Elliptic deformations of quantum Virasoro and W algebras

Luc Frappat

Laboratoire d'Annecy-le-Vieux de Physique Théorique, France

work in collaboration with J. Avan and E. Ragoucy

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① Introduction

Notion discussed in e.g. [Curtright Zachos 1990]

Extended center of $U_q(\hat{g})$ algebra at critical level [Reshetikhin Semenov-Tjan-Shanskii 1990]

Poisson algebra $W_q(\hat{g})$ [Frenkel Reshetikhin 1995]

Quantized version of Virasoro [Shiraishi et al. (SKAO) 1995]

Extension to q-W_N algebras [Shiraishi et al. (AKOS), Feigin Frenkel 1995]

Operators acting on eigenvectors for Ruijsenaars–Schneider models, hence connection with Macdonald / Koornwinder polynomials [SKAO 1995]

ZF algebra for large N limit of XYZ spin chain [Lukyanov 1995]

Symmetries for restricted SOS models [Lukyanov Pugai 1996]

Extension of AGT correspondence to 5D N=2 superconformal gauge theories, elliptic Vir symmetry in 6D [Awata Yamada 2009, Taki 2014, Nieri 2015]

Connection to q-Painlevé [Bershtein Shchechkin 2017]

DVA structure:

$$f(w/z)\,T(z)\,T(w) - T(w)\,T(z)\,f(z/w) = \frac{(p^{-1}-1)(pq^2-1)}{q^2-1}\Big[\delta\big(\frac{q^2 w}{z}\big) - \delta\big(\frac{w}{q^2 z}\big)\Big]$$

structure function given by

$$f(x) = \exp\Big(\sum_{\ell=1}^{\infty} \frac{(1-p^{-\ell})(1-(pq^2)^{\ell})}{1+q^{2\ell}}\,\frac{x^{\ell}}{\ell}\Big) \quad \text{and} \quad \delta(x) = \sum_{\ell\in\mathbb{Z}} x^{\ell}$$

Quantization using current algebra construction of Vir and W_N

Alternative: direct embedding into larger algebraic structures?
Started in previous papers [Avan et al. (AFRS) 1997-99]

General idea: SKAO formula has 2 parameters p, q and constituent blocks of elliptic functions; ratio of structure function $f(x)/f(x^{-1})$ is ratio of elliptic Jacobi Theta functions.

Suggests quantization of classical DVA/DWA naturally inserted into elliptic quantum algebra instead of quantum affine algebra

② The elliptic quantum algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$

Original proposition for gl(2) [Foda et al. 1994]
Drinfel'd twist and extension to gl(N) [Jimbo et al. (JKOS) 1997]
Justification of Quasi-Hopf structure [JKOS, Arnaudon et al. (ABRR) 1997]

Algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$ is defined in the FRT formalism:

$$\widehat{R}_{12}(z/w)\, L_1(z)\, L_2(w) = L_2(w)\, L_1(z)\, \widehat{R}^*_{12}(z/w)$$

- Lax matrix encapsulates generators:

$$L(z) = \sum_{1\le i,j\le N} L_{ij}(z)\, e_{ij} \quad \text{with} \quad L_{ij}(z) = \sum_{n\in\mathbb{Z}} L_{ij}[n]\, z^{-n}$$

- Let $\bar{R}(z)$ be Belavin's $\mathbb{Z}_N$-symmetric R-matrix: [Belavin 1981]

$$\bar{R}(z) = \rho(z) \sum_{(\alpha_1,\alpha_2)\in\mathbb{Z}_N\times\mathbb{Z}_N} W_{(\alpha_1,\alpha_2)}(z,q,p)\; I_{(\alpha_1,\alpha_2)} \otimes I^{-1}_{(\alpha_1,\alpha_2)}$$

where $\quad I_{(\alpha_1,\alpha_2)} = g^{\alpha_2}\, h^{\alpha_1}, \quad g_{ij} = \omega^i \delta_{ij}, \quad h_{ij} = \delta_{i+1,j}, \quad \omega = e^{2i\pi/N}$

