Asymptotics of Quantum Hurwitz numbers

J. Harnad

Centre de recherches mathématiques
Université de Montréal

Department of Mathematics and Statistics
Concordia University

CRM workshop
Algebraic methods in integrable systems
July 16-22, 2018

*Based in part on joint work with: J. Ortmann

1. **Classical Hurwitz numbers**
   - Group theoretical/combinatorial meaning
   - Geometric meaning: simple Hurwitz numbers
   - Simple double Hurwitz numbers (Okounkov)
2. **KP and 2D Toda τ-functions as generating functions**
   - Hypergeometric τ-function as generating function
   - Change of basis: Frobenius character formula
   - Weighted Hurwitz numbers
   - Geometric weighted Hurwitz numbers: weighted coverings
   - τ-functions as generating functions
   - Two examples: simple branching and Belyi curves
3. **Quantum Hurwitz numbers and semiclassical asymptotics**
   - Quantum dilogarithm as weight generating function
   - Stable large *n* limits
   - Classical limit of the generating function
   - Classical limit of Hurwitz numbers
   - Semiclassical asymptotics of quantum Hurwitz numbers
   - Zero temperature limit

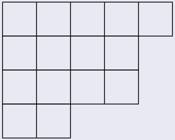
Factorization of elements in S_n

Question: What is the number $n!F(\mu^{(1)}, \dots \mu^{(k)})$ of distinct ways the identity element $\mathbf{I} \in S_n$ in the symmetric group in n elements can be written as a product

$$\mathbf{I} = h_1 h_2 \cdots h_k$$

of k elements $h_i \in S_n$ of cycle type $h_i \in \mathrm{cyc}(\mu^{(i)})$ for a given sequence of partitions $\{\mu^{(i)} \in \mathcal{P}_n\}_{i=1,\dots,k}$ of n?

Young diagram of a partition. Example $\mu = (5, 4, 4, 2)$



Representation theoretic answer (Frobenius-Schur)

The **Frobenius-Schur** formula expresses this in terms of characters:

$$F(\mu^{(1)}, \dots \mu^{(k)}, \mu) = \sum_{\lambda, |\lambda|=n} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{z_{\mu^{(i)}}}$$

where $h_\lambda = \left(\det \frac{1}{(\lambda_i - i + j)!}\right)^{-1}$ is the **product of the hook lengths** of the partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda} > 0$, where $\chi_\lambda(\mu^{(i)})$ is the **irreducible character** of representation λ evaluated in the conjugacy class $\mu^{(i)}$, and

$$z_\mu := \prod_i i^{m_i(\mu)}(m_i(\mu))! = |\mathrm{aut}(\mu)|$$

is the **order of the stabilizer** of an element of $\mathrm{cyc}(\mu)$ **($m_i(\mu) =$# parts μ_j of μ equal to i)**.

Geometric meaning: simple Hurwitz numbers

Hurwitz numbers: Let $H(\mu^{(1)}, \dots, \mu^{(k)})$ be the number of inequivalent branched n-sheeted covers of the Riemann sphere, with k branch points, and ramification profiles $(\mu^{(1)}, \dots, \mu^{(k)})$ at these points.

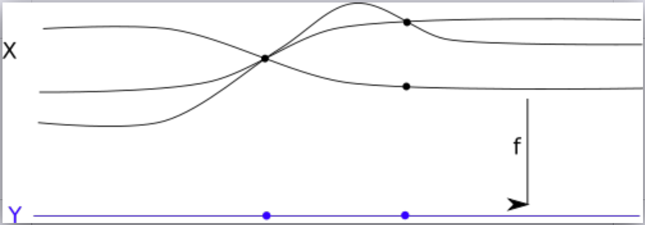
The **Euler characteristic** of the covering curve is given by the **Riemann-Hurwitz formula**:

$$2 - 2g = 2n - d, \quad d := \sum_{i=1}^{l} \ell^*(\mu^{(i)}),$$
$$g = \text{genus of covering curve},$$

where $\ell^*(\mu) := |\mu| - \ell(\mu)$ is the **colength** of the partition.
The **Monodromy Representation** shows these two enumerative invariants are identical.

