Progress in the classification of second order superintegrable systems

Jonathan Kress

Joint work Konrad Schöbel, Andreas Vollmer and Jeremy Nugent

School of Mathematics and Statistics
University of New South Wales, Australia

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Outline

1. What is a superintegrable system
2. Nondegenerate second order superintegrability in 2D
3. Conformal superintegrability and a geometric definition of superintegrability
4. Towards a classification in *n*D

Classical integrability

- $2n$-dimensional phase space with coordinates $(\mathbf{x}, \mathbf{p})$
- Hamiltonian

$$H = g + V = \sum_{i,j=1}^{n} g^{ij}(\mathbf{x}) p_i p_j + V(\mathbf{x}).$$

- Poisson bracket $\{\ ,\ \}$
- Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}, \qquad \dot{x}_i = \frac{\partial H}{\partial p_i}$$

- $L(\mathbf{x}, \mathbf{p})$ such that $\{H, L\} = 0$ is an integral or symmetry.
- n mutually commuting integrals $H = L_1,\ L_i(\mathbf{x}, \mathbf{p})$,

$$\{L_i, L_j\} = 0, \qquad i, j = 1 \dots n.$$

Classical superintegrability

- $2n$-dimensional phase space with coordinates $(\mathbf{x}, \mathbf{p}) = (x_i, p_i)$
- Hamiltonian
$$H = \sum_{i,j=1}^{n} g^{ij}(\mathbf{x}) p_i p_j + V(\mathbf{x}).$$
- More than n integrals $H = L_1,\ L_i(x_i, p_i)$,
$$\{L_i, L_i\} = 0, \qquad i = 1 \ldots 2n - 1.$$
- Maximally superintegrable systems have $2n - 1$ independent integrals.
- Trajectories lie in 1D submanifolds.

Quantum superintegrability

- n-dimensional manifold with coordinates $\mathbf{x} = (x_1, \ldots, x_n)$
- Hamiltonian

$$H = \triangle + V(\mathbf{x}),$$

- Integrable: n algebraically independent differential operators H, L_i such that

$$[H, L_i] = 0, \qquad [L_i, L_j] = 0, \quad \text{for all } i, j = 1, \ldots, n-1.$$

- The L_i are symmetries.
- Superintegrable: H has $2n - 1$ independent symmetries
- Associated with the possibility of solving the eigenvalue problem $H\Psi = E\Psi$ exactly, analytically and algebraically.

Superintegrability

- Also know as *degenerate integrability* or *non-Abelian integrability*.
- Symmetries form a non-trivial algebra (eg Racah algebra).
- Second order superintegrability leads to multiseparability and interesting connections with othogonal polynomials.
- Inönü-Wigner and Bôcher contractions of 2D second order superintegrable systems reproduce the Askey-Wilson scheme of othogonal polynomials.

Example: Kepler

The Kepler Hamiltonian in 3 dimensions is

$$H = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}$$

with $\{H, M_i\} = \{H, L_i\} = 0$ for $i = 1, 2, 3$, where

$$M_1 = yp_z - zp_y, \qquad L_1 = p_y M_3 - p_z M_2 + \frac{\alpha x}{2r},$$
$$M_2 = zp_x - xp_z, \qquad L_2 = p_z M_1 - p_x M_3 + \frac{\alpha y}{2r},$$
$$M_3 = xp_y - yp_x, \qquad L_3 = p_x M_2 - p_y M_1 + \frac{\alpha z}{2r}.$$

H is maximally superintegrable due to Laplace-Runge-Lenz vector.

At most $2 \times 3 - 1 = 5$ constants can be functionally independent.

$$M_1 L_1 + M_2 L_2 + M_3 L_3 = 0 \quad \text{and} \quad L_1^2 + L_2^2 + L_3^2 - (M_1^2 + M_2^2 + M_3^2) H = \frac{\alpha^2}{4}.$$

The symmetry algebra closes quadratically.

