The classification of RCFTs of WZW type:

What follows A-D-E?

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Conformal field theory (CFT)

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- Full CFT contains subtheory of holomorphic quantum fields (a chiral CFT).
- Full CFT contains subtheory of anti-holomorphic quantum fields (also a chiral CFT).
- Full CFT can be recovered by splicing together those two chiral halves.

Chiral conformal field theory

Mathematically, a chiral CFT is a vertex operator algebra (VOA) (Wightman axioms).

(There are other, presumably equivalent, approaches, e.g. conformal nets of factors (Haag–Kastler axioms)).

All-important are the representations of chiral CFT.

Sadly, the language of representation theory is categories.
So categories enter into any serious treatment.
Fortunately, my talk won't be very serious...

Rational conformal field theory

Simplest case: the VOA has a semi-simple representation theory. These CFT or VOAs are called rational.
The corresponding categories of representations are called modular tensor categories (MTC).

This representation theory is like that of finite groups, except MUCH MUCH BETTER:

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- but in addition give knot invariants in 3-manifolds; finite-dimensional representations of all mapping class groups (e.g. $\mathrm{SL}_2(\mathbb{Z})$); each gives zillions of modular (automorphic) forms; Verlinde's formula; ...

Examples

It's conjectured that every MTC comes from a rational VOA.
In any case, there are buckets and buckets of rational VOAs: e.g.

- ones associated to even positive-definite lattices L (bosons compactified on $\mathbb{R}^n/L$)

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- the Virasoro minimal models (Ising, Yang-Lee,...)
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- the monstrous moonshine VOA $V^\natural$
- ...but almost every rational VOA doesn't have anything to do with Lie algebras or finite groups: VOAs are quantum field theories, so belong to quantum (i.e. 21st century) mathematics, not classical math

The Lie algebra VOAs

Nevertheless, this talk we'll focus on RCFT coming from affine Kac-Moody algebras.
Choose a simple Lie algebra $\mathfrak{g}$, and any integer $\kappa \in \mathbb{Z}_{>0}$
so $\mathfrak{g} = A_r$ $(r \geq 1)$, B_r $(r \geq 3)$, C_r $(r \geq 2)$, D_r $(r \geq 4)$, E_6, E_7, E_8, F_4, G_2
Let $\mathcal{V}(\mathfrak{g}, \kappa)$ be the corresponding rational VOA.
It has an irrep for each level κ highest weight λ. Approx $\kappa^{rank}/rank!$ of these λ.

Get character $\chi_\lambda(\tau)$.
These $\chi_\lambda(\tau)$ form the components of a vector-valued modular function:

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda,\mu}\,\chi_\mu(\tau)\,,\ \chi_\lambda(\tau+1) = T_{\lambda,\lambda}\,\chi_\lambda(\tau)$$

The baby example: A_1

Take Lie algebra $\mathfrak{g} = A_1$ and level $\kappa \geq 3$.
Then highest weights are $\lambda = 1, 2, \ldots, \kappa - 1$
The $(\kappa - 1)$-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ is:

$$S_{\lambda,\mu} = \sqrt{\frac{2}{\kappa}} \sin(\pi \frac{\lambda\mu}{\kappa})$$

$$T_{\lambda\lambda} = e^{\pi i \lambda^2/(2\kappa) - \pi i/4}$$

S is numerator of Weyl character formula: alternating sum over Weyl group $Sym(2)$.
T is 2nd Casimir (exponentiated).

Next simplest example: A_2

By comparison, take $\mathfrak{g} = A_2$ and level $\kappa \geq 4$; then highest weights are $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_i \in \mathbb{Z}_{>0}$ and $\lambda_1 + \lambda_2 < \kappa$,

$$S_{\lambda,\mu} = \frac{-i}{\sqrt{3}\kappa} \times$$

$$\big\{e_\kappa(2\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+2\lambda_2\mu_2)+e_\kappa(-\lambda_1\mu_1-2\lambda_1\mu_2+\lambda_2\mu_1-\lambda_2\mu_2)$$

$$+e_\kappa(-\lambda_1\mu_1+\lambda_1\mu_2-2\lambda_2\mu_1-\lambda_2\mu_2)-e_\kappa(-\lambda_1\mu_1-2\lambda_1\mu_2-2\lambda_2\mu_1-\lambda_2\mu_2)$$

$$+e_\kappa(2\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1-\lambda_2\mu_2)-e_\kappa(-\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+2\lambda_2\mu_2)\big\}$$

$$T_{\lambda\lambda} = e_\kappa(-\lambda_1^2 - \lambda_1\lambda_2 - \lambda_2^2 + \kappa)$$

for $e_\kappa(x) = \exp(\frac{-2\pi i x}{3\kappa})$. (Weyl group is $Sym(3)$)

