The higher rank q-deformed Bannai-Ito and Askey-Wilson algebra

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Bannai-Ito algebra

Definition

$$\{\Gamma_{12}, \Gamma_{23}\} = \Gamma_{13} + \omega_{13}, \quad \{\Gamma_{23}, \Gamma_{13}\} = \Gamma_{12} + \omega_{12}, \quad \{\Gamma_{13}, \Gamma_{12}\} = \Gamma_{23} + \omega_{23}.$$

S. Tsujimoto, L. Vinet, A. Zhedanov,
Dunkl shift operators and Bannai-Ito polynomials.
Adv. Math. **229** (2012), 2123-2158.

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- Bispectral properties of Bannai-Ito polynomials
- Symmetry algebra of 3D Dirac-Dunkl operator
- Connection with Leonard pairs

Bannai-Ito algebra

Some generalizations:

- Extension to higher rank

  H. De Bie, V.X. Genest, L. Vinet,
  The $\mathbb{Z}_2^n$ Dirac-Dunkl operator and a higher rank Bannai-Ito algebra.
  Adv. Math. **303** (2016), 390-414.

- q-deformation

  V.X. Genest, L. Vinet, A. Zhedanov,
  The quantum superalgebra $\mathfrak{osp}_q(1|2)$ and a q-generalization of the Bannai-Ito polynomials.
  Comm. Math. Phys., **344** (2016), 465-481.

Our aim: combining these

The rank 1 q-Bannai-Ito algebra

Quantum algebra $\mathfrak{osp}_q(1|2)$

$$\{A_+, A_-\} = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}, \qquad KA_+ = q^{\frac{1}{2}} A_+ K, \qquad KA_- = q^{-\frac{1}{2}} A_- K,$$
$$\{P, A_\pm\} = 0, \qquad [P, K] = 0, \qquad P^2 = 1.$$

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- Casimir element: $\Gamma^q = \left(-A_+ A_- + \frac{q^{-1/2} K^2 - q^{1/2} K^{-2}}{q - q^{-1}} \right) P$
- Hopf algebraic structure: $\Delta : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)^{\otimes 2}$
  $\Delta(A_\pm) = A_\pm \otimes KP + K^{-1} \otimes A_\pm, \quad \Delta(K) = K \otimes K, \quad \Delta(P) = P \otimes P.$

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$$\Gamma^q_{\{1\}} = \Gamma^q \otimes 1 \otimes 1, \quad \Gamma^q_{\{2\}} = 1 \otimes \Gamma^q \otimes 1, \quad \Gamma^q_{\{3\}} = 1 \otimes 1 \otimes \Gamma^q,$$
$$\Gamma^q_{\{1,2\}} = \Delta(\Gamma^q) \otimes 1, \quad \Gamma^q_{\{2,3\}} = 1 \otimes \Delta(\Gamma^q)$$
$$\Gamma^q_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Gamma^q)$$

The rank 1 q-Bannai-Ito algebra

q-anticommutator: $\{A, B\}_q = q^{1/2}AB + q^{-1/2}BA$

$$\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q = \Gamma^q_{\{1,3\}} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}}\right)$$

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$$
\begin{aligned}
\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q &= \Gamma^q_{\{1,3\}} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}} \right) \\
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\{\Gamma^q_{\{1,3\}}, \Gamma^q_{\{1,2\}}\}_q &= \Gamma^q_{\{2,3\}} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \left(\Gamma^q_{\{2\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{1\}}\Gamma^q_{\{1,2,3\}} \right)
\end{aligned}
$$

$$\Big\downarrow \; q \to 1$$

$$
\begin{aligned}
\{\Gamma_{\{1,2\}}, \Gamma_{\{2,3\}}\} &= \Gamma_{\{1,3\}} + 2\left(\Gamma_{\{1\}}\Gamma_{\{3\}} + \Gamma_{\{2\}}\Gamma_{\{1,2,3\}}\right) \\
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\{\Gamma_{\{1,3\}}, \Gamma_{\{1,2\}}\} &= \Gamma_{\{2,3\}} + 2\left(\Gamma_{\{2\}}\Gamma_{\{3\}} + \Gamma_{\{1\}}\Gamma_{\{1,2,3\}}\right)
\end{aligned}
$$

