_J \ u_1 \wedge \cdots \hat{u}_i \cdots \wedge u_k. \end{aligned} \tag{38}$$

Fock representation

Verifies the Canonical Anticommutation Relation (CAR):

$$\left\{a_J(u), a_J^\dagger(v)\right\} = \langle u, v\rangle_J \mathbb{1}, \quad \{a_J(u), a_J(v)\} = 0 = \left\{a_J^\dagger(u), a_J^\dagger(v)\right\} \tag{39}$$

Its span is a representation of the algebra A_{CAR} over the Hilbert space $\mathcal{F}_J(V)$. Consider the usual complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and the polarization: $V^{\mathbb{C}} = W_J \oplus \overline{W}_J$, where $W_J = \left\{v - \mathrm{i}Jv \in V^{\mathbb{C}} \mid v \in V\right\}$, and $\overline{W}_J = W_{-J}$. Define the inner folowing inner product in $V^{\mathbb{C}}$:

$$\langle\!\langle w, z\rangle\!\rangle = 2g^{\mathbb{C}}(\overline{w}, z) \quad w, z \in V^{\mathbb{C}}. \tag{40}$$

With this inner product $\overline{W}_J = W_J^{\perp}$. Morover $\overline{W}_J \cap W_J = \varnothing$.

$P_J := \frac{1}{2}(\mathbb{1} - iJ)$ the projection in $EndV^{\mathbb{C}}$ with range W_J. Defines an unitary isomorphism between $(V_J, \langle\cdot,\cdot\rangle_J)$ and $(W_J, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$.

Fock representation

It is possible get a representation of the Clifford algebra $\mathbb{C}l(V)$ on $\mathcal{F}_J(V)$ from the A_{CAR} representation by

$$c_J(v) = a_J(v) + a_J^\dagger(v) \tag{41}$$

Notice that $a_J(v) = c_J(P_{-J}v)$ and $a_j^\dagger(v) = c_J(P_J v)$.

The representation can be understood as GNS representation of the CAR Algebra for a quasifree state ω_J with two point function

$$\omega_J(a_J(u)a_J^\dagger(v)) = \langle u, v\rangle_J = \langle\!\langle P_J u, P_J v\rangle\!\rangle. \tag{42}$$

Let be $|0\rangle = 1 \in \mathbb{C} = \bigwedge^0 V_J$ the vacuum vector. The Fock vacuum is defined by:

$$\mathbb{C}l(V) \ni a \mapsto \omega_J(a) = \langle 0|c_J(a)|0\rangle \in \mathbb{C}. \tag{43}$$

If $u \in V^{\mathbb{C}}$, we have $c_J(u)|0\rangle = 0 \Leftrightarrow u \in \overline{W}_J$.

Fock representation

If a representation $\pi : \mathbb{C}l \to \mathcal{L}(\mathcal{H})$ has a cyclic vector $|\Phi\rangle \in \mathcal{H}$, that satisfy the vacuum condition $\pi(v + iJv)|\Phi\rangle = 0 \; \forall v + iJv \in W_{-J}$, then π is unitarily equivalent to the Fock representation c_J.

- J an orthogonal complex structure over (V, g), and $h \in O(V, g)$ an orthogonal element $(g(hu, hv) = g(u, v))$, this can be extented to an automorphism θ_h of $\mathbb{C}l(V)$ and it is called Bogoliubov automorphism.
- We can define $K = hJh^{-1}$ that is also a new complex orthogonal structure, moreover this action is transitive.
- If the $dimV = 2m$, the manifold of orthogonal complex structures can be identify with $O(2m)/U(m)$.

Emerge the question, when two complex structure gives arise inequivalent representations?

Theorem (Shale-Stinespring theorem)
Let J, K be two orthogonal complex structures. Them π_J and π_K are unitarily equivalent iff $J - K$ is Hilbert-Schmidt.

- $h \in O(V, g)$ decompose in linear and antilinear part:

$$h = p_h + q_h, \qquad p_h := \tfrac{1}{2}(h - JhJ), \quad q_h := \tfrac{1}{2}(h + JhJ) \tag{44}$$

- $O_J(V) = \{h \in O(V) \mid [J, h] \text{ es Hilbert-Schmidt (H-S)}\}$
- Equivalence problem = implementability problem = cyclic vector which satisfies the vacuum condition

$$\big(a_J(p_h v) + a_J^*(q_h v)\big)\Phi = 0 \tag{45}$$

$$\Phi = \zeta \wedge f_{T_l} \equiv u_1 \wedge \cdots \wedge u_n \wedge f_{T_l}, \quad l = rh \in O_J(V), \tag{46}$$

where $T_l = q_l p_l^{-1}$ and $n = \dim ker p_h$.

Index

Let $u(h)$ be the spin representation and let χ_J be the grading operator in $\mathcal{F}_J(V)$. If $u(h)$ implements θ_h on $\mathcal{F}_J(V)$ then so does $\chi_J u(h)\chi_J$, then

- $\chi_J^2 = \mathbb{1}$ and $\chi_J u(h)\chi_J = \pm u(h)$
- The parity of $u(h)$ is the parity of $\zeta \wedge f_{T_I}$, that is $n = dimkerp_h$.
- $\chi_J \mu(h)\chi_J := \iota(h)\mu(h)$, where ι is the sign homomorphism:

$$\begin{aligned} \iota : O_J(V) &\to \mathbb{Z}_2 \\ h &\mapsto \iota(h) := (-1)^{\mathrm{dimker}(p_h)} = (-1)^n \end{aligned} \tag{47}$$