Weights

$$W_{(\alpha_1,\alpha_2)}(z,q,p) = \frac{\vartheta \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}(zq^{\frac{1}{N}},p)}{N\vartheta \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}(q^{\frac{1}{N}},p)} \qquad (\gamma_i = \tfrac{1}{2} + \tfrac{\alpha_i}{N})$$

Gauge transformation

$$R(z) = (g^{\frac{1}{2}} \otimes g^{\frac{1}{2}})\bar{R}(z)(g^{-\frac{1}{2}} \otimes g^{-\frac{1}{2}})$$

Specific normalization

$$\widehat{R}(z) = \tau(q^{\frac{1}{2}}z^{-1})R(z) \quad \text{with} \quad \tau(z) = z^{\frac{2}{N}-2}\frac{\Theta_{q^{2N}}(qz^2)}{\Theta_{q^{2N}}(qz^{-2})}$$

Jacobi Theta function

$$\Theta_a(z) = (z;a)_\infty(az^{-1};a)_\infty(a;a)_\infty \quad \text{with} \quad (z;a)_\infty = \prod_{n\geq 0}(1-za^n)$$

Set $\widehat{R}^*(z) = \widehat{R}(z)|_{p\to p^*=pq^{-2c}}$

Properties of R

- Unitarity:

$$\widehat{R}_{12}(z)\,\widehat{R}_{21}(z^{-1}) = \tau(q^{\frac{1}{2}}z^{-1})\tau(q^{\frac{1}{2}}z) := \mathcal{U}(z)$$

- Regularity:

$$R_{12}(1) = P_{12}$$

- Crossing-unitarity:

$$\left(\widehat{R}_{12}(x)^{t_2}\right)^{-1} = \left(\widehat{R}_{12}(q^N x)^{-1}\right)^{t_2}$$

- Antisymmetry:

$$\widehat{R}_{12}(-z) = (g^{-1}\otimes \mathbb{1})\,\widehat{R}_{12}(z)\,(g\otimes \mathbb{1})$$

- Quasi-periodicity:

$$\widehat{R}_{12}(-zp^{\frac{1}{2}}) = M_1^{-1}\,\widehat{R}_{21}(z^{-1})^{-1}\,M_1 \quad \text{with} \quad M = g^{\frac{1}{2}}hg^{\frac{1}{2}}$$

Quantum determinant

Let $A_N^{(N)}$ be the antisymmetrizer on $(\mathbb{C}^N)^{\otimes N}$. Then

$$L_1(z)\dots L_N(zq^{1-N})\,A_N^{(N)} = \mathrm{qdet}\,L(z)\,A_N^{(N)}$$

The *quantum determinant* $\mathrm{qdet}\,L(z)$ lies in the center of $\mathcal{A}_{q,p}(\widehat{gl}_N)$

$$\begin{aligned}\mathrm{qdet}\,L(z) &= \mathrm{tr}_{1\dots N}\left(L_1(z)\dots L_N(zq^{1-N})A_N^{(N)}\right)\\ &= \sum_{\sigma\in\mathfrak{S}_N}\mathrm{sgn}(\sigma)L_{1\sigma(1)}(z)L_{2\sigma(2)}(\frac{z}{q})\dots L_{N\sigma(N)}(zq^{1-N})\end{aligned}$$

Define

$$\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N) = \mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_N)/\langle \mathrm{qdet}\,L(z) - q^{c/2}\rangle$$

③ Deformed Virasoro algebras (DVA)

Step 1: Closure conditions

Define quadratic functionals of Lax matrices and conditions on parameters p, q, c such that quadratic functionals close exchange algebra

Step 2: Abelianity conditions

Define second (analytical) set of conditions on p, q, c such that the exchange algebra becomes abelian

Step 3: Poisson structures

Expand around set of conditions to get the Poisson structures. Closure algebra then automatically yields a quantization of the Poisson structure

Q1: Does one get classical DVA Poisson?
Q2: Does it quantize to standard DVA?