The rank 1 q-Bannai-Ito algebra

$$\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q = \Gamma^q_{\{1,3\}} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}}\right)$$
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Generalization to higher rank: Find Γ^q_A for all $A \subset [n] = \{1, 2, \ldots, n\}$:

- Known for sets of consecutive numbers
  $\Gamma^q_{[n]} = (\underbrace{1 \otimes \cdots \otimes 1}_{n-2 \text{ times}} \otimes \Delta) \ldots (1 \otimes \Delta)\Delta(\Gamma^q)$
- What about *sets with holes*?

Higher rank q-Bannai-Ito-algebra

Extension morphism $\tau : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)^{\otimes 2}$

$$
\begin{aligned}
\tau(A_-K) &= K^2P \otimes A_-K, \\
\tau(A_+K) &= (K^{-2}P \otimes A_+K) + q^{-1/2}(q - q^{-1})(A_+^2 P \otimes A_-K) \\
&\quad + q^{-1/2}(q^{1/2} - q^{-1/2})(A_+K^{-1}P \otimes K^2P) \\
&\quad - q^{-1/2}(q - q^{-1})(A_+K^{-1}P \otimes \Gamma^q), \\
\tau(K^2P) &= 1 \otimes K^2P - (q - q^{-1})(A_+K \otimes A_-K), \\
\tau(\Gamma^q) &= 1 \otimes \Gamma^q.
\end{aligned}
$$

Constructing Γ_A^q

$$\Gamma_A^q = \underbrace{1\otimes\cdots\otimes 1}_{\min(A)-1 \text{ times}} \otimes \Gamma^q_{A-\min(A)+1} \otimes \underbrace{1\otimes\cdots\otimes 1}_{n-\max(A) \text{ times}}$$

$$\Gamma_A^q = \left(\overrightarrow{\prod_{k=2}^{n}} \tau^A_{k-1,k}\right)(\Gamma^q)$$

$$\tau^A_{k-1,k} = \begin{cases} \underbrace{1\otimes\cdots\otimes 1}_{k-2 \text{ times}}\otimes\Delta & \text{if } k\in A,\ k-1\in A, \\ (\underbrace{1\otimes\cdots\otimes 1}_{k-1 \text{ times}}\otimes\tau)(\underbrace{1\otimes\cdots\otimes 1}_{k-2 \text{ times}}\otimes\Delta) & \text{if } k\notin A,\ k-1\in A, \\ \underbrace{1\otimes\cdots\otimes 1}_{k-2 \text{ times}}\otimes\Delta\otimes 1 & \text{if } k\notin A,\ k-1\notin A, \\ \mathrm{id} & \text{if } k\in A,\ k-1\notin A. \end{cases}$$

Constructing Γ_A^q

Example: $n = 7$ and $A = \{2, 5, 6\}$:

$$\Gamma^q$$

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$$\Big\downarrow 1\otimes$$
$$1 \otimes \textcircled{2}$$

Constructing Γ_A^q

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$$\begin{array}{l} \Gamma^q \\ \big\downarrow 1 \otimes \\ 1 \otimes \textcircled{2} \\ \big\downarrow 1 \otimes \Delta \\ 1 \otimes \textcircled{2} \otimes \textcircled{5} \end{array}$$

Constructing Γ_A^q

Example: $n = 7$ and $A = \{2, 5, 6\}$:

$$\Gamma^q$$
$$\downarrow 1\otimes$$
$$1 \otimes \textcircled{2}$$
$$\downarrow 1 \otimes \Delta$$
$$1 \otimes \textcircled{2} \otimes \textcircled{5}$$
$$\downarrow 1 \otimes 1 \otimes \tau$$
$$1 \otimes \textcircled{2} \otimes \textcircled{3} \otimes \textcircled{5}$$