Recovering full CFT from chiral halves

Understanding how the full theory is recovered from the chiral halves is theme of much of Fuchs–Runkel–Schweigert and collaborators.
But they need an input (*module category*). My talk is about the possible inputs, when the chiral algebras are affine Kac-Moody.
Ingredients of the passage from chiral to full CFT:

- The space of states $\mathcal{H}$ of full CFT carries an action of $\mathcal{V}_L \otimes \mathcal{V}_R$:

$$\mathcal{H} = \oplus_{\lambda,\overline{\mu}} \lambda \otimes \overline{\mu}$$

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$$\mathcal{H} = \oplus_{\lambda,\overline{\mu}} \lambda \otimes \overline{\mu}$$
- The boundary states carry a representation of fusion ring of MTC:
$$\lambda.x = \sum_y \mathcal{N}_{\lambda x}^{y} y$$

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Module categories

Mathematically, the parametrization of possible full CFTs from a given VOA is by the so-called module categories of the given modular tensor category.
So the chiral theory (MTC) is like a commutative ring, and the full CFT (module category) is like a module for that ring. A module category is NOT the category of modules of the VOA; it's a module for the category of modules of the VOA.
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- an extension $\mathcal{V}_L^e$ of the left chiral algebra (VOA) $\mathcal{V}_L$;
- an extension $\mathcal{V}_R^e$ of the right chiral algebra (VOA) $\mathcal{V}_R$;
- an equivalence $MTC(\mathcal{V}_L^e) \leftrightarrow MTC(\mathcal{V}_R^e)$

Modular invariant partition functions

Most useful for constraining module categories for affine algebras is the action of $\mathcal{V}(\mathfrak{g},\kappa)\otimes\mathcal{V}(\mathfrak{g},\kappa)$ on $\mathcal{H}$.
The modular invariant partition function (graded trace of $\mathcal{H}$) is:

$$\mathcal{Z}(\tau)=\sum_{\lambda,\mu}\mathcal{Z}_{\lambda,\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^*$$

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- $\mathcal{Z}_{\rho,\rho}=1$ where ρ is Weyl vector ($\rho=1$ resp $(1,1)$ for A_1, A_2)
- $\mathcal{Z}S=S\mathcal{Z}$, $\mathcal{Z}T=T\mathcal{Z}$

First era: Ancient times

The first attack on classifying modular invariants for affine algebras was in 1980s.

Cappelli-Itzykson–Zuber (*CMP*, 1987) classified these for $\mathfrak{g} = A_1$. This implied the classification for the Virasoro minimal models.

This is one of most celebrated results in CFT.

Nevertheless, it was the era of stone knives and menacing glares

The answer for $\mathfrak{g} = A_1$

$$\forall \kappa : \sum |\chi_\lambda|^2 \qquad \forall\ \kappa : \ \text{(string on } SU(2))$$

$$4|\kappa : |\chi_1|^2 + \chi_2\, \chi^*_{\kappa-2} + |\chi_3|^2 + \cdots + |\chi_{\kappa-1}|^2 : \ \mathrm{SO}(3)$$

$$\kappa = 6, 10, \ldots : \ |\chi_1 + \chi_{\kappa-1}|^2 + |\chi_3 + \chi_{\kappa-3}|^2 + \cdots + 2|\chi_{\frac{\kappa-1}{2}}|^2 : \ SO(3)$$

$$\kappa = 12 : |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 : \quad Sp(4)$$

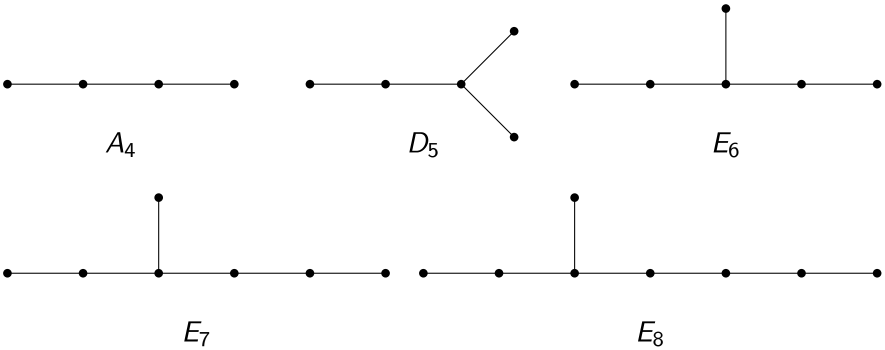
$$\kappa = 18 : |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + \chi_9\, (\chi_3 + \chi_{15})^* + cc$$

$$\kappa = 30 : |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 : \ \ G_2$$

Kirillov-Ostrik (2002) got the classification of module categories

A-D-E!