  This map is continuous, so it is an index map.
- If $K = hJh^{-1}$, with $h \in O_J(V)$ then $\iota(h) = (-1)^{\frac{1}{2}\mathrm{dimker}(J+K)}$

Relation between the approaches

- Begin with a complex Hilbert space $\mathcal{H}$ and consider the conjugate Hilbert $\overline{\mathcal{H}}$ where the multiplication by scalars are defined by $\alpha \cdot \psi = \overline{\alpha}\psi \ \forall \psi$, and inner product by $\langle \phi, \psi \rangle_{\overline{\mathcal{H}}} := \overline{\langle \phi, \psi \rangle}_{\mathcal{H}}$.
- Now consider the complex Hilbert space $\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{H}}$ and the projection P on the first component.
- Define the conjugation operation T on $\mathcal{H}$ and Γ on $\mathcal{K}$ given by $\Gamma(x \oplus y) = Ty \oplus Tx$, then P is a basis projection $\mathbb{1} - P = \Gamma P \Gamma$ on $\mathcal{K}$.
- Finally define the real vector space $V = Re\mathcal{K}_\Gamma$ and define the complex structure in V by $J = 2P - \mathbb{1}$ and the real inner product $g(\phi, \psi) = Re\langle \phi, \psi \rangle_{\mathcal{K}}$, then $V_J \cong \mathcal{H}$.
- We have the following equivalences: $\mathbb{C}l(V) \cong A_{CAR}(\mathcal{H}) \cong A_{SD}(\mathcal{K}, \Gamma)$

Example of 2 sites

Initial hamiltonian for 2 sites chain in the fermionic algebra is:

$$H_\lambda = -(a_1^\dagger a_2 + a_2^\dagger a_1 + a_1^\dagger a_2^\dagger - a_1 a_2) + 2\lambda(a_1^\dagger a_1 + a_2^\dagger a_2), \tag{48}$$

In the Araki's formalism it can be writen by:

$$H = \frac{1}{2}(a^\dagger, a)\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{49}$$

Where $a = (a_1, a_2)$, $A = 2\lambda\mathbb{1} - \sigma_x$ and $B = -i\sigma_y$.

Consider de complex estructure on $V = \mathbb{R}^4$ defined by $J = -\mathrm{i}\sigma_x \otimes \sigma_y$, then $V_J = \mathcal{H} = \mathbb{C}^2$ and $\mathcal{K} = \mathbb{C}^4$.

Define $\alpha = \sqrt{1 + 4\lambda^2}$, $\beta_\pm = \sqrt{(\alpha - 1)/2\alpha}$, and $\sigma = sgn(\lambda - 1)$, then:

$$\Phi = \begin{pmatrix} \beta_+ & \beta_- \\ -\beta_- & \beta_+ \end{pmatrix}, \ \Psi = \begin{pmatrix} \sigma\beta_- & \sigma\beta_+ \\ -\beta_+ & \beta_- \end{pmatrix}, \tag{50}$$

Example 2 sites

So the orthogonal transformation h that induce the Bogoliubov transformation:

$$h = \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}, \tag{51}$$

Define $g = \frac{1}{2}(\Phi + \Psi)$ and $f = \frac{1}{2}(\Phi - \Psi)$, then the linear and antilinear part of h are respectively:

$$p_h = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \; q_h = \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix}, \tag{52}$$

Define the new complex estructure $K = hJh^{-1}$, now we can calculate the kernel of $J + K$ and then the index by:

$$\iota(h) = (-1)^{\frac{1}{2}\mathrm{dim}\ker(J+K)} = det(h) = \sigma. \tag{53}$$

Perspectives

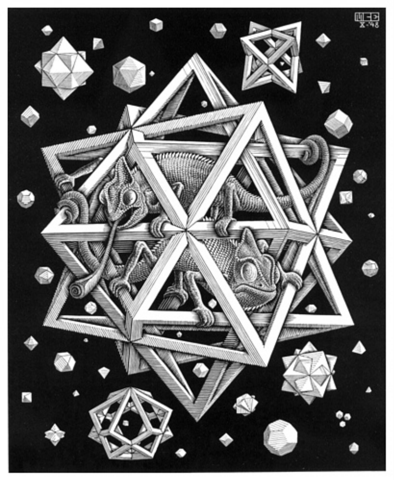
Work in progress

- To verify the validity of the local index formula proposed in the reference of Kauffman et al.[1] for concrete physical models.
- To explore the effect of perturbations and disorder in the context of non-commutative geometry approaches for concrete physical models.
- To explore the bulk-edge correspondence in the non-commutative framework.

1. Ralph M. Kaufmann, Dan Li & Birgit Wehefritz-Kaufmann, Topological insulators and K-theory, arXiv: 1510.08001 (2016)

- S. Sachdev, *Quantum phase transitions* (Cambridge University Press, 2011)
- H. Araki and D.E. Evans, On a C*-algebra approach to phase transition in the two-dimensional Ising model, *Commun. Math. Phys.* 91, 489-503 (1983).
- H. Araki., On Quasifree States of CAR and Bogoliubov Automorphisms, *Publ. RIMS Kyoto Univ.* 6, 385-442 (1970).
- H. Araki and T. Matsui, Ground states of the XY-model, *Commun. Math. Phys.* 101, 213-245 (1985)
- D.E. Evans and Y. Kawahigashi, *Quantum symmetries on operator algebras* (Oxford University Press, 1998)
- J.M. Gracia-Bondía, J.C. Varilly and H. Figueroa, *Elements of noncommutative geometry* (Birkhoauser, 2001)

Thank you!



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