Theorem: Closure relations

On the surface $\mathscr{S}_{mn}$ defined by $(-p^{\frac{1}{2}})^m(-p^{*\frac{1}{2}})^n = q^{-N}$, the generators

$$t_{mn}(z) = \mathrm{tr}\left(M^{-m}\, L((-p^{*\frac{1}{2}})^n z)\; M^{-n}\, L(z)^{-1}\right)$$

realize an exchange algebra with the generators $L(w)$ of $\mathcal{A}_{q,p}(\widehat{gl}_N)$:

$$t_{mn}(z)\, L(w) \;=\; \frac{\mathcal{F}_{-m}(z/w)}{\mathcal{F}^*_n(z/w)}\; L(w)\, t_{mn}(z)$$

where

$$\mathcal{F}_a(x) = \begin{cases} \displaystyle\prod_{k=0}^{a-1} \mathcal{U}\big((-p^{\frac{1}{2}})^k x\big) & \text{for } a > 0 \\ \displaystyle\prod_{k=1}^{|a|} \mathcal{U}\big((-p^{\frac{1}{2}})^{-k} x\big)^{-1} & \text{for } a < 0 \end{cases}$$

and $\mathcal{F}^*_a(x) = \mathcal{F}_a(x)\big|_{p\to p^*}$, $\mathcal{F}_0(x) = 1$

Remarks:

1. When $m = 1$, $n = -1$, closure relation yields $c = -N$ and leads directly to commutation of $t_{mn}(z)$ with $L(w)$

   ▶ extended center at critical level $c = -N$

2. Extends in a weaker form when $m + n = 0$, $m \neq 1$, n odd: implies commutation of generators $t_{mn}(z)$ with themselves when $c = N/n$, but extra relation between p and q (see below).

3. At $N = 2$, $t_{02}(z) = \mathrm{tr}\left(L(q^{-2}z)L(z)^{-1}\right)$ lies in the center of the elliptic quantum algebra, no closure condition required!
   Related to the qdet through

   ▶ Liouville formula: $\frac{1}{2}\,\mathrm{tr}\left(L(q^{-2}z)L(z)^{-1}\right) = \dfrac{\mathrm{qdet}\,L(q^{-1}z)}{\mathrm{qdet}\,L(z)}$

Quadratic subalgebras

On the surface $\mathscr{S}_{mn}$, the generators $t_{mn}(z)$ close a quadratic subalgebra in $\mathcal{A}_{q,p}(\widehat{gl}_N)$:

$$t_{mn}(z)\, t_{mn}(w) = \mathcal{Y}_{mn}(z/w)\, t_{mn}(w)\, t_{mn}(z)$$

with

$$\mathcal{Y}_{mn}(x) = \frac{\mathcal{F}^*_n(x)\mathcal{F}^*_{-n}(x)}{\mathcal{F}_m(x)\mathcal{F}_{-m}(x)}$$

In terms of the $\mathcal{U}(z)$ function:

$$\mathcal{Y}_{mn}(x) = \frac{\displaystyle\prod_{k=1}^{|m|} \mathcal{U}\big((-p^{\frac{1}{2}})^{-k}x\big) \prod_{k'=0}^{|n|-1} \mathcal{U}\big((-p^{*\frac{1}{2}})^{k'}x\big)}{\displaystyle\prod_{k=0}^{|m|-1} \mathcal{U}\big((-p^{\frac{1}{2}})^{k}x\big) \prod_{k'=1}^{|n|} \mathcal{U}\big((-p^{*\frac{1}{2}})^{-k'}x\big)}.$$

Classical limit (N=2)