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{L_i, M_j\} = \epsilon_{ijk} L_k \quad \text{and} \quad \{L_i, L_j\} = \epsilon_{ijk} H M_k.$$

Example: Deformations of Kepler

A 4 parameter (semi-degenerate) deformation of Kepler is also superintegrable with second order constants:

$$H = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \delta.$$

A 5 parameter (non-degenerate) deformation of Kepler is also superintegrable but requires a fourth order constant (Verrier & Evans 2008):

$$H = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} + \eta.$$

The symmetry algebras of these systems close polynomially but only if an additional functionally dependent symmetry is included.

These have quantum counter parts.

Example: Smorodinsky-Winternitz system (1965)

$$H = p_x^2 + p_y^2 + \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \delta$$

$$L_1 = p_x^2 + \alpha x^2 + \frac{\beta}{x^2}, \qquad L_2 = (xp_y - yp_x)^2 + \beta\frac{y^2}{x^2} + \gamma\frac{x^2}{y^2}.$$

$$\{H, L_1\} = \{H, L_2\} = 0.$$

With $R = \{L_1, L_2\}$ the Poisson algebra closes quadratically.

$$\begin{aligned}
\{R, L_1\} &= 8L_1^2 - 8(H-\delta)L_1 + 16\alpha L_2, \\
\{R, L_2\} &= -16L_1L_2 + 8(H-\delta)L_2 - 16(\beta+\gamma)L_1 + 16(H-\delta)\beta, \\
R^2 &= -16L_1^2L_2 + 16(H-\delta)L_1L_2 - 16\alpha L_2^2 - 16(\beta+\gamma)L_2^2 \\
&\quad + 32\beta(H-\delta)L_1 - 16\beta(H-\delta)^2 + 16\alpha\beta\gamma.
\end{aligned}$$

More examples

- Oscillators with rational frequency ratios

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \omega_1^2 x^2 + \omega_2^2 y^2 , \quad \frac{\omega_1}{\omega_2} \in \mathbb{Q}$$

- Calogero-Moser (Wojciechowski, 1983)

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j=1}^{n} \frac{1}{(x_i - x_j)^2}$$

- Toda lattice (non-polynomial symmetries) (Agrotis, Damianou, Sophocleous, 2005)

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}}$$

- A non-separable system (Post and Winternitz, 2011)

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\alpha y}{x^{2/3}}$$

2D and 3D **second order** superintegrable systems

This talk: second order superintegrability

- 2D free systems ($V = 0$) found by Koenigs (1872).
- 2D non-degenerate[1] systems are known.
- 2D degenerate systems are parameter restrictions of non-degenerate systems.

- 3D non-degenerate systems on conformally flat spaces are known.
- 3D degenerate systems — examples are known.[2]

- nD — examples are known.

Quadratically closed symmetry algebras exist in all dimensions.

[1] Potentials depending on the maximum number of parameters - see later.
[2] Miller and Escobar Ruiz 2017 — large family of semi-degenerate systems

2D complex Euclidean second order systems

Consider 2D complex Euclidean space with complex variables

$$z = x + iy,\ \bar{z} = x - iy, \qquad p_z = \frac{1}{2}(p_x - ip_y),\ p_{\bar{z}} = \frac{1}{2}(p_x + ip_y),$$

and look for Hamiltonians of the form

$$H = 4p_z p_{\bar{z}} + V(z, \bar{z})$$

admitting two independent second order constants

$$\begin{aligned} L_1 &= a_1^{(1)} p_z^2 + a_2^{(1)} p_z p_{\bar{z}} + a_3^{(1)} p_{\bar{z}}^2 + W^{(1)} \\ L_2 &= a_1^{(2)} p_z^2 + a_2^{(2)} p_z p_{\bar{z}} + a_3^{(2)} p_{\bar{z}}^2 + W^{(2)}, \end{aligned}$$

where $a_j^{(i)}$ and $W^{(i)}$ are functions of z and $\bar{z}$.