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$$
\begin{array}{l}
\Gamma^q \\
\big\downarrow 1 \otimes \\
1 \otimes \textcircled{2} \\
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1 \otimes \textcircled{2} \otimes \textcircled{5} \\
\big\downarrow 1 \otimes 1 \otimes \tau \\
1 \otimes \textcircled{2} \otimes \textcircled{3} \otimes \textcircled{5} \\
\big\downarrow 1 \otimes 1 \otimes \Delta \otimes 1 \\
1 \otimes \textcircled{2} \otimes \textcircled{3} \otimes \textcircled{4} \otimes \textcircled{5}
\end{array}
$$

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Example: $n = 7$ and $A = \{2, 5, 6\}$:

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\begin{array}{l}
\Gamma^q \\
\big\downarrow 1\otimes \\
1 \otimes 2 \\
\big\downarrow 1 \otimes \Delta \\
1 \otimes 2 \otimes 5 \\
\big\downarrow 1 \otimes 1 \otimes \tau \\
1 \otimes 2 \otimes 3 \otimes 5 \\
\big\downarrow 1 \otimes 1 \otimes \Delta \otimes 1 \\
1 \otimes 2 \otimes 3 \otimes 4 \otimes 5 \\
\big\downarrow 1 \otimes 1 \otimes 1 \otimes 1 \otimes \Delta \\
1 \otimes 2 \otimes 3 \otimes 4 \otimes 5 \otimes 6
\end{array}
$$

Constructing Γ_A^q

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$$
\begin{array}{l}
\Gamma^q \\
\big\downarrow 1\otimes \\
1 \otimes ② \\
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\end{array}
$$

Higher rank q-Bannai-Ito algebra

Algebra relations: $A, B \subseteq \{1,2,\dots,n\}, C = (A \cup B) \setminus (A \cap B)$

$$\begin{aligned}
\{\Gamma_A^q, \Gamma_B^q\}_q &= \Gamma_C^q + (q^{1/2} + q^{-1/2}) \left(\Gamma_{A\cap B}^q \Gamma_{A\cup B}^q + \Gamma_{A\setminus(A\cap B)}^q \Gamma_{B\setminus(A\cap B)}^q \right), \\
\{\Gamma_B^q, \Gamma_C^q\}_q &= \Gamma_A^q + (q^{1/2} + q^{-1/2}) \left(\Gamma_{B\cap C}^q \Gamma_{B\cup C}^q + \Gamma_{B\setminus(B\cap C)}^q \Gamma_{C\setminus(B\cap C)}^q \right), \\
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\end{aligned}$$

Conditions:

- $\max(A \setminus (A \cap B)) < \min(A \cap B)$ and $\max(A \cap B) < \min(B \setminus (A \cap B))$
  e.g. $A = \{1, 2, 4, 6\}, B = \{4, 6, 7\}$.
- A has no *holes*

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  e.g. $A = \{1, 2, 4, 6\}, B = \{4, 6, 7\}$.
- A has no *holes*

Definition

$\mathcal{A}_n^q = \langle \Gamma_A^q : A \subseteq \{1, 2, \ldots, n\} \rangle$

Higher rank q-Bannai-Ito algebra

Algebra relations: $A, B \subseteq \{1, 2, \ldots, n\}, A \subseteq B$

$$[\Gamma_A^q, \Gamma_B^q] = 0.$$

Condition:

A and B have no *holes*

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Condition:

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Maximal Abelian subalgebra: $\langle \Gamma_{[2]}^q, \Gamma_{[3]}^q, \ldots, \Gamma_{[n-1]}^q \rangle$

Corollary

$\mathcal{A}_n^q$ has rank $n - 2$.

The Universal Askey-Wilson algebra $AW(3)_Q$

Q-commutator: $[A, B]_Q = QAB - Q^{-1}BA$

Definition

$$\frac{[A_0, A_1]_Q}{Q^2 - Q^{-2}} + A_2 = \frac{\alpha_2}{Q + Q^{-1}}, \quad \frac{[A_1, A_2]_Q}{Q^2 - Q^{-2}} + A_0 = \frac{\alpha_0}{Q + Q^{-1}},$$
$$\frac{[A_2, A_0]_Q}{Q^2 - Q^{-2}} + A_1 = \frac{\alpha_1}{Q + Q^{-1}}.$$