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = F(\mu^{(1)}, \dots, \mu^{(k)}).$$

Example: 3-sheeted branched cover with ramification profiles (3) and $(2,1)$



Simple single Hurwitz numbers (Pandharipande)

In particular, choosing only simple ramifications $\mu^{(i)} = (2, (1)^{n-2})$ at $d = k - 1$ points and one further arbitrary one μ at a single point, say, 0, we have the **single simple Hurwitz number**:

$$H^d(\mu) := H((2, (1)^{n-1}), \ldots, (2, (1)^{n-1}), \mu).$$

By the **Frobenius-Schur formula** this is

$$H^d(\mu) = \sum_{\lambda, |\lambda|=|\mu|} \frac{\chi_\lambda(\mu)}{z_\mu h_\lambda} (\mathrm{cont}_\lambda)^d,$$

where the **content sum** of the Young diagram associated to λ is defined as

$$\mathrm{cont}(\lambda) := \sum_{(ij)\in\lambda} (j - i) = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i(\lambda_i - 2i + 1) = \frac{\chi_\lambda((2, (1)^{n-2}) h_\lambda}{z_{(2,(1)^{n-2})}}$$

Simple double Hurwitz numbers (Okounkov)

The **simple (double) Hurwitz number** (Okounkov (2000)), defined as

$$\mathrm{Cov}_d(\mu,\nu) = H^d_{\exp}(\mu,\nu)) := H((2,(1)^{n-1}),\ldots,(2,(1)^{n-1}),\mu,\nu)$$

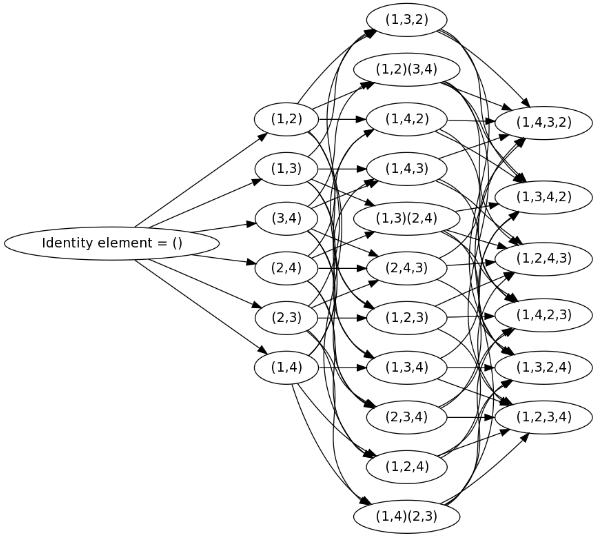
have the ramification types (μ,ν) at two points, say $(0,\infty)$, and simple ramification $\mu^{(i)} = (2,(1)^{n-2})$ at d other branch points.

Combinatorial meaning: paths in the Cayley graph

Combinatorially, this equals the number of d-step paths in the **Cayley graph** of S_n generated by **transpositions**, starting at an element $h \in \mathrm{cyc}(\mu)$ and ending in the conjugacy class $cyc(\nu)$.

Example: Cayley graph for S_4 generated by all transpositions

Transpositioncayleyons4.png 867×779 pixels 14-08-23 10:17 PM



Hypergeometric τ-function as generating function for simple single and double Hurwitz numbers: (Okounkov, Pandharipande)