An immediate consequence of

$$\{H, L_1\} = \{H, L_2\} = 0$$

is that the $a_j^{(1)}$ and $a_j^{(2)}$ are the components of a Killing tensor.

Nondegenerate systems

Integrability conditions for $W^{(i)}$ (Bertrand-Darboux equations) are:

$$\begin{aligned}
-2a_1^{(1)} V_{zz} + 2a_3^{(1)} V_{\bar{z}\bar{z}} &= \left(a_{2\bar{z}}^{(1)} - 2a_{1z}^{(1)}\right) V_z - \left(a_{2z}^{(1)} - 2a_{3\bar{z}}^{(1)}\right) V_{\bar{z}},\\
-2a_1^{(2)} V_{zz} + 2a_3^{(2)} V_{\bar{z}\bar{z}} &= \left(a_{2\bar{z}}^{(2)} - 2a_{1z}^{(2)}\right) V_z - \left(a_{2z}^{(2)} - 2a_{3\bar{z}}^{(2)}\right) V_{\bar{z}}.
\end{aligned}$$

Solve for V_{zz} and $V_{\bar{z}\bar{z}}$ in terms of $V_{z\bar{z}}$, V_z and $V_{\bar{z}}$.

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix}$$

Nondegenerate: all derivatives of V at $\mathbf{x}_0$ are determined by the 4 parameters $V_{z\bar{z}}(\mathbf{x}_0)$, $V_z(\mathbf{x}_0)$, $V_{\bar{z}}(\mathbf{x}_0)$ and $V(\mathbf{x}_0)$.

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-2a_1^{(2)} V_{zz} + 2a_3^{(2)} V_{\bar z\bar z} &= \left(a_{2\bar z}^{(2)} - 2a_{1z}^{(2)}\right) V_z - \left(a_{2z}^{(2)} - 2a_{3\bar z}^{(2)}\right) V_{\bar z}.
\end{aligned}
$$

Solve for V_{zz} and $V_{\bar z\bar z}$ in terms of $V_{z\bar z}$, V_z and $V_{\bar z}$.

$$
\begin{bmatrix} V_{zz} \\ V_{\bar z\bar z} \end{bmatrix} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar z} \end{bmatrix}
$$

Nondegenerate: all derivatives of V at $\mathbf{x}_0$ are determined by the 4 parameters $V_{z\bar z}(\mathbf{x}_0)$, $V_z(\mathbf{x}_0)$, $V_{\bar z}(\mathbf{x}_0)$ and $V(\mathbf{x}_0)$.

Integrability of $a_j^{(i)}$

For each constant

$$L = a_1^{(i)} p_z^2 + a_2^{(i)} p_z p_{\bar{z}} + a_3^{(i)} p\bar{z}^2 + W,$$

coefficients of cubic terms in $\{H, L\} = 0$ and integrability conditions for W lead to

$$\begin{aligned}
\partial_z a_1^{(i)} &= -\tfrac{2}{3} C^{11} a_1^{(i)} + \tfrac{2}{3} C^{21} a_3^{(i)} \\
\partial_{\bar{z}} a_1^{(i)} &= 0 \\
\partial_z a_2^{(i)} &= \tfrac{2}{3} C^{22} a_3^{(i)} - \tfrac{2}{3} C^{12} a_1^{(i)} \\
\partial_{\bar{z}} a_2^{(i)} &= \tfrac{2}{3} C^{11} a_1^{(i)} - \tfrac{2}{3} C^{21} a_3^{(i)} \\
\partial_z a_3^{(i)} &= 0 \\
\partial_{\bar{z}} a_3^{(i)} &= -\tfrac{2}{3} C^{22} a_3^{(i)} + \tfrac{2}{3} C^{12} a_1^{(i)}
\end{aligned}$$

All derivatives of the $a_j^{(i)}$ at a point are determined by their values and the $C^{k\ell}$ at that point.

[Greatly simplified prolongation of Killing tensor equations.]

