Quasi-Hopf algebras for extended W-algebras in Logarithmic CFTs

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Introduction: 2d CFTs and VOAs

VOAs

A good algebraic concept of "quantum fields" (i.e. vertex operators)
Examples: Virasoro algebras, affine Lie algebras, W-algebras, ...

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This data is encoded in braided tensor category + modular.

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Different spicies

- Rational CFTs: with **finite semisimple** braided categories $\mathbf{Rep}\,\mathrm{VOA}$
- Non-rational CFTs: **non**-finite and **non**-semisimple braided categories

Introduction: duality

Logarithmic Conformal Field Theory in 2D

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Vertex Operator Algebras with log's

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↓ representation theory [HLZ'10-11]

Non-semisimple braided tensor categories

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Non-semisimple braided tensor categories

↓ Tannaka-Krein duality (if $\exists$ fiber functor)

quasi-triangular Hopf algebras

Introduction: duality

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↓ representation theory [HLZ'10-11]

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↓ usually NO fiber functor

???

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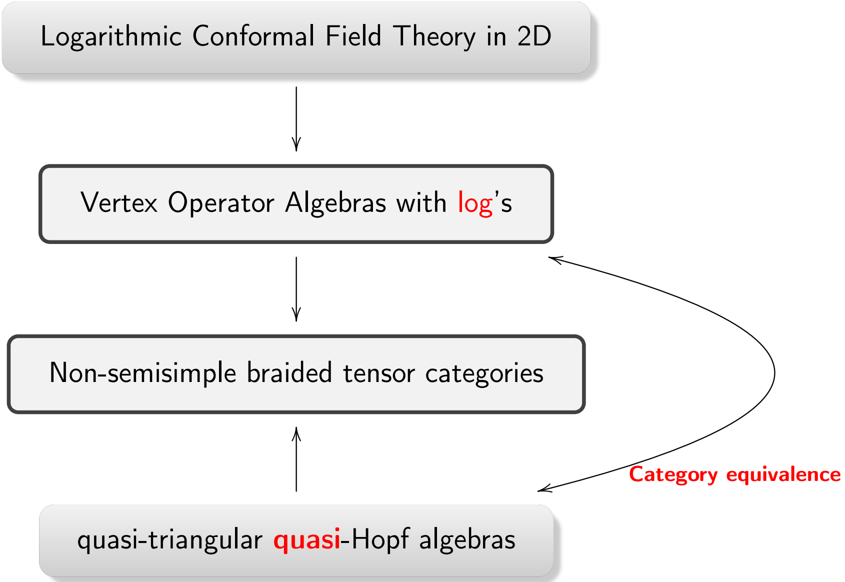
Non-semisimple braided tensor categories

↓ if ∃ quasi-fiber functor

quasi-triangular **quasi**-Hopf algebras

Introduction: duality



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$\mathcal{O}_p$ is a braided tensor category for $p \in \mathbb{C} - \mathbb{Q}_{\geq 0}$.

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Let $\mathcal{C}_p$ be the representation category of the Lusztig's specialisation of $U_q^{\mathbb{Q}}\,\mathfrak{g}$ at $\mathfrak{q} = e^{-\mathrm{i}\pi/p}$ (with "divided powers"). *It is a braided tensor category.*

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Theorem (Kazhdan–Lusztig 1994)

*For $p \in \mathbb{C} - \mathbb{Q}_{\geq 0}$, the categories $\mathcal{O}_p$ and $\mathcal{C}_p$ are **equivalent** as braided tensor categories, **including roots of unity cases** $p \in \mathbb{Q}_{<0}$*

Lusztig's specialisation $U^L_{\mathfrak{q}}\,\mathfrak{g}$ at roots of unity

Drinfeld–Jimbo's quantum algebra $U^{\mathbb{Q}}_q\,\mathfrak{g}$ over $\mathbb{Q}(q)$ (for $\mathfrak{g} = s\ell(n, \mathbb{C})$) is generated by $K_i^{\pm 1}$ and E_i, F_i with $1 \leq i \leq n-1$

$$K_i K_j = K_j K_i \quad - \quad \textit{Cartan subalgebra: morally } K_i \ = \ q^{H_i}$$