Define

$$\tau^{mKP(\gamma,\beta)}(N,\mathbf{t}) := \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^{\exp}(N,\beta)) h_\lambda^{-1} s_\lambda(\mathbf{t})$$

$$\tau^{2DToda(\gamma,\beta)}(N,\mathbf{t},\mathbf{s}) := \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^{\exp}(N,\beta) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$

where $$r_\lambda^{\exp}(N,\beta) := \prod_{(ij)\in\lambda} r_{N+j-i}^{\exp}(\beta), \quad r_j^{\exp}(\beta) := e^{j\beta}$$

and

$$\mathbf{t} = (t_1, t_2, \dots), \quad \mathbf{s} = (s_1, s_2, \dots)$$

are the KP and 2D Toda flow variables.
For $N = 0$, we have

$$r_\lambda^{\exp}(0,\beta) = e^{\beta\,\mathrm{cont}(\lambda)}$$

mKP Hirota bilinear relations for $\tau_g^{mKP}(N, \mathbf{t}),\ \mathbf{t} := (t_1, t_2, \dots),\ N \in \mathbf{Z}$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t},z)} \tau_g^{mKP}(N, \mathbf{t} - [z^{-1}]) \tau_g^{mKP}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}]) = 0$$

$$\xi(\delta\mathbf{t}, z) := \sum_{i=1}^{\infty} \delta t_i\, z^i, \quad [z^{-1}]_i := \frac{1}{i}\, z^{-i}, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots)$$

$2D$ Toda Hirota bilinear relations for $\tau_g^{2Toda}(N, \mathbf{t}, \mathbf{s}),\ \mathbf{s} := (s_1, s_2, \dots)$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t},z)} \tau_g^{2Toda}(N, \mathbf{t} - [z^{-1}], \mathbf{s}) \tau_g^{2Toda}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}], \mathbf{s}) =$$

$$\oint_{z=0} z^{N-N'} e^{-\xi(\delta\mathbf{s},z)} \tau_g^{2Toda}(N+1, \mathbf{t}, \mathbf{s} - [z]) \tau_g^{2Toda}(N'-1, \mathbf{t}, \mathbf{s} + \delta\mathbf{s} + [z])$$

$$[z]_i := \frac{1}{i} z^i, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots),\ \delta\mathbf{s} := (\delta s_1, \delta s_2, \dots)$$

Change of basis: Frobenius character formula

Using the **Frobenius character formula**:

$$s_\lambda(\mathbf{t}) = \sum_{\mu,\, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu(\mathbf{t})$$

where we restrict to

$$it_i := p_i, \quad is_i := p_i'$$

and the p_μ's are the **power sum symmetric functions**

$$p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}, \quad p_i := \sum_{a=1}^{\infty} x_a^i, \quad p_i' := \sum_{a=1}^{\infty} y_a^i,$$

Generating functions for single and double simple Hurwitz numbers (Okounkov, Pandharipande)

$$\tau^{(\gamma,\beta)}(\mathbf{t}) := \tau^{KP(\gamma,\beta)}(0,\mathbf{t}) = \sum_{\lambda} \gamma^{|\lambda|} h_\lambda^{-1} e^{\beta\,\mathrm{cont}(\lambda)} s_\lambda(\mathbf{t})$$

$$= \sum_{n=0}^{\infty} \gamma^n \sum_{d=0}^{\infty} \frac{\beta^d}{d!} \sum_{\mu, |\mu|=n} H^d_{\exp}(\mu, 0) p_\mu(\mathbf{t})$$

$$\tau^{2D(\gamma,\beta)}(\mathbf{t},\mathbf{s}) := \tau^{2DToda(\gamma,\beta)}(0,\mathbf{t},\mathbf{s}) = \sum_{\lambda} \gamma^{|\lambda|} e^{\beta\,\mathrm{cont}(\lambda)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$

$$= \sum_{n=0}^{\infty} \gamma^n \sum_{d=0}^{\infty} \frac{\beta^d}{d!} \sum_{\mu,\nu, |\mu|=\nu|=n} H^d_{\exp}(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s})$$

These are therefore **generating functions** for the **simple single and double Hurwitz numbers**.