Involutive system for C^{ij}

Integrability for $a_j^{(i)}$ give expressions for derivatives of the C^{ij}.

With two new symbols $C_z^{21} = \partial_z C^{21}$ and $C_{\bar{z}}^{12} = \partial_{\bar{z}} C^{12}$ we have

$$
\begin{aligned}
\partial_z C^{11} &= \tfrac{2}{3} C^{22} C^{12} + \tfrac{2}{3}(C^{11})^2 - C_{\bar{z}}^{12} & \qquad \partial_{\bar{z}} C^{11} &= \tfrac{2}{3} C^{12} C^{21} \\
\partial_z C^{22} &= \tfrac{2}{3} C^{12} C^{21} & \partial_{\bar{z}} C^{22} &= \tfrac{2}{3} C^{11} C^{21} + \tfrac{2}{3}(C^{22})^2 - C_z^{21} \\
\partial_z C^{12} &= \tfrac{2}{3} C^{11} C^{12} & \partial_{\bar{z}} C^{21} &= \tfrac{2}{3} C^{22} C^{21} \\
\partial_{\bar{z}} C_z^{21} &= \tfrac{2}{3} C_z^{21} C^{22} + \tfrac{4}{9} C^{12} C^{21} & \partial_z C_{\bar{z}}^{12} &= \tfrac{2}{3} C_{\bar{z}}^{12} C^{11} + \tfrac{4}{9} C^{12} C^{21}
\end{aligned}
$$

$$
\begin{aligned}
\partial_z C_z^{21} &= \tfrac{8}{9} C^{22} C^{12} C^{21} - \tfrac{4}{3} C^{21} C_{\bar{z}}^{12} + \tfrac{4}{9} C^{21} (C^{11})^2 + \tfrac{2}{3} C^{11} C_z^{21} \\
\partial_{\bar{z}} C_{\bar{z}}^{12} &= \tfrac{8}{9} C^{11} C^{12} C^{21} - \tfrac{4}{3} C^{12} C_z^{21} + \tfrac{4}{9} C^{12} (C^{22})^2 + \tfrac{2}{3} C^{22} C_{\bar{z}}^{12}.
\end{aligned}
$$

Integrability conditions

Integrability conditions are:

$$0 = 9C^{21}C^{12}_{\bar{z}} - 3C^{11}C^{21}_{z} - 2C^{22}C^{12}C^{21} - 2C^{21}(C^{11})^2$$

$$0 = 9C^{12}C^{21}_{z} - 3C^{22}C^{12}_{\bar{z}} - 2C^{11}C^{12}C^{21} - 2C^{12}(C^{22})^2$$

$$0 = 9C^{12}_{\bar{z}}C^{21}_{z} - 3C^{11}C^{21}C^{12}_{\bar{z}} - 3C^{22}C^{12}C^{21}_{z} + (C^{12})^2(C^{21})^2 - 2C^{11}C^{22}C^{12}C^{21}$$

Classification

Define some numbers a_{ij} by

$$\begin{aligned} D &= \det \begin{bmatrix} a^1_{(1)} & a^1_{(2)} \\ a^3_{(1)} & a^3_{(2)} \end{bmatrix} = \sum a_{ij} z^i \bar{z}^j \\ A_z &= a_{21}\bar{z}^2 + 2a_{20}\bar{z} + a_{30} \\ B_{\bar{z}} &= a_{12}z^2 + 2a_{02}z + a_{03}. \end{aligned}$$

Then

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -D_z & A_z \\ B_{\bar{z}} & -D_{\bar{z}} \end{bmatrix}$$