The sexiest aspect of their work: these fall into an A-D-E pattern



(just as finite subgroups of SU(2) do, as do simple singularities, ...)
Their A-D-E is easiest to see using the fusion ring representation

Second era: Middle ages

So what comes after A-D-E???
What happens with e.g. $\mathfrak{g} = A_2$??

The proof of CIZ for $\mathfrak{g} = A_1$ was long and technical, and crashes when applied to higher rank examples.

New ideas were needed. These came in the 1990s, with the discovery of gunpowder!

$\mathcal{Z}T = T\mathcal{Z}$ is easy to understand: you get the selection rule

$$\mathcal{Z}_{\lambda\mu} \neq 0 \Rightarrow T_{\lambda\lambda} = T_{\mu\mu}$$

The problem is to extract something useful from $\mathcal{Z}S = S\mathcal{Z}$...

Galois!

The entries $S_{\lambda\mu}$ lie in a cyclotomic field $\mathbb{Q}[\xi_N]$, where $\xi_N = \exp(2\pi i/N)$.
The automorphisms $\mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$ form a group $\mathbb{Z}_N^*$:
any integer ℓ coprime to N defines an automorphism σ_ℓ sending $poly(\xi_N)$ to $poly(\xi_N^\ell)$.
Geometry of Weyl groups implies (Gannon, 1993)

$$S_{\lambda\mu} = \epsilon_\ell(\lambda) S_{\lambda^\ell,\mu} = \epsilon_\ell(\mu) S_{\lambda,\mu^\ell}$$

for some signs $\epsilon_\ell(\lambda)$ and some permutation $\lambda \mapsto \lambda^\ell$ of level κ highest weights
In fact the analogous result holds for any MTC (Coste-Gannon, 1994).

Galois selection rule

Who cares????
Hit $\mathcal{Z} = S\mathcal{Z}S^\dagger$ with the ℓth automorphism: get

$$\mathcal{Z}_{\lambda\mu} = \sigma_\ell(\mathcal{Z}_{\lambda\mu}) = \sum_{\nu,\psi} \sigma_\ell(S_{\lambda\nu})\,\sigma_\ell(\mathcal{Z}_{\nu\psi})\,\sigma_\ell(S_{\psi\mu})^*$$

$$= \sum_{\nu,\psi} \epsilon_\ell(\lambda)\,S_{\lambda^\ell,\nu}\,\mathcal{Z}_{\nu\psi}\,\epsilon_\ell(\mu)\,S^*_{\psi\mu^\ell} = \epsilon_\ell(\lambda)\,\epsilon_\ell(\mu)\,\mathcal{Z}_{\lambda^\ell,\mu^\ell}$$

Hence we get the Galois selection rule:

$$\mathcal{Z}_{\lambda\mu} \neq 0 \Rightarrow \epsilon_\ell(\lambda) = \epsilon_\ell(\mu) \quad \forall \ell \in \mathbb{Z}_N^*$$

This is especially constraining for the affine algebras

Is it any good??? The baby case $\mathfrak{g} = A_1$

It yields a 1-page proof of modular invariants for $\mathfrak{g} = A_1$: the key Lemma is:
When $\kappa \neq 12, 30$,

$$\epsilon_\ell(1) = \epsilon_\ell(a) \;\; \forall \ell \in \mathbb{Z}_{2\kappa}^* \;\Rightarrow\; a \in \{1, \kappa - 1\}$$

Proof: $\epsilon_\ell(1)\epsilon_\ell(a) = sign(\sin(\pi\ell/\kappa)\sin(\pi\ell a/\kappa))$ equals $+1$ iff $\cos(\pi\ell(a-1)/\kappa) > \cos(\pi\ell(a+1)/\kappa)$. etc etc

This means that (except for $\kappa = 12, 30$) the only possible VOA extensions of $\mathcal{V}(A_1, \kappa)$ are simple-current extensions=completely obvious, completely understood.