Generalizations: Generating function for weights

Choose a **weight generating function**

$$G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i$$

For Okounkov-Pandharipande's **simple single and double Hurwitz numbers**: $G(z) = e^z$.
If $G(z)$ is expressible as an **infinite (or finite) product expansion**

$$G(z) := \prod_{i=1}^{\infty}(1 + zc_i), \quad \text{or} \quad G(z) := \prod_{i=1}^{\infty}(1 - zc_i)^{-1}, \quad \mathbf{c} = (c_1, c_2, \dots),$$

the g_i's are the **elementary** or **complete symmetric functions**

$$g_i = e_i(\mathbf{c}), \quad \text{or } g_i = h_i(\mathbf{c}).$$

of the weight determining parameters $\mathbf{c} = (c_1, c_2, \dots)$.

Suppose the **generating function** $G(z)$ and its **dual** $\tilde{G}(z) := \frac{1}{G(-z)}$ can be represented as infinite (or finite) products

$$G(z) = \prod_{i=1}^{\infty}(1 + zc_i), \quad \tilde{G}(z) = \prod_{i=1}^{\infty}\frac{1}{1 - zc_i}.$$

Define the **weight for a branched covering having a pair of branch points with ramification profiles of type (μ, ν), and k additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$** to be:

$$W_G(\mu^{(1)}, \dots, \mu^{(k)}) := \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{1\leq i_1<\cdots<i_k} c_{i_\sigma(1)}^{\ell^*(\mu^{(1)})}\cdots c_{i_\sigma(k)}^{\ell^*(\mu^{(k)})},$$

$$W_{\tilde{G}}(\mu^{(1)}, \dots, \mu^{(k)})\frac{(-1)^{\ell^*(\lambda)}}{k!}\sum_{\sigma\in S_k}\sum_{1\leq i_1\leq\cdots\leq i_k} c_{i_\sigma(1)}^{\ell^*(\mu^{(1)})}, \cdots c_{i_\sigma(k)}^{\ell^*(\mu^{(k)})},$$

where the partition λ of length k has **parts $(\lambda_1, \dots, \lambda_k)$ equal to the colengths $(\ell^*(\mu^{(1)}), \dots, \ell^*(\mu^{(k)}))$**, arranged in weakly decreasing order, and $|\mathrm{aut}(\lambda)|$ is the product of the factorials of the multiplicities of the parts of λ.

Definition (Weighted geometrical Hurwitz numbers)

The **weighted geometrical Hurwitz numbers** for n-sheeted branched coverings of the Riemann sphere, having a pair of branch points with ramification profiles of type (μ, ν), and k additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$ are defined to be

$$H^d_G(\mu,\nu) := \sum_{k=0}^{\infty} \sum'_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{k} \ell^*(\mu^{(i)})=d}} W_G(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

$$H^d_{\tilde{G}}(\mu,\nu) := \sum_{k=0}^{\infty} \sum'_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{d} \ell^*(\mu^{(i)})=d}} W_{\tilde{G}}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

where $\sum'$ denotes the sum over all partitions other than the cycle type of the identity element.

Hypergeometric τ-function

Define the τ-functions of **hypergeometric type**

$$\tau^{\gamma G,\beta}(\mathbf{t},\mathbf{s}) := \sum_{\lambda} \gamma^{|\lambda|} r^G_\lambda(\beta) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$

where the coefficients $r^G_\lambda(\beta)$ are defined by the:

Content product formula

$$r^G_\lambda(\beta) := \prod_{(ij)\in\lambda} G((j-i)\beta), \quad r^G_j(\beta)$$

Again use the

Frobenius character formula

$$s_\lambda(\mathbf{t}) = \sum_{\mu,\,|\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu(\mathbf{t})$$

Theorem (Hypergeometric τ-functions as generating function for weighted branched covers)

$$\tau^{\gamma G,\beta}(\mathbf{t},\mathbf{s}) = \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^G(\beta) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$
$$= \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu,\\ |\mu|=|\nu|}} \gamma^{|\mu|} H_G^d(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}) \beta^d$$

is the generating function for the **weighted Hurwitz numbers** $H_G^d(\mu,\nu)$ *for n-fold branched coverings of the sphere, with a pair of specified branch points having ramification profiles* (μ,ν) *and genus given by the* **Riemann-Hurwitz formula**

$$2-2g = \ell(\mu) + \ell(\nu) - d.$$

Example: Okounkov's simple double Hurwitz numbers

$$G(z) = \exp(z), \quad \exp_j = \frac{1}{j!}$$
$$r_j^{\exp}(\beta) = \exp(j\beta), \quad r_\lambda^{\exp}(\beta) = \prod_{(ij)\in\lambda} \exp(\beta(j-i)),$$