A. Zhedanov,
"Hidden symmetry" of the Askey-Wilson polynomials.
Theor. Math. Phys. **89** (1991), 1146–1157.

P. Terwilliger,
The universal Askey-Wilson algebra.
SIGMA **7** (2011), Paper 069.

The Universal Askey-Wilson algebra $AW(3)_Q$

Quantum algebra $\mathcal{U}_Q(\mathfrak{sl}_2)$

$$\kappa_+ J_+ = Q^2 J_+ \kappa_+, \quad \kappa_+ J_- = Q^{-2} J_- \kappa_+, \quad [J_+, J_-] = \frac{\kappa_+ - \kappa_-}{Q - Q^{-1}}, \quad \kappa_+ \kappa_- = 1.$$

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- Casimir element: $\Lambda = (Q - Q^{-1})^2 J_+ J_- + Q^{-1}\kappa_+ + Q\kappa_-$.
- Hopf algebraic structure:
  $\Delta(J_+) = J_+ \otimes 1 + \kappa_+ \otimes J_+, \quad \Delta(J_-) = J_- \otimes \kappa_- + 1 \otimes J_-, \quad \Delta(\kappa_\pm) = \kappa_\pm \otimes \kappa_\pm.$

The Universal Askey-Wilson algebra $AW(3)_Q$

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- Casimir element: $\Lambda = (Q - Q^{-1})^2 J_+ J_- + Q^{-1}\kappa_+ + Q\kappa_-$.
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  $\Delta(J_+) = J_+ \otimes 1 + \kappa_+ \otimes J_+, \quad \Delta(J_-) = J_- \otimes \kappa_- + 1 \otimes J_-, \quad \Delta(\kappa_\pm) = \kappa_\pm \otimes \kappa_\pm.$

$$\Lambda_{\{1\}} = \Lambda \otimes 1 \otimes 1, \quad \Lambda_{\{2\}} = 1 \otimes \Lambda \otimes 1, \quad \Lambda_{\{3\}} = 1 \otimes 1 \otimes \Lambda,$$
$$\Lambda_{\{1,2\}} = \Delta(\Lambda) \otimes 1, \quad \Lambda_{\{2,3\}} = 1 \otimes \Delta(\Lambda)$$
$$\Lambda_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Lambda)$$

Connection with the Askey-Wilson algebra

$$Q = iq^{\frac{1}{2}}$$

q-Bannai-Ito algebra $\mathcal{A}_3^q$	**Askey-Wilson algebra** $AW(3)_Q$
$\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q - \Gamma^q_{\{1,3\}} = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}}\right)$ + cyclic permutations	$\frac{[\Lambda_{\{1,2\}}, \Lambda_{\{2,3\}}]_Q}{Q^2 - Q^{-2}} + \Lambda_{\{1,3\}} = \frac{\Lambda_{\{1\}}\Lambda_{\{3\}} + \Lambda_{\{2\}}\Lambda_{\{1,2,3\}}}{Q + Q^{-1}}$ + cyclic permutations

Connection with the Askey-Wilson algebra

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q-Bannai-Ito algebra $\mathcal{A}_3^q$	Askey-Wilson algebra $AW(3)_Q$
$\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q - \Gamma^q_{\{1,3\}} = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}}\right)$ + cyclic permutations	$\frac{[\Lambda_{\{1,2\}}, \Lambda_{\{2,3\}}]_Q}{Q^2 - Q^{-2}} + \Lambda_{\{1,3\}} = \frac{\Lambda_{\{1\}}\Lambda_{\{3\}} + \Lambda_{\{2\}}\Lambda_{\{1,2,3\}}}{Q + Q^{-1}}$ + cyclic permutations
Realized in $\mathfrak{osp}_q(1\|2)^{\otimes 3}$: $\Gamma^q_{\{1,2\}} = \Delta(\Gamma^q) \otimes 1$, $\Gamma^q_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Gamma^q), \dots$	Realized in $\mathcal{U}_Q(\mathfrak{sl}_2)^{\otimes 3}$: $\Lambda_{\{1,2\}} = \Delta(\Lambda) \otimes 1$, $\Lambda_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Lambda), \dots$

