Quantum Entropic Ambiguity and Tomita-Takesaki Theory

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- In order to define an appropriate notion of entanglement measure, the von Neumann entropy, for a general quantum system [1]

Partial trace $\longleftrightarrow$ Restriction of a state to a subalgebra

- GNS representation provides an entropy, which in general is not unique since the decomposition of GNS space is not unique [2]
- Goal: Explain this ambiguity in terms of Tomita-Takesaki theory and relate it with anomalies in quantum systems [3]

[1] Balachandran, Govindarajan, de Queiroz, Reyes-Lega (2013)
[2] Balachandran, de Queiroz, Vaidya (2012)
[3] Balachandran, de Queiroz, Vaidya (2013)

Outline

GNS construction

$$(\mathcal{A}, \omega) \to (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega) \tag{1}$$

$\mathcal{A}$: (unital) C^*-algebra
$\omega : \mathcal{A} \to \mathbb{C}$: state $\quad \omega(a^* a) \geq 0$, $\omega(a^*) = \overline{\omega(a)}$, $\omega(\mathbb{1}_\mathcal{A}) = 1$, $a \in \mathcal{A}$
$\omega(a) = \langle a \rangle_\omega$

- "Inner product" $a, b \in \mathcal{A}$

$$\omega(a^* b) = \langle a|b\rangle\,, \quad \omega(a^* a) = 0 \nRightarrow a = 0 \tag{2}$$

- Null space: Gelfand ideal

$$\mathcal{N}_\omega = \{a \in \mathcal{A} \mid \omega(a^* a) = 0\} \tag{3}$$

- GNS Hilbert space

$$\mathcal{H}_\omega = \overline{\mathcal{A}/\mathcal{N}_\omega}, \quad \omega(a^* b) = \langle [a]|[b]\rangle \tag{4}$$

- Unique cyclic representation $\pi_\omega : \mathcal{A} \to \mathcal{H}_\omega$ with cyclic vector $\Omega_\omega = |[\mathbb{1}_\mathcal{A}]\rangle$, $\overline{\pi_\omega(\mathcal{A})} = \mathcal{H}_\omega$

$$\pi_\omega(a)\,|[b]\rangle = |[ab]\rangle\,, \quad \omega(a) = \langle [\mathbb{1}_\mathcal{A}]|\pi_\omega(a)|[\mathbb{1}_\mathcal{A}]\rangle \tag{5}$$

GNS space

- Decomposition into irreducibles (non-unique)

$$\mathcal{H}_\omega = \bigoplus_{\alpha=1}^{N} \mathcal{H}_\omega^\alpha \tag{6}$$

- $\{e_i^{(\alpha)}\}_i$: orthonormal basis for $\mathcal{H}_\omega^\alpha$.
- Projectors

$$P_\alpha = \sum_{i=1}^{N} |e_i^{(\alpha)}\rangle\langle e_i^{(\alpha)}| \in \pi_\omega(\mathcal{A})' \tag{7}$$

$P_\alpha^2 = P_\alpha,\ P_\alpha^* = P_\alpha,\ \sum_\alpha P_\alpha = 1.$

Density matrix

$$
\begin{aligned}
\omega(a) &= \langle [\mathbb{1}_{\mathcal{A}}] | \pi_\omega(a) | [\mathbb{1}_{\mathcal{A}}] \rangle \\
&= \langle [\mathbb{1}_{\mathcal{A}}] | \pi_\omega(a) \sum_\alpha P_\alpha | [\mathbb{1}_{\mathcal{A}}] \rangle \\
&= \langle [\mathbb{1}_{\mathcal{A}}] | \pi_\omega(a) \sum_\alpha P_\alpha \sum_n |n\rangle\langle n| \, P_\alpha | [\mathbb{1}_{\mathcal{A}}] \rangle \\
&= \sum_n \langle n | \sum_\alpha P_\alpha | [\mathbb{1}_{\mathcal{A}}] \rangle \, \langle [\mathbb{1}_{\mathcal{A}}] | \pi_\omega(a) P_\alpha | n \rangle \\
&= \sum_n \langle n | \sum_\alpha P_\alpha | [\mathbb{1}_{\mathcal{A}}] \rangle \, \langle [\mathbb{1}_{\mathcal{A}}] | P_\alpha \pi_\omega(a) | n \rangle \\
&= \operatorname{tr} \left(\sum_\alpha P_\alpha \, |[\mathbb{1}_{\mathcal{A}}]\rangle\langle[\mathbb{1}_{\mathcal{A}}]| \, P_\alpha \pi_\omega(a) \right) \\
&\equiv \operatorname{tr} \left(\rho_\omega \, \pi_\omega(a) \right)
\end{aligned}
\tag{8}
$$