Example: Belyi curves: strongly monotone paths

$$G(z) = E(z) := 1 + z,$$
$$e_1 = 1, \quad e_j = 0 \text{ for } j > 1, \quad r_\lambda^E(\beta) = \prod_{((ij)\in\lambda} (1 + \beta(j-i)),$$

Weight generating functions for *simple* Quantum Hurwitz numbers

$$G(z) = E(q,z) := \prod_{k=0}^{\infty}(1+q^k z) = \sum_{k=0}^{\infty} E_k(q) z^k = e^{-\mathrm{Li}_2(q,-z)},$$

$$\mathrm{Li}_2(q,z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)} \text{ (quantum dilogarithm)}$$

$$E_i(q) := \prod_{j=0}^{i} \frac{q^j}{1-q^j},$$

$$r_j^{E(q)}(\beta) = \prod_{k=0}^{\infty}(1+q^k \beta j), \quad r_\lambda^{E(q)}(\beta) = \prod_{k=0}^{\infty} \prod_{(ij)\in\lambda}^{n} (1+q^k \beta(j-i)).$$

Symmetrized monotone monomial sums

Using the sums:

$$\sum_{\sigma\in S_k} \sum_{0\le i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$

$$= \sum_{\sigma\in S_k} \frac{x_{\sigma(1)}^{k-1} x_{\sigma(2)}^{k-2}\cdots x_{\sigma(k-1)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$

$$\sum_{\sigma\in S_k} \sum_{1\le i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$

$$= \sum_{\sigma\in S_k} \frac{x_{\sigma(1)}^{k} x_{\sigma(2)}^{k-1}\cdots x_{\sigma(k)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$

Theorem (Quantum Hurwitz numbers (cont'd))

$$\tau^{\gamma E(q,)\beta}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} \beta^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^d_{E(q)}(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}), \quad \textit{where}$$

$$H^d_{E(q)}(\mu,\nu) := \sum_{d=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}}' W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

$$\textit{with} \quad W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{k!} \sum_{\sigma\in S_k} \sum_{0\le i_1<\cdots<i_k}^{\infty} q^{i_1 \ell^*(\mu^{(\sigma(1)})} \cdots q^{i_k \ell^*(\mu^{(\sigma(k)})}$$

$$= \frac{1}{k!} \sum_{\sigma\in S_k} \frac{q^{(k-1)\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k-1))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})}) \cdots (1-q^{\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k))})})}$$

are the **weighted (quantum) Hurwitz numbers** *that count the number of branched coverings with genus g given by the* **Riemann-Hurwitz formula***:* $\quad 2-2g = \ell(\lambda)+\ell(\mu)-k$. *and sum of colengths d.*

Bosonic gases and Planck's distribution law

A slight modification consists of replacing the generating function $E(q,z)$ by

$$E'(q,z) := \prod_{k=1}^{\infty}(1+q^k z).$$

The effect of this is simply to replace the weighting factors

$$\frac{1}{1-q^{\ell^*(\mu)}} \quad \text{by} \quad \frac{1}{q^{-\ell^*(\mu)}-1}.$$

If we identify

$$q := e^{-\beta\hbar\omega_0}, \quad \beta = 1/k_B T,$$

where ω_0 is the lowest frequency excitation in a **gas of identical bosonic particles** and assume the energy spectrum of the particles consists of integer multiples of $\hbar\omega_0$

$$\epsilon_j = j\hbar\omega_0, \quad \epsilon(\mu) := \ell^*(\mu)\hbar\omega_0.$$

Expectation values of Hurwitz numbers

The relative probability of occupying the energy level ϵ_k is

$$\frac{q^k}{1-q^k} = \frac{1}{e^{\beta\epsilon_k}-1},$$

the **energy distribution of a bosonic gas**.