Modular invariants for $\mathfrak{g} = A_2$

Using Galois, the A_2 modular invariant classification is difficult but manageable: (Gannon, *CMP* 1994):

- Get 4 infinite series for each κ, built from extended Dynkin diagram symmetries (simple-currents, contragredient) (these are completely obvious, completely understood);

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- get exceptional extensions of $\mathcal{V}(A_2, \kappa)$ at $\kappa = 8, 12, 24$ (analogous to the $\mathcal{E}_6, \mathcal{E}_8$ of $\mathfrak{g} = A_1$);
- get exceptional MTC equivalence at $\kappa = 12$ (analogous to $\mathcal{E}_7$ of $\mathfrak{g} = A_1$).

Interesting answer???

No A-D-E.

Some effort (not superconvincing) to relate it to finite subgroups of SU(3) (Hanany-He, *JHEP* 1999).

Most spectacular: connections to Jacobians of Fermat curves $x^\kappa + y^\kappa = z^\kappa$ (Bauer-Coste-Itzykson-Ruelle, *J Geom Phys* 1997)

This connection is still mysterious, 20 years later!

What next???

That's about the state of the art!

Galois selection rule continues to be effective at higher rank, but hard to prove theorems about all κ.
But can get modular invariant classifications for any $\mathfrak{g}$ at 'small' level (e.g. $\mathfrak{g} = E_8$ at level $\kappa \leq 415$)

The possible isomorphisms between relevant MTC has been done (Gannon, *Invent* 1995; Gannon-Ruelle-Walton, *CMP* 1996).
Answer: almost come from Dynkin diagram symmetries like simple-currents

So what remains is understanding the possible extensions of the VOAs $\mathcal{V}(\mathfrak{g}, \kappa)$

New ideas were needed. I went on to other things.

Third era: Modern times

We need to understand better, i.e. constrain better, the possible VOA extensions.
Think finite group extensions: the representation theory involves the interplay of restrictions and inductions of representations. But up to this point, the work on modular invariants used restrictions but not induction.
Induction here is more complicated than for finite groups, but it was understood in 1990s in the work of subfactors.
In early 2000s, Ocneanu announced (without proof, without details) that induction applied to $\mathfrak{g} = A_1$ implies there are no possible extensions of $\mathcal{V}(A_1, \kappa)$ for level $\kappa > 30$ (except the well-understood simple-current extensions).
Much more important, he suggested (without giving any details) that his argument can be extended to higher rank.

Understanding how induction is involved turns out to be the missing idea!!! It leads to the Modern era: the era of rocketry!!

Schopieray's breakthrough

Ocneanu's work has remained hidden. It seems clear he never worked out the bounds for higher rank $\mathfrak{g}$.
The breakthrough came last year: PhD work of Andrew Schopieray, a student of Victor Ostrik (currently a post-doc in Sydney).
Schopieray made the argument (Ocneanu's argument, or something pretty close) explicit, for the rank 2 algebras $\mathfrak{g}$, and worked out the explicit bounds K.

Theorem (Schopieray, 2017) The only possible extensions of $\mathcal{V}(C_2, \kappa)$ are simple-current ones, if $\kappa \geq 19\,896$. The only possible extensions of $\mathcal{V}(G_2, \kappa)$ are simple-current, for $\kappa \geq 18\,271\,135$.

The argument extends to arbitrary $\mathfrak{g}$, but it will be harder and harder to work out the explicit bound. The bound grows like $10^{number\ of\ positive\ roots}$. So for $\mathfrak{g} = E_8$, it will be bigger than a google. Even for say G_2, it is too big to be helpful.

Current status

But Schopieray emphasizes that a key idea is induction.
Recently, I found how to combine induction to Galois to get bounds that are *much* smaller.
For $\mathfrak{g} = C_2$, I find that there are precisely 61 levels κ (not 20000) that could possibly have nonsimple-current extensions. For G_2, my number is 129 (not 18 million). For A_8 my number is 881; for E_8 it is 9163 (compared to a google).
My bound grows like rank-cubed, not 10^{rank^2}
Recall: Galois is effective at handling "small" levels. So modular invariant classifications for small rank algebras $\mathfrak{g}$ are now immanent!

C_2 modular invariant classification

Using my new bound, I completed the modular invariant classification for C_2 and G_2:

The answer for C_2:

- identity matrix $\forall \kappa$;

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- exceptional isomorphism for $\kappa = 11$.

G_2 modular invariant classification

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