The integrability conditions require that D factors as

$$D(z,\bar{z}) = (a_1 z + c_1)(a_2 z + b_2 \bar{z} + c_2)(b_3 \bar{z} + c_3)$$

or

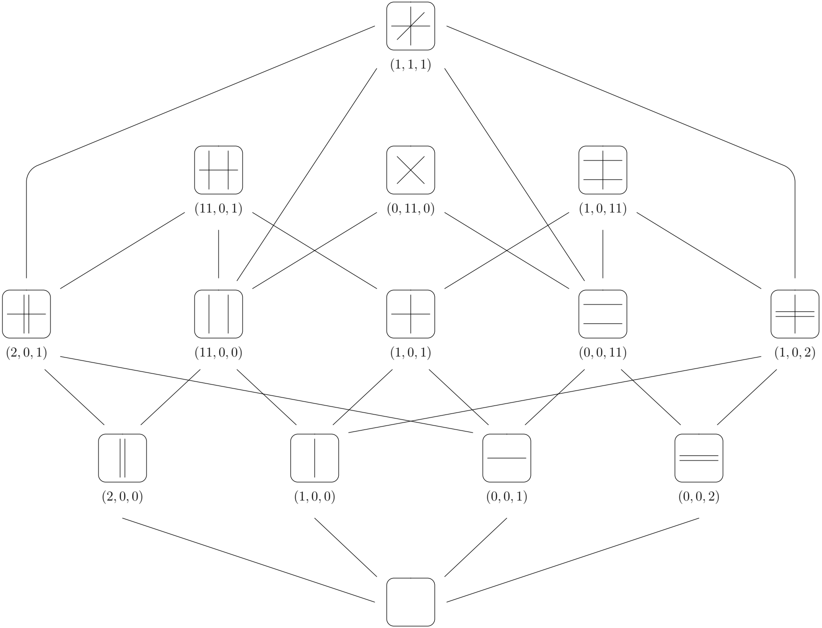
$$D(z,\bar{z}) = (a_1 z + b_1 \bar{z} + c_1)(a_3 z + b_3 \bar{z} + c_3).$$

class	$D(z,\bar z)$	superintegrable potentials		
$(1,1,1)$	$z(z+\bar z)\bar z$	$\frac{1}{\sqrt{z\bar z}}$	$\frac{1}{\sqrt{z\bar z}}\frac{1}{(\sqrt{z}+\sqrt{\bar z})^2}$	$\frac{1}{\sqrt{z\bar z}}\frac{1}{(\sqrt{z}-\sqrt{\bar z})^2}$
$(11,0,1)$	$z(z+1)\bar z$	$\frac{\bar z}{\sqrt{(\bar z+1)(\bar z-1)}}$	$\frac{1}{\sqrt{z(\bar z+1)}}$	$\frac{1}{\sqrt{z(\bar z-1)}}$
$(2,0,1)$	$z^2\bar z$	$\frac{1}{\sqrt{z\bar z}}$	$\frac{1}{\bar z\sqrt{z\bar z}}$	$\frac{1}{\bar z^2}$
$(0,11,0)$	$(z+\bar z)(z-\bar z)$	$z\bar z$	$\frac{1}{x^2}$	$\frac{1}{y^2}$
$(11,0,0)$	$z(z+1)$	$z\bar z$	$\frac{\bar z}{\sqrt{\bar z^2-1}}$	$\frac{2z\bar z^2-z}{\sqrt{\bar z^2-1}}$
$(2,0,0)$	z^2	$z\bar z$	$\frac{1}{\bar z^2}$	$\frac{z}{\bar z^3}$
$(1,0,1)$	$z\bar z$	$\frac{1}{\sqrt{z\bar z}}$	$\frac{1}{\sqrt{z}}$	$\frac{1}{\sqrt{\bar z}}$

2D second order superintegrable systems

class	$D(z,\bar{z})$	superintegrable potentials		
$(0,1,0)$	$z+\bar{z}$	x^2+4y^2	$\frac{1}{x^2}$	y
$(1,0,0)$	z	$\frac{z}{\sqrt{\bar{z}}}$	$\frac{1}{\sqrt{\bar{z}}}$	z
$(1,0,0)$	z	$\frac{1}{\sqrt{\bar{z}}}$	x	$\frac{z+3\bar{z}}{\sqrt{\bar{z}}}$
$(0,0,0)$	1	$z\bar{z}$	z	$\bar{z}$
$(0,0,0)$	1	$\bar{z}^3+3z\bar{z}$	$\bar{z}^2+z$	$\bar{z}$