We may associate the branch points to the states of the gas and view the Hurwitz numbers $H(\mu^{(1)}, \dots \mu^{(l)})$ as **random variables**, with the state energies proportional to the sums over the colengths

$$\epsilon(\mu^{(i)}) := \epsilon_{\ell^*(\mu^{(i)})} = \ell^*(\mu^{(i)})\beta\hbar\omega_0,$$

and weight

$$\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}} = \frac{1}{e^{\beta\epsilon_{\ell^*(\mu^{(i)})}}-1}$$

Probability measure for quantum Hurwitz numbers

We can interpret $H^d_{E'(q)}(\mu,\nu)$ as an expectation value. For $k \in \{1,\dots,d\}$ consider the (finite) set of k-tuples

$$\mathfrak{M}^{(n)}_{d,k} = \left\{ (\mu^{(1)},\dots,\mu^{(k)}) \ : \ \sum_{j=1}^{k} \ell^*(\mu^{(j)} = d \right\}$$

and their disjoint union $\quad \mathfrak{M}^{(n)}_{d} := \coprod_{k=1}^{d} \mathfrak{M}^{(n)}_{d,k}$

The (unnormalized weight) is

$$W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) = \frac{1}{k!} \sum_{\sigma\in S_k} \frac{1}{(e^{\beta\epsilon(\mu^{(\sigma(1))})}-1)\cdots(e^{\beta\sum_{i=1}^{k}\epsilon(\mu^{(\sigma(i))})}-1)}$$

Expectation values of Hurwitz numbers

Since $W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)})$ is always real, positive and normalizable, define a probability measure $\theta^{(n,d)}_{E'(q)}$ on $\mathfrak{M}^{(n)}_d$ by

$$\theta^{(n,d)}_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{\widetilde{Z}^{(n,d)}_{E'(q)}} W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)}),$$

where the *partition function* $\widetilde{Z}^{(n,d)}_{E'(q)}$ is

$$\widetilde{Z}^{(n,d)}_{E'(q)} = \sum_{k=1}^{d} \sum_{\mathfrak{M}^{(n)}_{d,k}} W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)}).$$

Definition ($\Lambda_d^{(n)}$, $\mathcal{P}_{n,k}$)

For $n, d \in \mathbb{Z}_{>0}$ define the function $\Lambda_d^{(n)} : \mathfrak{M}_d^{(n)} \longrightarrow \mathcal{P}_d$ as follows:

$$\Lambda_d^{(n)} : \left(\mu^{(1)}, \dots, \mu^{(k)}\right) \longmapsto \lambda$$

where λ is the unique partition of d such that

$$\{\lambda_1, \dots, \lambda_k\} = \left\{\ell^*\left(\mu^1\right), \dots, \ell^*\left(\mu^{(k)}\right)\right\}.$$

Let

$$\mathcal{P}_{n,k} := \text{set of integer partitions of } n \text{ with } k \text{ parts}$$

Definition (Push-forward $\widetilde{\xi}_{E'(q)}^{(n,d)}$ of $\theta_{E'(q)}^{(n,d)}$ as a measure on $\mathcal{P}_d$)

$$\widetilde{\xi}_{E'(q)}^{(n,d)} := \left(\Lambda_d^{(n)}\right)_* \theta_{E'(q)}^{(n,d)}$$

The normalized quantum Hurwitz numbers are expectation values

$$\bar{H}^d_{E'(q)}(\mu,\nu) := \frac{1}{\tilde{Z}^{(n,d)}_{E'(q)}} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

where $W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) = \frac{1}{k!}\sum_{\sigma\in S_k} W(\mu^{(\sigma(1)}) \cdots W(\mu^{(\sigma(1)},\cdots,\mu^{(\sigma(k)})$

$$W(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{e^{\beta\sum_{i=1}^k \epsilon(\mu^{(i)})}-1},$$

$$\tilde{Z}^{(n,d)}_{E'(q)} := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}).$$

is the **canonical partition function** for total energy $d\hbar\omega$.