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$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$$

$$[E_i, F_j] = \delta_{i,j} \cdot \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

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and two sets of *Serre-relations* for any $i \neq j$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q X_i^{1-a_{ij}-r} X_j X_i^r = 0, \qquad X = E, F.$$

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and the Hopf-algebra structure

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

$$S(E_i) = -E_i K_i^{-1} \qquad S(F_i) = -K_i F_i$$

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We need a basis over $\mathbb{Z}[q, q^{-1}]$

We can not specialise q in $U^{\mathbb{Q}}_q\,\mathfrak{g}$ to a root of unity because the coefficients are in $\mathbb{Q}(q)$ and wrong things might happen like $\frac{0}{0}$

Lusztig's specialisation $U_{\mathfrak{q}}^L \mathfrak{g}$ at roots of unity

Integral form is a $\mathbb{Z}[q, q^{-1}]$-subalgebra in $U_q^{\mathbb{Q}} \mathfrak{g}$ which is after extension of scalars $\otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ isomorphic to $U_q^{\mathbb{Q}} \mathfrak{g}$ as $\mathbb{Q}(q)$-algebras.

Lusztig's integral form $U_q^{\mathbb{Z}} \mathfrak{g}$

$U_q^{\mathbb{Z}} \mathfrak{g}$ is a $\mathbb{Z}[q, q^{-1}]$-subalgebra in $U_q^{\mathbb{Q}} \mathfrak{g}$ generated by K_i and

$$E_i^{(r)} = \frac{E_i^r}{[r]!}, \qquad F_i^{(r)} = \frac{F_i^r}{[r]!}, \qquad r > 0$$

- *It is a Hopf algebra over* $\mathbb{Z}[q, q^{-1}]$

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Lusztig's specialisation $U_{\mathfrak{q}}^{L}\,\mathfrak{g}$

$$U_{\mathfrak{q}}^{L}\,\mathfrak{g} := U_q^{\mathbb{Z}}\,\mathfrak{g} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_{\mathfrak{q}}$$

where $\mathbb{C}_{\mathfrak{q}} = \mathbb{C}$ with the $\mathbb{Z}[q,q^{-1}]$-module structure defined by evaluating $q \mapsto \mathfrak{q} \in \mathbb{C}^{\times}$ at ℓ-th root of unity.

Small quantum group $\overline{U}_{\mathfrak{q}}\,\mathfrak{g}$

For $\mathfrak{q} \in \mathbb{C}^{\times}$ a primitive $2p$-th root of unity, we have relations in $U^L_{\mathfrak{q}}\,\mathfrak{g}$

$$E_i^p = F_i^p = 0, \qquad K_i^{2p} = \mathbf{1}$$

Definition

The *small quantum group* $\overline{U}_{\mathfrak{q}}\,\mathfrak{g}$ is a fin-dim Hopf sub-algebra in $U^L_{\mathfrak{q}}\,\mathfrak{g}$ generated by K_i and E_i, F_i

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"Factorisation" of $U_{\mathfrak{q}}^{L}\,\mathfrak{g}$

The *small quantum group* is the kernel of the Frobenius map in the exact sequence of Hopf algebras:

$$\overline{U}_{\mathfrak{q}}\,\mathfrak{g} \hookrightarrow U_{\mathfrak{q}}^{L}\,\mathfrak{g} \xrightarrow{\text{Frob}} U\,\mathfrak{g}$$

$$\text{Frob}: \qquad E_i^{(p)} \mapsto e_i, \quad F_i^{(p)} \mapsto f_i$$

Small quantum group $\overline{U}_{\mathfrak{q}} s\ell(2)$

The **small** $\overline{U}_{\mathfrak{q}} s\ell(2)$, with $\mathfrak{q} = e^{i\pi/p}$ and integer $p \geq 2$

$$\mathsf{K}\mathsf{E}\mathsf{K}^{-1} = \mathfrak{q}^2\mathsf{E} \,, \qquad \mathsf{K}\mathsf{F}\mathsf{K}^{-1} = \mathfrak{q}^{-2}\mathsf{F} \,, \qquad [\mathsf{E},\mathsf{F}] = \frac{\mathsf{K} - \mathsf{K}^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}$$