Connection with the Askey-Wilson algebra

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q-Bannai-Ito algebra $\mathcal{A}_3^q$	**Askey-Wilson algebra** $AW(3)_Q$
$\{\Gamma^q_{\{1,2\}}, \Gamma^q_{\{2,3\}}\}_q - \Gamma^q_{\{1,3\}} = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\left(\Gamma^q_{\{1\}}\Gamma^q_{\{3\}} + \Gamma^q_{\{2\}}\Gamma^q_{\{1,2,3\}}\right)$ + cyclic permutations	$\frac{[\Lambda_{\{1,2\}}, \Lambda_{\{2,3\}}]_Q}{Q^2 - Q^{-2}} + \Lambda_{\{1,3\}} = \frac{\Lambda_{\{1\}}\Lambda_{\{3\}} + \Lambda_{\{2\}}\Lambda_{\{1,2,3\}}}{Q + Q^{-1}}$ + cyclic permutations
Realized in $\mathfrak{osp}_q(1\|2)^{\otimes 3}$: $\Gamma^q_{\{1,2\}} = \Delta(\Gamma^q) \otimes 1$, $\Gamma^q_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Gamma^q), \dots$	Realized in $\mathcal{U}_Q(\mathfrak{sl}_2)^{\otimes 3}$: $\Lambda_{\{1,2\}} = \Delta(\Lambda) \otimes 1$, $\Lambda_{\{1,2,3\}} = (1 \otimes \Delta)\Delta(\Lambda), \dots$

Different τ-morphism required

Connection with Askey-Wilson algebra

Extension morphism $\tau : \mathcal{U}_Q(\mathfrak{sl}_2) \to \mathcal{U}_Q(\mathfrak{sl}_2)^{\otimes 2}$

$$\begin{aligned}
\tau(J_+\kappa_-) &= \kappa_- \otimes J_+\kappa_-, \\
\tau(J_-) &= \kappa_+ \otimes J_- - Q^{-3}(Q - Q^{-1})^2 J_-^2\kappa_+ \otimes J_+\kappa_- \\
&\quad + Q^{-1}(Q + Q^{-1})J_-\kappa_+ \otimes \kappa_- - Q^{-1}J_-\kappa_+ \otimes \Lambda \\
\tau(\kappa_-) &= 1 \otimes \kappa_- - Q^{-1}(Q - Q^{-1})^2 J_- \otimes J_+\kappa_- \\
\tau(\Lambda) &= 1 \otimes \Lambda.
\end{aligned}$$

$$\Lambda_A = \left(\overrightarrow{\prod_{k=2}^{n}} \tau^A_{k-1,k} \right) (\Lambda)$$

Higher rank Askey-Wilson algebra

Definition

$$AW(n)_Q = \langle \Lambda_A : A \subseteq \{1,2,\dots,n\}\rangle$$

Algebra relations: $A, B \subseteq [n], C = (A \cup B) \setminus (A \cap B)$

$$\frac{[\Lambda_A, \Lambda_B]_Q}{Q^2 - Q^{-2}} + \Lambda_C = \frac{\Lambda_{A\cap B}\Lambda_{A\cup B} + \Lambda_{A\setminus(A\cap B)}\Lambda_{B\setminus(A\cap B)}}{Q + Q^{-1}},$$
$$\frac{[\Lambda_B, \Lambda_C]_Q}{Q^2 - Q^{-2}} + \Lambda_A = \frac{\Lambda_{B\cap C}\Lambda_{B\cup C} + \Lambda_{B\setminus(B\cap C)}\Lambda_{C\setminus(B\cap C)}}{Q + Q^{-1}},$$
$$\frac{[\Lambda_C, \Lambda_A]_Q}{Q^2 - Q^{-2}} + \Lambda_B = \frac{\Lambda_{C\cap A}\Lambda_{C\cup A} + \Lambda_{C\setminus(C\cap A)}\Lambda_{A\setminus(C\cap A)}}{Q + Q^{-1}}.$$

Conditions:

- $\max(A \setminus (A \cap B)) < \min(A \cap B)$ and $\max(A \cap B) < \min(B \setminus (A \cap B))$
- A has no *holes*