Theorem (Stable large n limits)

Let $n, d \in \mathbf{N}^+$ with $n \geq 2d$. Then

1. *The partition function $\widetilde{Z}^{(n,d)}_{E'(q)}$ does not depend on n:*

$$\widetilde{Z}^{(n,d)}_{E'(q)} = Z^{(d)}_{E'(q)}$$

2. *The probability measure $\widetilde{\xi}^{(n,d)}_{E'(q)}$ does not depend on n: for any $\lambda \in \mathcal{P}_d$,*

$$\widetilde{\xi}^{(n,d)}_{E'(q)}(\lambda) = \xi^{(d)}_{E'(q)}(\lambda)$$

Classical limit of the generating function

Choosing

$$q = e^{-\epsilon}$$

with ϵ a small positive number, and taking the limit $\epsilon \to 0^+$ of the scaled **quantum dilogarithm function** $Li_2(q, \epsilon z)$ gives

$$\lim_{\epsilon \to 0^+} Li_2(q, \epsilon z) = z.$$

It follows that the generating function $E'(q, z)$ has as scaled limit the generating function for the Okounkov-Pandharipande simple (single and double) Hurwitz numbers

$$\lim_{\epsilon \to 0^+} E(q, z) = \lim_{\epsilon \to 0^+} E'(q, z) = \lim_{\epsilon \to 0^+} H(q, z) = e^z$$

We begin by stating the classical limits.

Definition (Dirac measure)

The *Dirac measure* δ_x at $x \in S$ on a measurable space (S, Σ) is defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \Sigma$.

Definition (Notation for partitions)

For $\lambda \in \mathcal{P}_d$,

$$\lambda = \left(1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots\right).$$

We also use the following notation for partitions with at most two different part lengths: $\ell \in \mathbb{Z}_{>0}$ and 1: we write

$$\ell_n^m := \left(1^{n-\ell m}, \ell^m\right)$$

Theorem

Let $d \in \mathbb{Z}_{>0}$. As $q \longrightarrow 1^-$, the sequence of measures $\left(\xi^{(d)}_{E'(q)}\right)_{q<1}$, on $\mathcal{P}_d$ converges weakly to the Dirac measure $\delta_{(1^d)}$ at $(1^d) \in \mathcal{P}_d$.

Corollary

If $d \geq 2n$ then the sequence of measures $\theta^{(n,d)}_{E'(q)}$, on $\mathfrak{M}^{(n)}_d$ converges weakly, as $q \longrightarrow 1^-$, to the Dirac measure at $(\underbrace{2^1_n, \dots, 2^1_n}_{d \text{ terms}})$

Remark

Observe that the limiting measure in the Corollary corresponds to the Okounkov / Pandharipande measure .

Definition

Let

$$q := e^{-\epsilon}, \quad \epsilon := \hbar\omega_0$$

and

$$w_0(\lambda) := \sum_{\sigma \in S_{\ell(\lambda)}} \frac{1}{\prod_{j=1}^{\ell(\lambda)} \sum_{i=1}^{j} \lambda_{\sigma(i)}}$$

$$w_1(\lambda) := \frac{1}{2} \sum_{\sigma \in S_{\ell(\lambda)}} \sum_{r=1}^{\ell(\lambda)} \frac{\sum_{i=1}^{r} \lambda_{\sigma(i)}}{\prod_{j=1}^{\ell(\lambda)} \sum_{i=1}^{j} \lambda_{\sigma(i)}}$$

Theorem (Semiclassical asymptotics of quantum weights)

For any $\lambda \in \mathcal{P}_d$ we have

$$\epsilon^{-\ell(\lambda)} w_{E'(e^{-\epsilon})}(\lambda) = w_0(\lambda) + \epsilon w_1(\lambda) + O\left(\epsilon^2\right)$$