and the additional relations

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The comultiplication, counit and antipode are given by

$$\Delta(\mathsf{E}) = \mathbf{1} \otimes \mathsf{E} + \mathsf{E} \otimes \mathsf{K} \,, \qquad \Delta(\mathsf{F}) = \mathsf{K}^{-1} \otimes \mathsf{F} + \mathsf{F} \otimes \mathbf{1} \,, \qquad \Delta(\mathsf{K}) = \mathsf{K} \otimes \mathsf{K} \,.$$

$$\epsilon(\mathsf{E}) = 0 \,, \qquad \epsilon(\mathsf{F}) = 0 \,, \qquad \epsilon(\mathsf{K}) = 1 \,,$$

$$S(\mathsf{E}) = -\mathsf{E}\mathsf{K}^{-1} \,, \qquad S(\mathsf{F}) = -\mathsf{K}\mathsf{F} \,, \qquad S(\mathsf{K}) = \mathsf{K}^{-1} \,.$$

This defines a Hopf $\mathbb{C}$-algebra of dimension $\dim \overline{U}_{\mathfrak{q}} s\ell(2) = 2p^3$

KL equivalence again

Theorem (Kazhdan–Lusztig 1994)

*For $p \in \mathbb{Q}_{<0}$, the categories $\mathcal{O}_p = \mathbf{Rep}\,\widehat{s\ell(2)}_{p-2}$ and $\mathcal{C}_p = \mathbf{Rep}\,U^L_{\mathfrak{q}}\,s\ell(2)$ at $\mathfrak{q} = e^{-\mathrm{i}\pi/p}$ are **equivalent** as braided tensor categories*

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How about $\overline{U}_{\mathfrak{q}} s\ell(2) \subset U^L_{\mathfrak{q}}\, s\ell(2)$?

What is the KL-like equivalence for the **small** quantum group $\overline{U}_{\mathfrak{q}} s\ell(2)$?

VOA for $\overline{U}_{\mathfrak{q}} s\ell(2)$: triplet $\mathcal{W}_p$

One of the best investigated Logarithmic CFTs are provided by ***triplet W-algebras*** $\mathcal{W}_p$ with $c = 1 - \frac{6(p-1)^2}{p}$ for integer $p \geq 2$
– introduced by [Kausch'91]

Definition [FHST'03]

Consider rank-one lattice $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2p$ and corresponding lattice VOA $V_L = \oplus_{n\in\mathbb{Z}} F_{\alpha n}$. Then, the triplet algebra is defined as

$$\mathcal{W}_p = \mathrm{Ker}_{V_L} \mathcal{S}$$

and $\mathcal{S}$ is the "short" screening operator acting on V_L and $[\mathcal{S}, \mathrm{Vir}_c] = 0$

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In terms of generators, $\mathcal{W}_p$ comes with 4 distinguished vectors:

- one generator is the usual Virasoro vector (of weight 2) or generating function $T(z) = \sum_n L_n z^{-n-2}$

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- and the remaining 3 are certain Virasoro singular vectors of conformal weight $2p-1$ or $W^{\alpha}(z) = \sum_n W^{\alpha}_n z^{-n-(2p-1)}$

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Properties [FHST'03, CF'05, AM'07]

1. The triplet vertex algebra $\mathcal{W}_p$ is C_2-cofinite
2. It has precisely $2p$ inequivalent irreducible modules

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3. Its category of modules is non-semisimple and contains ***logarithmic modules*** – with non-diagonalizable action of $L_0 \in \mathrm{Vir}_c$

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- $\overline{U}_{\mathfrak{q}} s\ell(2)$ has precisely $2p$ inequivalent irreducible modules, as $\mathcal{W}_p$ has
- same structure of indecomposable blocks as $\mathcal{W}_p$ has

Small quantum group $\overline{U}_{\mathfrak{q}}s\ell(2)$

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Theorem (FGST'05)