For $N = 2$ an explicit factorization is available:

$$\mathcal{Y}_{mn}(x) = \frac{g_{mn}(x^2)}{g_{mn}(x^{-2})} \quad \text{with} \quad g_{mn}(z) = \frac{g^{(|m|)}(z) \left(\prod_{k=1}^{|m|-1} g^{(k)}(z) \right)^2}{g^{*(|n|)}(z) \left(\prod_{k=1}^{|n|-1} g^{*(k)}(z) \right)^2}$$

where

$$g^{(k)}(z) = \exp\left(\sum_{\ell=1}^{\infty} \frac{(1-p^{-k\ell})(1-(p^k q^2)^\ell)}{1+q^{2\ell}} \, \frac{z^\ell}{\ell} \right)$$

$$g^{*(k)}(z) = g^{(k)}(z)\Big|_{p \to p^*}$$

Scaling limit can be defined: $p = 1 + \varepsilon$ and $q = 1 + \eta\,\varepsilon$ with $\varepsilon \to 0$

$$g_{mn}(z) = 1 - \varepsilon^2 \Big\{ \beta_m - (1-2\eta c)^2 \beta_n - 2\eta^2 c(1-2\eta c) n(n-2) \Big\} \frac{z}{(1-z)^2} + o(\varepsilon^2)$$

where

$$\beta_\ell = \frac{|\ell|(|\ell|-1)(2|\ell|-1)}{6} + \eta\ell(\ell-2).$$

z-dependence coincides (up to a surface dependent coefficient) with the scaling limit of SKAO structure function, which yielded the undeformed Virasoro algebra.

Also true for $N > 2$, with no full square expression for g_{mn} and more complicated factor for scaling limit.

- Quadratic algebras characterized as Deformed Virasoro Algebras.

But... DVA of SKAO elusive due to square, and delicate issue with central extension to get exact scaling limit (however 1st term of g_{mn} anyway fully controls centerless part of limit).

Abelian subalgebras in $\mathcal{A}_{q,p}(\widehat{gl}_N)$

On the surface $\mathscr{S}_{mn}$, the generators $t_{mn}(z)$ realize an abelian subalgebra in $\mathcal{A}_{q,p}(\widehat{gl}_N)$ when one of the following conditions is satisfied:

	c	$-p^{\frac{1}{2}}$	$-p^{*\frac{1}{2}}$
$\lvert m\rvert, \lvert n\rvert > 1$	$\frac{N}{nm}(\lambda' m - \lambda n)$	$q^{-N\lambda/m}$	$q^{-N\lambda'/n}$
where $\lambda, \lambda' \in \mathbb{Z} \setminus \{0\}$ and $\lambda + \lambda' = 1$			
$\lvert n\rvert = 1, \lvert m\rvert > 1$	$Nn\big(1 - \lambda(m+n)\big)$	$q^{-N\lambda}$	$q^{-Nn(1-\lambda m)}$
where $\lambda \in \mathbb{Z}/2$ or $\lambda \in \mathbb{Z}/u$, u being any divisor of m or $m+n$			
$\lvert m\rvert = 1, \lvert n\rvert > 1$	$Nm\big(\lambda'(n+m) - 1\big)$	$q^{-Nm(1-\lambda' n)}$	$q^{-N\lambda'}$
where $\lambda' \in \mathbb{Z}/2$ or $\lambda' \in \mathbb{Z}/u'$, u' being any divisor of n or $n+m$			
$m + n = 0$ $n > 0$ and odd	N/n	$q^{-(n-1)N/2n}$	$q^{-(n+1)N/2n}$

Poisson structures

Fix closure condition (surface $\mathscr{S}_{mn}$), then set quasi-abelianity relations as $p^{1-\varepsilon} = q^{\alpha N \ell}$ ($\ell \in \mathbb{Z}$ and α from previous tableau), and expand as

$$\{t(z), t(w)\}_\ell = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(t(z)t(w) - t(w)t(z)\big)$$

Theorem: Poisson structures

$$\{t(z), t(w)\}_\ell = f_\ell(z/w)\, t(z)t(w)$$

where

$$f_\ell(x) = 2N\ell(\ln q)\big(2I(x) - I(qx) - I(q^{-1}x) - (x \leftrightarrow x^{-1})\big)$$

$I(x)$ is given by expressions depending on the different cases of abelianity conditions, e.g. for $|n| = 1, |m| > 1$ ($\alpha = 1$) and ℓ even

$$I(x) = \frac{1}{2}\, m(m+1) \left[\sum_{s=1}^{\infty} \frac{x^2 q^{2Ns}}{1 - x^2 q^{2Ns}} + \frac{1}{2}\, \frac{x^2}{1 - x^2}\right]$$

exact structure of classical DVA!