Theorem (Semiclassical asymptotitcs of the partition function)

For $d \in \mathbb{Z}_{>0}$ and $q = e^{-\epsilon}$ we have

$$\epsilon^d Z^{(d)}_{E'(e^{-\epsilon})} = \frac{1}{d!} + \epsilon \frac{3-d}{4(d-1)!} + O\left(\epsilon^2\right),$$
$$\epsilon^d Z^{(d)}_{E(e^{-\epsilon})} = \frac{1}{d!} + \epsilon \frac{5+d}{4(d-1)!} + O\left(\epsilon^2\right),$$
$$\epsilon^d Z^{(d)}_{H(e^{-\epsilon})} = \frac{1}{d!} + \epsilon \frac{d+1}{(d-1)!} + O\left(\epsilon^2\right).$$

Theorem (Semiclassical asymptotics of the quantum Hurwitz numbers)

For any $\mu, \nu \in \mathcal{P}_n$ *and* $G \in \{E', E, H\}$ *we have*

$$
\begin{aligned}
\epsilon^d \; H^d_{E'(q)}(\mu,\nu) = & \frac{1}{d!} H(\underbrace{\mathbf{2}^1_n, \dots, \mathbf{2}^1_n}_{d \text{ times}}, \mu, \nu) \\
& + \frac{\epsilon}{(d-1)!} \Bigg[\Bigg[H(\underbrace{\mathbf{2}^1_n, \dots, \mathbf{2}^1_n}_{d-1 \text{ times}}, \mathbf{3}^1_n, \mu, \nu)) + H(\underbrace{\mathbf{2}^1_n, \dots, \mathbf{2}^1_n}_{d-1 \text{ times}}, \mathbf{2}^2_n, \mu, \nu)) \Bigg] \\
& + \frac{d-3}{4} H(\underbrace{\mathbf{2}^2, \dots, \mathbf{2}^2_n}_{d \text{ times}}, \mu, \nu) \Bigg] + O\left(\epsilon^2\right).
\end{aligned}
$$

Theorem (Zero temperture limit)

Let $d \in \mathbb{Z}_{>0}$. As $q \longrightarrow 0^+$, the sequence of measures $\left(\xi^{(d)}_{E'(q)}\right)$ on $\mathcal{P}_d$ converges weakly to the Dirac measure $\delta_{(d)}$ at $(d) \in \mathcal{P}_d$.

Corollary

If $n \geq 2d$ then the measure $\theta^{(n,d)}_{E'(q)}$ on $\mathfrak{M}^{(n)}_d$ converges weakly, as $q \longrightarrow 0^+$, to the uniform measure ν on $\mathfrak{M}^{(n)}_{d,1}$, the set of single partitions $\mu^{(1)}$ of n with colength d. That is, Belyi curves, with three branch points, having ramification profiles $(\mu^{(1)}, \mu, \nu)$

$$\nu(A) = \frac{\left|A \cap \mathfrak{M}^{(n)}_{d,1}\right|}{\left|\mathfrak{M}^{(n)}_{d,1}\right|}.$$

References

- A. Okounkov, "Toda equations for Hurwitz numbers", *Math. Res. Lett.* **7**, 447-453 (2000).
- M. Guay-Paquet and J. Harnad, "2D Toda τ-functions as combinatorial generating functions", *Lett. Math. Phys.* **105,** 827-852 (2015).
- M. Guay-Paquet and J. Harnad, "Generating functions for weighted Hurwitz numbers", *J. Math. Phys.* **58**, 083503 (2017).
- J. Harnad and A. Yu. Orlov, "Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths", *Commun. Math. Phys.* **338**, 267-284 (2015).
- J. Harnad, 'Weighted Hurwitz numbers and hypergeometric τ-functions: an overview", *AMS Proceedings of Symposia in Pure Mathematics* **93**, 289-333 (2016).
- J. Harnad and Janosch Ortmann, "Asymptotics of quantum weighted Hurwitz numbers", *J. Phys. A: Math. Theor.* **51**, 225201 (2018).