$SL(2,\mathbb{Z})$-action via modular transformations on the space of the $\mathcal{W}_p$ characters $\chi_i(\tau) = Tr_{\mathrm{Irr_i}} e^{2\pi i\tau(L_0 - c/24)}$ is equivalent to "Ljubashenko–Majid" $SL(2,\mathbb{Z})$-action on the centre of $\overline{U}_{\mathfrak{q}} s\ell(2)$ at $\mathfrak{q} = e^{i\pi/p}$

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Conjecture [FGST 2005]

The categories $\mathbf{Rep}\, \overline{U}_{\mathfrak{q}} s\ell(2)$ and $\mathbf{Rep}\, \mathcal{W}_p$ are equivalent as ribbon categories (or braided tensor categories with ribbon twist)

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NOT CORRECT

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There is nevertheless a close relation between $\mathbf{Rep}\,\overline{U}_{\mathfrak{q}} s\ell(2)$ and $\mathbf{Rep}\,\mathcal{W}_p$

Theorem (FGST 2005, Nagatomo-Tsuchiya 2009)

For $p \geq 2$, $\mathbf{Rep}\,\overline{U}_{\mathfrak{q}} s\ell(2)$ is equivalent to $\mathbf{Rep}\,\mathcal{W}_p$ as a $\mathbb{C}$-linear category.

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For $p \geq 3$, $\mathbf{Rep}\,\overline{U}_{\mathfrak{q}}s\ell(2)$ is not braidable since there are modules s.t.

$$U \otimes V \not\cong V \otimes U$$

For $p = 2$, less evident but $\overline{U}_{\mathfrak{i}}s\ell(2)$ has no universal R-matrix as well.

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It is the other version of the small quantum group with the relation $\mathsf{K}^p = \mathbf{1}$ (not $\mathsf{K}^{2p} = \mathbf{1}$) that is quasi-triangular and has the well-known universal R-matrix $R^{(\mathrm{st.})} \in U_{\geq 0} \otimes U_{\leq 0}$, and thus the braiding.

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Thus $\mathbf{Rep}\,\overline{U}_{\mathfrak{q}}s\ell(2)$ cannot be tensor equivalent to $\mathbf{Rep}\,\mathcal{W}_p$, as the latter is braided [HLZ] and the former not braidable!

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Task

to find a *suitable* modification of $\overline{U}_{\mathfrak{q}}s\ell(2)$ to make the representation categories equivalent as braided (ribbon) tensor categories.

Modification of $\overline{U}_{\mathfrak{q}}s\ell(2)$ – motivation

Let $\mathcal{C}$ be a finite tensor $\mathbb{C}$-linear category, e.g. with finitely many (iso classes of) simple objects X_i, $i \in I$, finite lengths, $\otimes$ is exact, etc.

- $Gr(\mathcal{C})$ – Grothendieck ring of $\mathcal{C}$ (encodes fusion rules of simple objects)

Definition: FPdim(X)

Frobenius-Perron dimension of $X \in \mathcal{C}$ is the maximal eigenvalue of the matrix of left multiplication by X in $Gr(\mathcal{C})$. $FPdim(X) \in \mathbb{R}_{>0}$

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Idea

Calculating *FPdim* of irreps of a VOA V (in many cases in LCFT they are indeed integer!) we can recover a hidden (quasi-)quantum group symmetry!

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Fusion rules for $\mathbf{Rep}\,\mathcal{W}_p$ are known [FHST, GR, TW]

$$\mathrm{X}^{\alpha}(s)\,\mathrm{X}^{+}(r) = \sum_{\substack{t=|s-r|+1\\ t\neq p,\ \mathrm{step}=2}}^{p-1-|p-s-r|} \mathrm{X}^{\alpha}(t) + \delta_{p,s,r}\mathrm{X}^{\alpha}(p) + \sum_{\substack{t=2p-s-r+1\\ \mathrm{step}=2}}^{p-1} 2\mathrm{X}^{\alpha}(t) + 2\mathrm{X}^{-\alpha}(p-t)$$

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It is thus clear from the start that $\mathbf{Rep}\,\mathcal{W}_p$ has to be equivalent to some quasi-Hopf algebra with underlying algebra $\overline{U}_{\mathfrak{q}} s\ell(2)$!