The $\mathbb{Z}_2^n$ q-Dirac-Dunkl model

q−shift operators :	$T_{q,i} f(x_1, \ldots, x_n)$	$= f(x_1, \ldots, qx_i, \ldots, x_n)$
Reflections :	$r_i f(x_1, \ldots, x_n)$	$= f(x_1, \ldots, -x_i, \ldots, x_n)$
Parameters :	$\mu_i > 0$	

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q-Dunkl operator

$$D_i^q = \frac{q^{\mu_i}}{q - q^{-1}} \left(\frac{T_{q,i} - r_i}{x_i} \right) - \frac{q^{-\mu_i}}{q - q^{-1}} \left(\frac{T_{q,i}^{-1} - r_i}{x_i} \right)$$

If $q \to 1$: $D_i^q \to \partial_{x_i} + \frac{\mu_i}{x_i}(1 - r_i)$

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Realization of $\mathfrak{osp}_q(1|2)$

$$A_+ \to x_i, \quad A_- \to D_i^q, \quad K \to q^{\frac{\mu_i}{2} + \frac{1}{4}} T_{q,i}^{1/2}, \quad P \to r_i$$

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- q-Dirac-Dunkl operator:

$$(1 \otimes \cdots \otimes 1 \otimes \Delta) \dots (1 \otimes \Delta)\Delta(A_-) \quad \to \quad D_{[n]}^q = \sum_{i=1}^{n} D_i^q R_{[n],i}^q$$

- q-position operator:

$$(1 \otimes \cdots \otimes 1 \otimes \Delta) \dots (1 \otimes \Delta)\Delta(A_+) \quad \to \quad X_{[n]}^q = \sum_{i=1}^{n} x_i R_{[n],i}^q$$

with $R_{[n],i}^q = \left(\prod_{j=1}^{i-1} q^{-\frac{\mu_j}{2}-\frac{1}{4}} (T_{q,j})^{-1/2} \right) \left(\prod_{j=i+1}^{n} q^{\frac{\mu_j}{2}+\frac{1}{4}} (T_{q,j})^{1/2} r_j \right).$

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- Spherical q-Dirac-Dunkl operator: Γ_A^q
  $\Rightarrow$ Realization of $\mathcal{A}_n^q$

The $\mathbb{Z}_2^n$ q-Dirac-Dunkl model

Theorem: $\mathcal{A}_n^q$ is algebra of symmetries

$$[D_{[n]}^q, \Gamma_A^q] = 0, \quad [X_{[n]}^q, \Gamma_A^q] = 0.$$

The $\mathbb{Z}_2^n$ q-Dirac-Dunkl model

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q-Dunkl monogenics: Polynomial null solutions of degree k

$$\mathcal{M}_k^q(\mathbb{R}^n) = \mathrm{Ker}(D_{[n]}^q) \cap \mathcal{P}_k(\mathbb{R}^n).$$

$\Rightarrow$ Action of $\mathcal{A}_n^q$ on $\mathcal{M}_k^q(\mathbb{R}^n)$

q-Dunkl monogenics

Fischer decomposition

$$\mathcal{P}_k(\mathbb{R}^n) = \bigoplus_{i=0}^{k} \left(X_{[n]}^q\right)^i \mathcal{M}_{k-i}^q(\mathbb{R}^n).$$

q-Dunkl monogenics

Fischer decomposition

$$\mathcal{P}_k(\mathbb{R}^n) = \bigoplus_{i=0}^{k} \left(X^q_{[n]}\right)^i \mathcal{M}^q_{k-i}(\mathbb{R}^n).$$

CK-isomorphism

The isomorphism between $\mathcal{P}_k(\mathbb{R}^{j-1})$ and $\mathcal{M}^q_k(\mathbb{R}^j)$ is given by

$$\mathbf{CK}^{\mu_j}_{x_j} = \sum_{\alpha=0}^{k} \frac{(-1)^{\frac{\alpha(\alpha+1)}{2}} q^{\frac{\alpha}{2}(\gamma_{[j]}+k-1)}}{[\mu_j,\alpha;q][\mu_j,\alpha-1;q]\ldots[\mu_j,1;q]} x_j^{\alpha} \left(D^q_{[j-1]}\right)^{\alpha},$$

with $[\mu_j; m, q] = \frac{(q^{\mu_j+m}-q^{\mu_j-m})-(-1)^m(q^{\mu_j}-q^{-\mu_j})}{q-q^{-1}}$.