K and Schöbel: arXiv:1602.07890


(1, 1, 1)
(11, 0, 1)
(0, 11, 0)
(1, 0, 11)
(2, 0, 1)
(11, 0, 0)
(1, 0, 1)
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(1, 0, 2)
(2, 0, 0)
(1, 0, 0)
(0, 0, 1)
(0, 0, 2)

Conformally superintegrable systems

A maximally conformally superintegrable Hamiltonian

$$H = \sum_{i,j=1}^{n} g^{ij}(\mathbf{x}) p_i p_j + V(\mathbf{x}),$$

has $2n - 1$ functionally independent conformal constants (polynomial in p_i) that commute with H up to a multiple of H,

$$\{H, L\} = \rho_L H,$$

where ρ_L a polynomial in p_i.

[Confomally commuting = R-commuting.]

L is constant on the hypersurface $H = 0$.

Superintegrable systems are conformally superintegrable with all $\rho_L = 0$.

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Coupling constant metamorphosis (Stäckel transform)

Suppose

$$H = H_0 + \alpha V \quad \text{and} \quad L = L_0 + \alpha W \quad \text{with} \quad \{H, L\} = \rho_L H.$$

Then coupling constant metamorphosis (Stäckel transform) gives

$$L' = L_0 - WH' \;\; \text{and} \;\; H' = \frac{H_0}{V} \quad \Rightarrow \quad \{H', L'\} = 0.$$

L_1 and L_2 are conformal constants for H with $\{L_1, L_2\} = \rho_{12} H$

$\downarrow$ CCM

L_1' and L_2' are constants for H' with $\{L_1', L_2'\} = 0$

conformally superintegrable system $\xrightarrow{\text{CCM}}$ superintegrable system

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conformally superintegrable system $\xrightarrow{\text{CCM}}$ superintegrable system

Superintegrable integral ⇒ conformally SI on flat space

A superintegrable system on a conformally flat space can written as

$$H = \frac{1}{\lambda}\sum_{i=1}^{n} p_i^2 + V.$$

Suppose L is a constant for H, that is,

$$\{H, L\} = 0.$$

If we define

$$\hat{H} = \sum_{i=1}^{n} p_i^2 + \lambda V$$

then

$$\{\hat{H}, L\} = \rho \hat{H}$$

where $\rho = \lambda^{-1}\{\lambda, L\}$.

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A more geometric perspective

One-to-one correspondence:

superintegrable systems on conformally flat spaces
$\updownarrow$
conformally superintegrable systems on flat space

A different perspective. Think of the potential V as a family of conformal scalings.

A non-degenerate superintegrable system is a subspace of second rank Killing tensors that remains Killing under a family of conformal scalings.

Algebraic superintegrability conditions in nD

Consider a second order maximally superintegrable system.

$$H = g + V = \sum \frac{1}{2} g_{ij}^{(\alpha)}(\mathbf{x}) p^i p^j + V$$

admitting $2n - 1$ functionally independent constants

$$L^{(\alpha)} = K^{(\alpha)} + V^{(\alpha)}, \qquad \alpha = 1 \ldots 2n-2,$$

where

$$K^{(\alpha)}(\mathbf{x}, \mathbf{p}) = \sum \frac{1}{2} K_{ij}^{(\alpha)}(\mathbf{x}) p^i p^j$$

Imposing

$$0 = \{H, L^{(\alpha)}\} = \{g + V, K + V^{(\alpha)}\} = \{g, K\} + \{g, V^{(\alpha)}\} + \{V, K\}$$

requires

$$\{g, K\} = 0 \qquad \text{(Killing tensor equation)}$$

$$\{g, V^{(\alpha)}\} + \{V, K\} = 0 \qquad (*)$$

Algebraic superintegrability conditions in nD

$(*)$ is $dV^{(\alpha)} = K^{(\alpha)} dV$ and has integrability condition

$$d(K^{(\alpha)} dV) = 0 \qquad \text{(Bertrand-Darboux)}$$

With equations from $2n-1$ linearly independent integrals and "solve" the Bertrand-Darboux equations to obtain.