quasi-Hopf algebras: definitions

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$$(\Delta \otimes \mathrm{id})(\Delta(a)) = \Phi \big((\mathrm{id} \otimes \Delta)(\Delta(a))\big) \Phi^{-1} \qquad \text{for all } a \in A$$

– "intertwiner" condition

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- antipode structure $(S, \boldsymbol{a}, \boldsymbol{b})$

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• The associativity isomorphisms $\alpha^{\mathbf{Rep}\,A}_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ for the tensor product of $\mathbf{Rep}\,A$ are related to the coassociator Φ of A by

$$\alpha^{\mathbf{Rep}\,A}_{U,V,W}(u \otimes v \otimes w) = \Phi.(u \otimes v \otimes w)$$

for any elements u, v, w in A-modules U, V, and W, respectively.
• The 3-cocycle condition gives the pentagon diagram.

Definition

A *quasi-triangular* quasi-Hopf is a quasi-Hopf A with an invertible element $R \in A \otimes A$ called *the universal R-matrix* s.t.

$$R\Delta(a) = \Delta^{\mathrm{op}}(a)R \qquad \text{for all } a \in A$$

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$$\Delta \otimes \mathrm{id}(R) = \Phi_{231}^{-1} R_{13} \Phi_{132} R_{23} \Phi^{-1}$$
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$\mathbf{Rep}\, A$ is then a *braided* monoidal category with the braiding $\sigma_{U,V}$

$$\sigma_{U,V}(u \otimes v) = \tau_{U,V}(R.(u \otimes v))$$

where τ is the symmetric braiding in vector spaces, $\tau_{U,V}(u \otimes v) = v \otimes u$

The quasi-Hopf structure on $\overline{U}_{\mathfrak{q}} s\ell(2)$, $\mathfrak{q} = e^{i\pi/p}$

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New coproduct

$$\Delta(E) = E \otimes K + (\boldsymbol{e}_0 + \mathfrak{q}\,\boldsymbol{e}_1)\mathbf{1} \otimes E$$
$$\Delta(F) = F \otimes 1 + (\boldsymbol{e}_0 + \mathfrak{q}^{-1}\boldsymbol{e}_1)K^{-1} \otimes F$$

The coproduct is **non-coassociative**, with the coassociator

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Antipode is also modified

$$S(E) = -EK^{-1}(\boldsymbol{e}_0 + q\, \boldsymbol{e}_1) \qquad S(F) = -KF(\boldsymbol{e}_0 + q^{-1}\boldsymbol{e}_1)$$

and evaluation $\boldsymbol{a} = \mathbf{1}$ and coevaluation elements $\boldsymbol{b} = \boldsymbol{e}_0 + K^{-1}\boldsymbol{e}_1$

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Let's denote such a **quasi**-Hopf algebra by $\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2)$

$\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2)$ is quasi-triangular!

universal R-matrix

$$R = \frac{1}{4p} \sum_{n=0}^{p-1} \sum_{s,r=0}^{2p-1} \frac{(q - q^{-1})^n}{[n]!} q^{\frac{n(n-1)}{2} - 2sr} \left(1 + q^{tr} + q^{-t(n+s)} + q^{\frac{1}{2}t^2 + tr - t(n+s)}\right)$$

$$\times K^s E^n \otimes K^r F^n$$

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$R \cdot \boldsymbol{e}_0 \otimes \boldsymbol{e}_0$ equals the standard R-matrix $R^{(\mathrm{st.})}$ for quotient with $K^p = \mathbf{1}$

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$$R = \frac{1}{4p} \sum_{n=0}^{p-1} \sum_{s,r=0}^{2p-1} \frac{(q-q^{-1})^n}{[n]!} q^{\frac{n(n-1)}{2} - 2sr} \left(1 + q^{tr} + q^{-t(n+s)} + q^{\frac{1}{2}t^2 + tr - t(n+s)}\right) \times K^s E^n \otimes K^r F^n$$

Non-degeneracy

$\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2)$ is factorisable – non-degenerate monodromy $R_{21} R$
– important for **Verlinde formula** analysis