Example: Finite dimensional case

- $\mathcal{A} = \mathcal{B}(\mathbb{C}^N) \equiv M_N(\mathbb{C})$
- $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(a) = \sum_j \lambda_j a_{jj} = \mathrm{tr}(\rho a), \quad \sum_j \lambda_j = 1, \quad 0 \leq \lambda_j \leq 1, \quad \rho = \sum_{i=1}^{N} \lambda_i \, |i\rangle\langle i|$$

$$\begin{aligned} \omega(a^* a) &= \mathrm{tr}(\rho a^* a) = \mathrm{tr}(\sqrt{\rho} a^* a \sqrt{\rho}) \\ &= \mathrm{tr}\left((a\sqrt{\rho})^* a\sqrt{\rho} \right) \\ &= \| a\sqrt{\rho} \|_{HS} = 0 \;\Rightarrow\; a\sqrt{\rho} = 0 \end{aligned} \tag{9}$$

- Taking $\lambda_i \in (0, 1)$, ρ is invertible

$$\mathcal{N}_\omega = \{0\} \tag{10}$$

- $\mathcal{H}_\omega = M_N(\mathbb{C})/\{0\} \simeq M_N(\mathbb{C})$

- Matrix unit $E_{ij} := |i\rangle\langle j| \in \mathcal{A}$

$$\langle [E_{ij}], [E_{kl}] \rangle = \omega(E_{ij}^* E_{kl}) = \delta_{ik}\delta_{jl}\lambda_j \tag{11}$$

- Orthonormal basis for $\mathcal{H}_\omega$

$$\left| e_i^{(\alpha)} \right\rangle := \frac{|[E_{i\alpha}\rangle]}{\sqrt{\lambda_\alpha}}, \quad i, \alpha \in \{1, \cdots, n\} \tag{12}$$

- Ordering the basis appropriately

$$\pi_\omega(a)\Omega_\omega = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} = a \otimes \mathbb{1}_{\mathbb{C}^N} \tag{13}$$

$$\mathcal{H}_\omega = \bigoplus_{\alpha=1}^{N} \mathcal{H}_\omega^\alpha \simeq \mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \tag{14}$$

- Density matrix
$$\rho_\omega := \sum_\alpha P_\alpha \left|[\mathbb{1}_\mathcal{A}]\right\rangle\!\left\langle[\mathbb{1}_\mathcal{A}]\right| P_\alpha \tag{15}$$
with spectrum
$$\mu_\alpha = \lambda_\alpha \tag{16}$$
- von Neumann entropy
$$S_{\mathrm{vN}}(\rho_\omega) \;=\; -\rho_\omega \log \rho_\omega \;=\; -\sum_\alpha \lambda_\alpha \log \lambda_\alpha \tag{17}$$
- But the decomposition
$$\mathcal{H}_\omega = \bigoplus_{\alpha=1}^{N} \mathcal{H}_\omega^\alpha \simeq \mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \tag{18}$$
is not unique $\rightarrow$ $S_{\mathrm{vN}}(\rho_\omega)$ can change $\rightarrow$ ambiguity
- $$\left|e_i^{(\alpha)}(u)\right\rangle := \sum_\beta \left|e_i^{(\beta)}\right\rangle u_{\beta\alpha}, \quad u \in \mathcal{A} \tag{19}$$

Tomita-Takesaki Theory

Definition

Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is called cyclic for $\mathcal{A}$ if the set $\{a\Omega \mid a \in \mathcal{A}\}$ is dense in $\mathcal{H}$. We say that $\Omega \in \mathcal{H}$ is separating for $\mathcal{A}$ if for any $a \in \mathcal{A}$, $a\Omega = 0$ implies $a = 0$.

Proposition (Lledó, 2004)

Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in H$. Then Ω is cyclic for $\mathcal{A}$ if and only if Ω is separating for $\mathcal{A}'$.