q-Dunkl monogenics

$$\mathcal{M}_k^q(\mathbb{R}^n) = \mathbf{CK}_{x_n}^{\mu_n} \mathcal{P}_k(\mathbb{R}^{n-1})$$

q-Dunkl monogenics

$$\begin{aligned}\mathcal{M}_k^q(\mathbb{R}^n) &= \mathbf{CK}_{x_n}^{\mu_n}\mathcal{P}_k(\mathbb{R}^{n-1})\\ &= \mathbf{CK}_{x_n}^{\mu_n}\left[\bigoplus_{i=0}^{k}\left(X_{[n-1]}^q\right)^{k-i}\mathcal{M}_i^q(\mathbb{R}^{n-1})\right]\end{aligned}$$

q-Dunkl monogenics

$$
\begin{aligned}
\mathcal{M}_k^q(\mathbb{R}^n) &= \mathbf{CK}_{x_n}^{\mu_n} \mathcal{P}_k(\mathbb{R}^{n-1}) \\
&= \mathbf{CK}_{x_n}^{\mu_n} \left[\bigoplus_{i=0}^{k} \left(X_{[n-1]}^q \right)^{k-i} \mathcal{M}_i^q(\mathbb{R}^{n-1}) \right] \\
&= \mathbf{CK}_{x_n}^{\mu_n} \left[\bigoplus_{i=0}^{k} \left(X_{[n-1]}^q \right)^{k-i} \mathbf{CK}_{x_{n-1}}^{\mu_{n-1}} \mathcal{P}_i(\mathbb{R}^{n-2}) \right]
\end{aligned}
$$

q-Dunkl monogenics

$$
\begin{aligned}
\mathcal{M}_k^q(\mathbb{R}^n) &= \mathbf{CK}_{x_n}^{\mu_n} \mathcal{P}_k(\mathbb{R}^{n-1}) \\
&= \mathbf{CK}_{x_n}^{\mu_n} \left[\bigoplus_{i=0}^{k} \left(X_{[n-1]}^q \right)^{k-i} \mathcal{M}_i^q(\mathbb{R}^{n-1}) \right] \\
&= \mathbf{CK}_{x_n}^{\mu_n} \left[\bigoplus_{i=0}^{k} \left(X_{[n-1]}^q \right)^{k-i} \mathbf{CK}_{x_{n-1}}^{\mu_{n-1}} \mathcal{P}_i(\mathbb{R}^{n-2}) \right] \\
&= \mathbf{CK}_{x_n}^{\mu_n} \left[\bigoplus_{i=0}^{k} \left(X_{[n-1]}^q \right)^{k-i} \mathbf{CK}_{x_{n-1}}^{\mu_{n-1}} \left[\bigoplus_{j=0}^{i} \left(X_{[n-2]}^q \right)^{i-j} \mathcal{M}_j^q(\mathbb{R}^{n-2}) \right] \right] \\
&= \ldots
\end{aligned}
$$

q-Dunkl monogenics

$$\mathcal{M}_k^q(\mathbb{R}^n) = \mathbf{CK}_{x_n}^{\mu_n}\left[\bigoplus_{i=0}^{k}\left(X_{[n-1]}^q\right)^{k-i}\mathbf{CK}_{x_{n-1}}^{\mu_{n-1}}\left[\bigoplus_{j=0}^{i}\left(X_{[n-2]}^q\right)^{i-j}\mathcal{M}_j^q(\mathbb{R}^{n-2})\right]\right]$$
$$= \dots$$

Basis functions

The functions $\psi_{\mathbf{j}}(x_1, x_2, \dots, x_n)$ defined by

$$\psi_{\mathbf{j}} = \mathbf{CK}_{x_n}^{\mu_n}\left[\left(X_{[n-1]}^q\right)^{j_{n-1}}\mathbf{CK}_{x_{n-1}}^{\mu_{n-1}}\left[\left(X_{[n-2]}^q\right)^{j_{n-2}}\dots\left[\left(X_{[2]}^q\right)^{j_2}\mathbf{CK}_{x_2}^{\mu_2}(x_1^{j_1})\right]\right]\right]$$

where $\mathbf{j} = (j_1, \dots, j_{n-1}) \in \mathbb{N}^{n-1}$ with $\sum_{i=1}^{n-1} j_i = k$, form a basis for $\mathcal{M}_k^q(\mathbb{R}^n)$.