Ribbon element

$$\boldsymbol{v} = \frac{1-\mathrm{i}}{2\sqrt{p}} \sum_{n=0}^{p-1} \sum_{j \in \mathbb{Z}_{2p}} \frac{(q-q^{-1})^n}{[n]!} q^{n(j-\frac{1}{2}) + \frac{1}{2}(j+p+1)^2} F^n E^n K^j$$

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Theorem (CGR'17)

$\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2)$ *is a factorisable ribbon quasi-Hopf algebra.*

Main result: equivalence to triplet W-algebra

Conjecture [CGR'17]

$\mathbf{Rep}\,\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2) \cong \mathbf{Rep}\,\mathcal{W}_p$ as $\mathbb{C}$-linear ribbon categories, for integer $p \geq 2$.

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singlet VOA $\mathcal{M}_p$

Recall S is "short" screening operator in the free boson CFT (V_L)

$$\mathcal{M}_p := \mathrm{Ker}_{F_0}\mathcal{S} = \left(\mathcal{W}_p\right)^{gl(1)} \subset \mathcal{W}_p$$

— an extension of Virasoro algebra by a single primary field of conformal dimension $2p-1$.

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This conjecture is motivated by results of many works
and agrees with everything we know!

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Assuming the above "unrolled vs singlet" conjecture holds we have

Theorem [CGR'17]

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Idea of the proof: use the theory of VOA extensions [CKM'17]

$$\begin{array}{ccccc}
\mathbf{Rep}\,\overline{U}_q^H s\ell(2) \ni A & \xrightarrow{\text{pass to local } A\text{-modules}} & A\text{-mod}^{\mathrm{loc}} & \xleftarrow[\mathcal{F}]{\sim} & \mathbf{Rep}\,\overline{U}_{\mathfrak{q}}^{(\Phi)} s\ell(2) \\
\sim \downarrow \text{conjecture} & & \sim \downarrow \text{consequence} & & \\
\mathbf{Rep}\,\mathcal{M}_p \ni \mathcal{W}_p & \xrightarrow{\text{pass to representations of } \mathcal{W}_p} & \mathbf{Rep}\,\mathcal{W}_p & &
\end{array}$$

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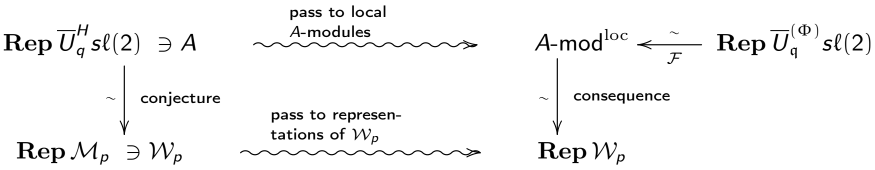

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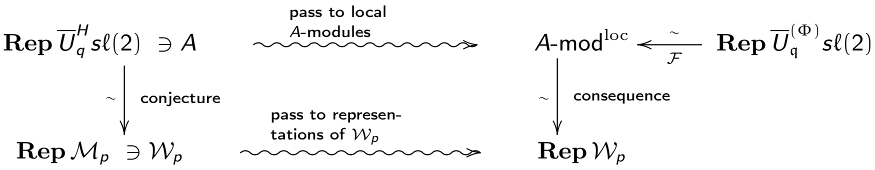

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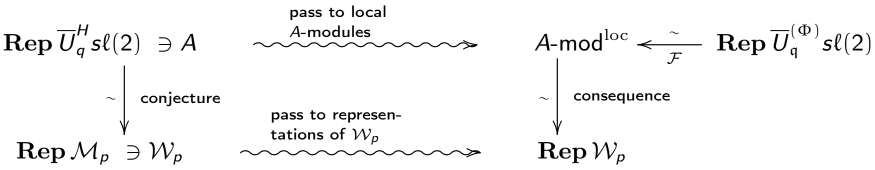

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5. There are several constructions related to CFT which have been formulated for Hopf-algebras: e.g. the construction of **mapping class group invariants** that can serve as bulk correlation functions in LCFT (Fuchs-Schweigert-Stigner'13). A generalization to quasi-Hopf setting is straightforward.
   So using quasi-Hopf algebras we can study bulk LCFTs.

Thank you!