Original DVA revisited

Original DVA exchange algebra NOT directly realized on a closure surface

Express $\mathcal{A}_{q,p}(\widehat{gl}_N)$ in terms of $L^{\pm}(z)$ generators instead of $L(z)$:

$$L^{+}(z) = L(q^{\frac{c}{2}} z) \quad \text{and} \quad L^{-}(z) = M\, L(-p^{\frac{1}{2}} z)\, M^{-1}$$

in this way the RLL relations take a form similar to the one of $\mathcal{U}_q(\widehat{gl}_N)$:

$$\widehat{R}_{12}(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, \widehat{R}^{*}_{12}(z/w)$$
$$\widehat{R}_{12}(q^{\frac{c}{2}} z/w)\, L_1^{+}(z)\, L_2^{-}(w) = L_2^{-}(w)\, L_1^{+}(z)\, \widehat{R}^{*}_{12}(q^{-\frac{c}{2}} z/w)$$

Consider now same form of relations but with *unitary* R-matrix $R(z)$:

$$R_{12}(z/w)\, \mathcal{L}_1^{\pm}(z)\, \mathcal{L}_2^{\pm}(w) = \mathcal{L}_2^{\pm}(w)\, \mathcal{L}_1^{\pm}(z)\, R^{*}_{12}(z/w)$$
$$R_{12}(q^{\frac{c}{2}} z/w)\, \mathcal{L}_1^{+}(z)\, \mathcal{L}_2^{-}(w) = \mathcal{L}_2^{-}(w)\, \mathcal{L}_1^{+}(z)\, R^{*}_{12}(q^{-\frac{c}{2}} z/w)$$

and don't assume any relation between $\mathcal{L}^{\pm}(z)$

$\Rightarrow$ Algebra NOT equivalent to $\mathcal{A}_{q,p}(\widehat{gl}_N)$, BUT

① $R(z)$ used in Foda et al. original paper to construct the VO

② Previous construction holds but now with different exchange functions

Write the corresponding generators $\tilde{t}_{mn}(z)$ in terms of $\mathcal{L}^{\pm}$:

$$\tilde{t}_{mn}(z) = \mathrm{tr}\left(M^{-m+1}\,\mathcal{L}^{+}\left((-p^{*\frac{1}{2}})^{n+1}q^{\frac{c}{2}}z\right)M^{-n-1}\,\mathcal{L}^{-}(z)^{-1}\right)$$

In the case $N=2$, the exchange algebra of

$$\tilde{t}_{2,-1}(z) = \mathrm{tr}\left(M^{-1}\,\mathcal{L}^{+}(q^{\frac{c}{2}}z)\,\mathcal{L}^{-}(z)^{-1}\right)$$

realizes exactly original quantum DVA

④ Generalization to q-deformed W_N algebras

Theorem: Closure relations for q-W_N generators

On the surface $\mathscr{S}_{mn}$, the generators

$$t^{(k)}_{mn}(z) = \mathrm{tr}\left(\prod_{i=1}^{\overleftarrow{k}} M_i^{-m} \, L_i((-p^{*\frac{1}{2}})^n z_i) \cdot \prod_{i=1}^{\overrightarrow{k}} M_i^{-n} \, L_i(z_i)^{-1} \cdot A_k^{(N)} \right)$$

realize an exchange algebra with the generators $L(w)$ of $\mathcal{A}_{q,p}(\widehat{gl}_N)$:

$$t^{(k)}_{mn}(z) \, L(w) \; = \; \prod_{i=1}^{k} \frac{\mathcal{F}_{-m}(z_i/w)}{\mathcal{F}^*_n(z_i/w)} \, L(w) \, t^{(k)}_{mn}(z)$$

where $z_i = q^{i-(k+1)/2} z$ and $A_k^{(N)}$ is the antisymmetrizer on $(\mathbb{C}^N)^{\otimes k}$