Tomita operator and modular objects

- Given the pair $(\mathcal{A}, \Omega)$, where Ω is cyclic and separating for $\mathcal{A}$, we can define the conjugation operators

$$\begin{aligned} S_0(a\Omega) &:= a^*\Omega, \quad a \in \mathcal{A} \\ F_0(a'\Omega) &:= (a')^*\Omega, \quad a' \in \mathcal{A}' \end{aligned} \tag{20}$$

Proposition (Bratelli, 1987)

$$S \equiv \overline{S_0} = F_0^*, \qquad F \equiv \overline{F_0} = S_0^*$$

- S: closed and antilinear operator $\rightarrow$ the Tomita operator of the pair $(\mathcal{A}, \Omega)$
- Polar decomposition

$$S = J\Delta^{1/2} \tag{21}$$

Δ (unique) positive selfadjoint operator $\rightarrow$ the modular operator
J (unique) antiunitary operator $\rightarrow$ the modular conjugation
$J^* = J,\ J^2 = 1$

Modular group

Definition

Let Δ be the modular operator, we construct a strongly continuous unitary group (via the functional calculus):

$$\Delta^{it} = \exp\big(it(\ln\Delta)\big), \quad t\in\mathbb{R} \tag{22}$$

It is called the modular group and

$$\sigma_t(a) := \Delta^{it} a \Delta^{-it}, \quad a\in\mathcal{A},\ t\in\mathbb{R} \tag{23}$$

gives a one parameter automorphism group on $\mathcal{A}$, the so-called modular automorphism group.

Tomita-Takesaki Theorem

$$J\mathcal{A}J = \mathcal{A}' \qquad \text{and} \qquad \sigma_t(\mathcal{A}) = \mathcal{A}, \quad t\in\mathbb{R} \tag{24}$$

Theorem

Let $\mathcal{A}$ be a von Neumann algebra and ω a faithful normal state:

$$\omega(a^* a) = 0 \to a = 0, \quad \omega(a) = \mathrm{tr}(\rho a) \tag{25}$$

Then, its GNS construction $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ satisfies

- π_ω *is faithful*
- π_ω *is a von Neumann algebra*
- Ω_ω *is separating for* $\pi_\omega(\mathcal{A})$

$$\pi_\omega(a)\Omega_\omega = 0 \Rightarrow \pi_\omega(a) = 0, \quad a \in \mathcal{A} \tag{26}$$

...returning to our example

$(\omega, \mathcal{A}) \to (\Omega_\omega, \mathcal{H}_\omega, \pi_\omega)$
π_ω: faithful
$\pi_\omega(\mathcal{A})$: von Neumann algebra
Ω_ω: separating (and cyclic by construction) for $\pi_\omega(\mathcal{A})$ (therefore for $\pi_\omega(\mathcal{A})'$)
$\Rightarrow$ we can apply Tomita-Takesaki theory

$$S e_i^{(\alpha)} = \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_j}} e_\alpha^{(i)}, \quad J e_i^{(\alpha)} = e_\alpha^{(i)}, \quad \Delta^{1/2} e_i^{(\alpha)} = \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_j}} e_i^{(\alpha)} \tag{27}$$

Commutant algebra action

- Let $u \in \mathcal{A}$ be any unitary element. It defines a new orthonormal, u-dependent basis

$$\left| e_i^{(\alpha)}(u) \right\rangle := J\pi_\omega(u)J \left| e_i^{(\beta)} \right\rangle \tag{28}$$

- It provides a new decomposition of the Hilbert space into irreducible subspaces:

$$\mathcal{H}_\omega = \bigoplus_{\alpha=1}^{N} \mathcal{H}_\omega^\alpha(u) \simeq \mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \tag{29}$$

- The projector

$$\begin{aligned} P_\alpha(u) &= \sum_{i=1}^{N} |e_i^{(\alpha)}(u)\rangle\langle e_i^{(\alpha)}(u)| \\ &= \left(J\pi_\omega(u)J\right) P_\alpha \left(J\pi_{\omega_\rho}(u^*)J\right) \end{aligned} \tag{30}$$

belongs to the commutant $\pi_\omega(\mathcal{A})'$.