Action of $\mathcal{A}_n^q$ on $\mathcal{M}_k^q(\mathbb{R}^n)$

Spherical q-Dirac-Dunkl equation

$$\Gamma_{[\ell]}^q \psi_{\mathbf{j}} = \lambda_\ell(\mathbf{j})\psi_{\mathbf{j}},$$

with eigenvalues $\lambda_\ell(\mathbf{j}) = (-1)^{|\mathbf{j}_{\ell-1}|}\left[|\mathbf{j}_{\ell-1}| + \gamma_{[\ell]} - \frac{1}{2}\right]_q$,

where $|\mathbf{j}_{\ell-1}| = \sum_{i=1}^{\ell-1} j_i$, $\gamma_{[\ell]} = \sum_{i=1}^{\ell}(\mu_i + \frac{1}{2})$ and $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$.

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$$\Gamma^q_{\{m,m+1\}}\psi_{\mathbf{j}} = B_{\mathbf{j}}\psi_{\mathbf{j}-\mathbf{h}_m} + A_{\mathbf{j}}\psi_{\mathbf{j}} + C_{\mathbf{j}}\psi_{\mathbf{j}+\mathbf{h}_m}, \quad \text{with } B_{\mathbf{j}}, C_{\mathbf{j}} \neq 0,$$

where $\mathbf{h}_m = (\underbrace{0,\dots,0}_{m-2\text{ entries}}, 1, -1, 0, \dots, 0)$.

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$$\Gamma_{\{m,m+1\}}^q \psi_{\mathbf{j}} = B_{\mathbf{j}}\psi_{\mathbf{j}-\mathbf{h}_m} + A_{\mathbf{j}}\psi_{\mathbf{j}} + C_{\mathbf{j}}\psi_{\mathbf{j}+\mathbf{h}_m}, \quad \text{with } B_{\mathbf{j}}, C_{\mathbf{j}} \neq 0,$$

where $\mathbf{h}_m = (\underbrace{0,\ldots,0}_{m-2 \text{ entries}},1,-1,0,\ldots,0)$.

$\Rightarrow$ Any $\psi_{\mathbf{j}}$ can be mapped to any $\psi_{\mathbf{k}}$

$\Rightarrow$ $\mathcal{A}_n^q$ acts on $\mathcal{M}_k^q(\mathbb{R}^n)$ irreducibly

Further research

Basis functions: $\psi_{\mathbf{j}} = \mathbf{CK}^{\mu_n}_{x_n}\left[\left(X^q_{[n-1]}\right)^{j_{n-1}} \dots \left[\left(X^q_{[2]}\right)^{j_2} \mathbf{CK}^{\mu_2}_{x_2}(x_1^{j_1})\right]\right]$

Permutation π of $\{1,\dots,n\}$,
applied to x_i, r_i and μ_i

New basis: $\phi_{\mathbf{j}} = \mathbf{CK}^{\mu_{\pi(n)}}_{x_{\pi(n)}}\left[\left(X^q_{\pi([n-1])}\right)^{j_{n-1}} \dots \left[\left(X^q_{\pi([2])}\right)^{j_2} \mathbf{CK}^{\mu_{\pi(2)}}_{x_{\pi(2)}}(x_{\pi(1)}^{j_1})\right]\right]$

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Basis functions: $\psi_{\mathbf{j}} = \mathbf{CK}_{x_n}^{\mu_n}\left[\left(X_{[n-1]}^{q}\right)^{j_{n-1}}\dots\left[\left(X_{[2]}^{q}\right)^{j_2}\mathbf{CK}_{x_2}^{\mu_2}(x_1^{j_1})\right]\right]$

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Connection coefficients: Askey-Wilson/q-Racah polynomials?

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