Quadratic subalgebras for q-W_N generators

On the surface $\mathscr{S}_{mn}$, the generators $t^{(k)}_{mn}(z)$ satisfy the quadratic exchange relations:

$$t^{(k)}_{mn}(z)\, t^{(k')}_{mn}(w) = \prod_{i=(1-k)/2}^{(k-1)/2} \prod_{j=(1-k')/2}^{(k'-1)/2} \mathcal{Y}_{mn}(q^{i-j} z/w)\; t^{(k')}_{mn}(w)\, t^{(k)}_{mn}(z)$$

- abelianity conditions identical to the DVA case
- extended center at the critical level $c = -N$

$$\left[t^{(k)}_{cr}(z)\,,\, t^{(k')}_{cr}(w)\right] = 0 \qquad (k, k' = 1, \ldots, N-1)$$

- Poisson structures follow

Non-elliptic limit: $p \to 0$

Elliptic R-matrix degenerates to the R-matrix of $\mathcal{U}_q(\widehat{gl}_N)$ in (a twisted version of) the principal gradation

Write the generators $t^{(k)}_{mn}(z)$ in terms of $L^{\pm}(z)$ generators

$$L^{+}(z) = L(q^{\frac{c}{2}}z) \quad \text{and} \quad L^{-}(z) = M\, L(-p^{\frac{1}{2}}z)\, M^{-1}$$

Take the limit $p \to 0$, then $L^{\pm}(z) \to L^{\pm}_{\mathcal{U}_q(\widehat{gl}_N)}(z)$ [Foda et al. 1994]

Express the result in the homogeneous gradation

▶ one recovers the result on higher Sugawara operators for the quantum affine algebras of type A [Frappat, Jing, Molev, Ragoucy 2016]

Poisson structures

Poisson structures

$$\left\{t^{(k)}(z), t^{(k')}(w)\right\}_\ell = f_\ell^{(k,k')}(z/w)\, t^{(k)}(z) t^{(k')}(w)$$

where

$$f_\ell^{(k,k')}(x) = \sum_{i=(1-k)/2}^{(k-1)/2} \sum_{j=(1-k')/2}^{(k'-1)/2} f_\ell(q^{i-j}x)$$

$f_\ell(x)$ is given by expressions depending on the different cases of abelianity conditions (see DVA case)

Comparison with Frenkel–Reshetikhin Poisson W_N-algebra

Modes are defined by

$$t^{(k)}[n] = \oint_C \frac{dz}{2i\pi z} z^{-n} t^{(k)}(z)$$

Poisson brackets between modes are defined by double contour integrals

$$\{t^{(k)}[n], t^{(k')}[m]\} = \oint_{C_1} \frac{dz}{2i\pi z^{n+1}} \oint_{C_2} \frac{dw}{2i\pi w^{m+1}} f_\ell^{(k,k')}(z/w)\, t^{(k)}(z)\, t^{(k')}(w)$$

In the simplest case, one recovers the result of Frenkel and Reshetikhin (excluding the delta-type terms):

$$\{t^{(k)}[n], t^{(k')}[m]\} \propto \sum_{r\in\mathbb{Z}} \frac{[(N-\max(i,j))r]_q [\min(i,j)r]_q}{[Nr]_q} \times t^{(k)}[n-2r]\, t^{(k')}[m+2r]$$

⑤ Dynamical case: elliptic quantum algebra $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)$

Lax matrix depends now on spectral parameter z and dynamical parameter $\lambda \in \mathfrak{h}$, $\mathfrak{h} = (h, c, d)$ Cartan subalgebra of $\widehat{gl}_2$