$$\nabla_i \nabla_j V = T_{ij}{}^{m} \nabla_m V + \frac{1}{n} g_{ij} \Delta V$$

- $T_{ij}{}^{k}$ is determined by Killing tensor subspace.
- Higher derivatives of V are determined by ∇V and ΔV.
- If no further conditions relate ∇V and ΔV then V is **non-degenerate** and depends on $n+2$ parameters.

Prologation of superintegrable Killing tensor

- Substituting

$$\nabla_i \nabla_j V = T_{ij}{}^{m} \nabla_m V + \frac{1}{n} g_{ij} \Delta V$$

- into

$$d(KdV) = 0$$

gives

$$K_{ij,k} = \frac{1}{3} \begin{array}{|c|c|}\hline j & i \\ \hline k & \\ \cline{1-1}\end{array} T^{m}{}_{ji} K_{mk}.$$

This is a miraculously short prologation.

($\begin{array}{|c|c|}\hline j & i \\ \hline k & \\ \cline{1-1}\end{array}$ is a symmetriser that symmetrises on indices in rows and antisymetrises on indices in columns.)

Prologation of Killing tensor (Wolf '98)

$$
\begin{aligned}
K^0_{ab;c} &= K^1_{abc} \\
K^1_{abc;d} &= K^2_{abcd} + S_{ab}\,(K^0_{am} R^m{}_{bcd}) \\
K^2_{abcd;e} &= 2\,S_{ab}\,(S_{cde}\,(K^0_{cm} R^m{}_{bda;e} - K^0_{cm} R^m{}_{dea;b} + \\
&\qquad 2\,K^1_{acm} R^m{}_{deb} - K^1_{cdm} R^m{}_{bea}) \\
&\qquad + S_{cd}\,(\tfrac{1}{3}\,K^0_{am} R^m{}_{cdb;e} + \tfrac{2}{3}\,K^0_{am} R^m{}_{ecb;d} - \\
&\qquad \tfrac{1}{3}\,K^1_{abm} R^m{}_{dec} + 2\,K^1_{amc} R^m{}_{edb} + \\
&\qquad K^1_{ame} R^m{}_{cdb}\,))
\end{aligned}
$$

Prologation of Killing tensor (Wolf '98)