Dynamical R-matrix: [Felder 1994]

$$R(z,\lambda) = \rho(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & \bar{c}(z) & \bar{b}(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$b(z) = \frac{\Theta_p(q^2 w)\,\Theta_p(z)}{\Theta_p(w)\,\Theta_p(q^2 z)} \qquad \bar{b}(z) = \frac{\Theta_p(q^2 w^{-1})\,\Theta_p(z)}{\Theta_p(w^{-1})\,\Theta_p(q^2 z)}$$

$$c(z) = \frac{\Theta_p(q^2)\,\Theta_p(wz)}{\Theta_p(w)\,\Theta_p(q^2 z)} \qquad \bar{c}(z) = \frac{\Theta_p(q^2)\,\Theta_p(w^{-1} z)}{\Theta_p(w^{-1})\,\Theta_p(q^2 z)}$$

p, w related to q by $p = q^{2r}$, $w = q^{2s}$ if $\lambda = (s+1)h + (r+2)d + s'c$

RLL relations take the form

$$R_{12}(z_1/z_2, \lambda + h)\, L_1^{\pm}(z_1, \lambda)\, L_2^{\pm}(z_2, \lambda + h^{(1)}) = L_2^{\pm}(z_2, \lambda)\, L_1^{\pm}(z_1, \lambda + h^{(2)})\, R_{12}(z_1/z_2, \lambda)$$

$$R_{12}(q^c z_1/z_2, \lambda + h)\, L_1^{+}(z_1, \lambda)\, L_2^{-}(z_2, \lambda + h^{(1)}) = L_2^{-}(z_2, \lambda)\, L_1^{+}(z_1, \lambda + h^{(2)})\, R_{12}(q^{-c} z_1/z_2, \lambda)$$

and the matrix $R(z, \lambda)$ satisfies the dynamical Yang–Baxter equation

$$R_{12}(z_1/z_2, \lambda + h^{(3)})\, R_{13}(z_1, \lambda)\, R_{23}(z_2, \lambda + h^{(1)}) = R_{23}(z_2, \lambda)\, R_{13}(z_1, \lambda + h^{(2)})\, R_{12}(z_1/z_2, \lambda)$$

Unitarity:

$$R_{12}(z,\lambda)R_{21}(z^{-1},\lambda) = \rho(z)\,\rho(z^{-1}) := n(z)$$

Crossing relation:

$$\sigma_y^{(1)}\left(R_{12}^{t_1}(z^{-1}q^{-2},\lambda)\right)^{-sl_1}\sigma_y^{(1)}\,\frac{\Upsilon(\lambda+\sigma_z^{(2)})}{\Upsilon(\lambda)} = R_{12}^{-1}(z^{-1},\lambda)$$

implies a crossing-unitarity relation:

$$n(z)\left(\left(\widetilde{R}_{12}(z^{-1}q^{-4},\lambda)^{-sl_2}\right)^{t_1}\right)^{-1} = G_1(\lambda)^{-1}\left(\widetilde{R}_{21}^{t_1}(z,\lambda)\right)^{-sc_2}G_1(\lambda-h^{(2)})$$

where $\Upsilon(\lambda) = w^{-1/2}\,\Theta_p(w)$, $G(\lambda) = \Upsilon(\lambda)\Upsilon(\lambda+h)^{-1}$.

For any matrix M, shift-column (sc) and shift-line (sl) defined as:

$$\left(M\right)^{sc} = \left(e^{h\partial}M^t\right)^t e^{-h\partial} = \left(e^{h\partial}\left(M\,e^{-h\partial}\right)^t\right)^t$$
$$\left(M\right)^{sl} = \left(\left(e^{h\partial}M\right)^t e^{-h\partial}\right)^t = e^{h\partial}\left(M^t\,e^{-h\partial}\right)^t$$

'Dynamical centers' in $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)$

1. The generators

$$t(z,\lambda) = \mathrm{tr}\left(N(\lambda)\, e^{-h\partial}\, L^+(q^{-c}z,\lambda)^{-1}\, L^-(z,\lambda)\, e^{h\partial}\right)$$

obey the exchange relations at the critical level $c=-2$:

$$t(z_1,\lambda)\, L^\pm(z_2,\lambda) = L^\pm(z_2,\lambda)\, t(z_1,\lambda+h)$$

the diagonal matrix $N(\lambda) \in \mathrm{End}\, V \otimes \mathbb{C}(\mathfrak{h})$ is given by $N(\lambda) = \Upsilon(\lambda-h)\Upsilon(\lambda)^{-1}$

2. Abelian subalgebra of $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)$ at $c=-2$:

$$\left[t(z_1,\lambda)\,,\, t(z_2,\lambda)\right] = 0$$

Rmk: define $U_{q,p}(\widehat{gl}_2)$ generators $\widetilde{L}^\pm(z) = L^\pm(z,\lambda)\, e^{h\partial}$, then $t(z_1,\lambda)$ commute with $\widetilde{L}^\pm(z_2)$, hence extended center in $U_{q,p}(\widehat{gl}_2)$ at $c=-2$.

Comments on Vertex-IRF correspondence

Vertex-IRF correspondence:

$$S_1(z_1;p,w)\,S_2(z_2;p,wq^{h_1})\,R^{IRF}(z_1/z_2;p,w) = R^{8V}((z_1/z_2)^{\frac{1}{2}};p)\,S_2(z_2;p,w)\,S_1(z_1;p,wq^{h_2})$$

Induces a morphism form $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)$ to $\mathcal{A}_{q,p}(\widehat{gl}_N)$:

$$\phi \;:\; L^{IRF}(z) \;\to\; S(z;p,w)^{-1}\,L^{8V}(z^{\frac{1}{2}})\,S(z;p^*,wq^h)$$

Applying pullback ϕ^* to exchange $t_{mn}(z)L^{8V}(w)$ relation implies

$$s_{mn}(z_1;w)\,L^{IRF}(z_2) = \mathcal{F}_{mn}((z_1/z_2)^{\frac{1}{2}})\;L^{IRF}(z_2)\,\widetilde{s}_{mn}(z_1;w,h)$$

where

$$s_{mn}(z;w) = S(z;p,w)^{-1}\,\phi^*\big(t_{mn}(z^{\frac{1}{2}})\big)\,S(z;p,w)$$
$$\widetilde{s}_{mn}(z;w,h) = S(z;p^*,wq^h)^{-1}\,\phi^*\big(t_{mn}(z^{\frac{1}{2}})\big)\,S(z;p^*,wq^h)$$

but... $\phi^*\big(t_{mn}(z^{\frac{1}{2}})\big)$ may depend on w, hence $\widetilde{s}_{mn}(z;w,h) \neq s_{mn}(z;wq^h)$

⑥ Conclusion and open issues

- q-Vir and q-W_N structures obtained as exchange algebras constructed from traces of multilinears in the Lax matrices of the elliptic quantum algebras
- Extended centers at critical level $c = -N$ and abelianity conditions, both yielding Poisson structures
- Connection with standard deformed Vir and W_N Poisson structures
- Dynamical case included in the scheme although result is somewhat disappointing

- Central extension terms in DVA and DWA have to be obtained otherwise: not directly gotten in abstract construction
- Explicit realizations
- Standard DVA and DWA still absent from the scheme. Possible to obtain standard DVA using a *unitary* elliptic R-matrix, but then what about the Hopf structure of the algebra?
- Link with orthogonal polynomials to be explored

References

Deformed Virasoro algebras from elliptic quantum algebras,
J. Avan, L. Frappat, E. Ragoucy,
Commun. Math. Phys. 354 (2017) 753.

Dynamical centers for the elliptic quantum algebra $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$*,*
J. Avan, L. Frappat, E. Ragoucy,
J. Phys. A: Math. Theor. 50 (2017) 394002.

The quantum determinant of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$*,*
L. Frappat, D. Issing, E. Ragoucy,
arXiv: 1803.00311 [math.QA].

Deformed W*-algebras from elliptic quantum algebras,*
J. Avan, L. Frappat, E. Ragoucy,
in preparation.

Thank you for your attention!