$$
\begin{aligned}
0 = \ & S_{ab}\, S_{cd}\, A_{ef}\, [K^0_{am}(2R^m{}_{ecb;df} - 3/2R^m{}_{nfe}R^n{}_{cdb} - R^m{}_{nfd}R^n{}_{ceb} - R^m{}_{bnc}R^n{}_{efd} \\
& +R^m{}_{nfb}R^n{}_{dec} + R^m{}_{cnd}R^n{}_{efb} - 4R^m{}_{nfd}R^n{}_{ecb} - 2R^m{}_{ndb}R^n{}_{efc} + R^m{}_{fnd}R^n{}_{ceb} \\
& +R^m{}_{fnd}R^n{}_{ecb} - R^m{}_{fnb}R^n{}_{dec}) + K^0_{cm}(R^m{}_{bea;df} - R^m{}_{dea;bf} - R^m{}_{eda;bf} \\
& +R^m{}_{nfe}R^n{}_{bda} + R^m{}_{nfd}R^n{}_{bea} - R^m{}_{anb}R^n{}_{efd} - R^m{}_{nfb}R^n{}_{dea} + R^m{}_{and}R^n{}_{efb} \\
& +2R^m{}_{nfb}R^n{}_{eda} + 2R^m{}_{ndb}R^n{}_{efa} - R^m{}_{fnd}R^n{}_{bea} + R^m{}_{fnb}R^n{}_{dea} + 4R^m{}_{fnb}R^n{}_{eda} \\
& +3R^m{}_{dnb}R^n{}_{efa}) + K^0_{em}(R^m{}_{bca;df} - R^m{}_{cda;bf} + R^m{}_{nfd}R^n{}_{bca} - R^m{}_{nfb}R^n{}_{cda} \\
& -3R^m{}_{ndb}R^n{}_{fca} - R^m{}_{fnd}R^n{}_{bca} + R^m{}_{fnb}R^n{}_{cda} - 3R^m{}_{dnb}R^n{}_{fca}) - K^1_{abm}R^m{}_{dec;f} \\
& +2K^1_{acm}R^m{}_{deb;f} + 2K^1_{acm}R^m{}_{edb;f} + 2K^1_{aem}R^m{}_{cdb;f} + 6K^1_{amc}R^m{}_{edb;f} \\
& +2K^1_{ame}R^m{}_{cdb;f} - 2K^1_{ame}R^m{}_{fcb;d} - K^1_{cdm}R^m{}_{bea;f} - 2K^1_{cem}R^m{}_{bda;f} \\
& -K^1_{cme}R^m{}_{bda;f} - K^1_{cme}R^m{}_{bfa;d} + K^1_{cme}R^m{}_{dfa;b} + K^1_{cme}R^m{}_{fda;b} \\
& +K^1_{emf}R^m{}_{bca;d} - K^1_{emf}R^m{}_{cda;b} + 3K^2_{abcm}R^m{}_{efd} + K^2_{abem}R^m{}_{dfc} \\
& -6K^2_{acdm}R^m{}_{efb} - 2K^2_{acem}R^m{}_{dfb} + 4K^2_{acem}R^m{}_{fdb} + 6K^2_{aecm}R^m{}_{fdb} \\
& +2K^2_{aefm}R^m{}_{cdb} + K^2_{cdem}R^m{}_{bfa} - 2K^2_{cefm}R^m{}_{bda}\ ]
\end{aligned}
$$

Algebraic superintegrability conditions

- Starting with

$$K_{ij,k} = \frac{1}{3}\begin{array}{|c|c|}\hline j & i \\ \hline k \\ \cline{1-1}\end{array} T^{m}{}_{ji}\, K_{mk}$$

- differentiating twice gives

$$\begin{array}{|c|c|}\hline i & j \\ \hline\end{array} R^{m}{}_{ikl}\, g_j^n K_{mn} = \begin{array}{|c|}\hline k \\ \hline l \\ \hline\end{array}\left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) K_{mn}.$$

with $P_{ijk}{}^{mn}$ defined by $K_{ij,k} = P_{ijk}{}^{mn} K_{mn}$

- Now assume an abundance[3] of Killing tensors allows K_{mn} to be eliminated

$$\begin{array}{|c|c|}\hline i & j \\ \hline\end{array} R^{m}{}_{ikl}\, g_j^n = \begin{array}{|c|}\hline k \\ \hline l \\ \hline\end{array}\left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right).$$

This gives purely algebraic equations in T.

[3] $n(n+1)/2$

What now?

- Contracting

$$K_{ij,k} = \frac{1}{3} \begin{array}{|c|c|}\hline j & i \\ \hline k & \\ \cline{1-1} \end{array} T^{m}{}_{ji} K_{mk}$$

  gives a linear system for T

$$\lambda_{,i} = -\tfrac{2}{3}(T_{ijk} - g_{ij} t_k) K^{jk}, \qquad \lambda = g_{jk} K^{jk}$$

- This can be solved when

$$D = \det(\mathsf{K}_{11}, \ldots, \mathsf{K}_{jk}, \ldots, \mathsf{K}_{nn}), \qquad 1 \leq j \leq k \leq n,$$

  is non-vanishing.
- Known examples suggest D will factor into linear factors as it did in 2 dimensions.
- Conjecture: non-degenerate 2nd order superintegrable systems can be classified by hyperplanes arrangements.