Theorem

For any $u \in U(N)$, the projectors $P_\alpha(u)$, $\alpha = 1, \ldots, N$ induce a density matrix

$$\rho_\omega(u) := \sum_{\alpha=1}^{N} P_\alpha(u) \, |[\mathbb{1}_\mathcal{A}]\rangle\langle[\mathbb{1}_\mathcal{A}]| \, P_\alpha(u), \tag{31}$$

such that, for all $a \in \mathcal{A}$, the following identity holds:

$$\omega(a) = \mathrm{tr}_{\mathcal{H}_\omega} \Big(\rho_\omega(u)\pi_\omega(a)\Big). \tag{32}$$

$$\rho_\omega(u) = \sum_{\alpha=1}^{N} \Big(J\pi_\omega(u)J\Big) P_\alpha \Big(J\pi_{\omega_\rho}(u^*)J\Big) \, |[\mathbb{1}]\rangle\langle[\mathbb{1}]| \, \Big(J\pi_\omega(u)J\Big) P_\alpha \Big(J\pi_\omega(u^*)J\Big)$$

- Its spectrum $\mu_\alpha(u) = \sum_{i=1}^{N} \lambda_i |u_{i\alpha}|^2$ provide a new von Neumann entropy.
- It is u invariant when $\lambda_i = 1/N$, that is, the maximal entropy state.

Ambiguity

- Any unitary $J\pi_\omega(u)J$ in the commutant $\pi_\omega(\mathcal{A})'$ of the representation will induce a map $\rho_\omega \equiv \rho_\omega(\mathbb{1}) \mapsto \rho_\omega(u)$ leaving the state unchanged

$$\omega(a) = \mathrm{tr}_{\mathcal{H}_\omega}\left(\rho_\omega(\mathbb{1})\pi_\omega(a)\right) = \mathrm{tr}_{\mathcal{H}_\omega}\left(\rho_\omega(u)\pi_\omega(a)\right) \tag{33}$$

- But it generates an entropy ambiguity

$$\begin{aligned} S_{\mathrm{vN}}(\rho_\omega) = & -\mathrm{tr}_{\mathcal{H}_\omega}\left(\rho_\omega \log \rho_\omega\right) \\ \neq & \; S_{\mathrm{vN}}(\rho_\omega(u)) \end{aligned} \tag{34}$$

- Moreover

$$S_{\mathrm{vN}}(\rho_\omega(u)) \geq S_{\mathrm{vN}}(\rho_\omega(\mathbb{1})) \tag{35}$$

$$\Delta S_{\mathrm{vN}} \geq 0 \tag{36}$$

Completely positive map

Theorem

The map

$$\xi : \rho_\omega(\mathbb{1}) \mapsto \rho_\omega(u) \tag{37}$$

is completely positive

Kraus decomposition

$$B_k(u) := \frac{1}{\lambda_k} P_k(u) \, |[\mathbb{1}]\rangle\langle[\mathbb{1}]| \, P_k(\mathbb{1}) \tag{38}$$

$$\rho_\omega(u) = \sum_k B_k(u) \, \rho_\omega(\mathbb{1}) \, B_k(u)^* \tag{39}$$

Bipartite system

- Since

$$\pi_\omega(a) = a \otimes \mathbb{1}_{\mathbb{C}^N}, \quad \mathcal{H}_\omega = \left(\mathbb{C}^N\right)^{\oplus N} \simeq \mathbb{C}^N \otimes \mathbb{C}^N \tag{40}$$

- We construct the isomorphism

$$\phi : \mathcal{H}_\omega \to \mathbb{C}^N \otimes \mathbb{C}^N \equiv \mathcal{H} \otimes \mathcal{H} \tag{41}$$

$$\phi\Big(\pi_\lambda(a)\,|[\mathbb{1}]\rangle\Big) = (a \otimes \mathbb{1}_{\mathcal{H}})\Omega \tag{42}$$

$$\phi\Big(e_j^{(\alpha)}\Big) \;=\; |\alpha\rangle \otimes |j\rangle \tag{43}$$

Proposition

Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ *and*

$$\Omega = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |i\rangle \otimes |i\rangle \in \mathcal{H} \otimes \mathcal{H} \tag{44}$$

Ω *is the cyclic and separating vector for the algebra* $\mathcal{M} = \mathcal{A} \otimes \mathbb{1}_{\mathcal{H}}$*, therefore is cyclic and separating vector for its commutant* $\mathcal{M}' = \mathbb{1}_{\mathcal{H}} \otimes \mathcal{A}$ *.*

Proposition

$$S(|i\rangle \otimes |j\rangle) = \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_j}} |j\rangle \otimes |i\rangle \tag{45}$$

$$J(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle \tag{46}$$

$$\Delta^{1/2}(|i\rangle \otimes |j\rangle) = \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_j}} |i\rangle \otimes |j\rangle \tag{47}$$

Proposition

$$J\pi_\omega(a)J = J(a \otimes \mathbb{1}_{\mathbb{C}^N})J = \mathbb{1}_{\mathbb{C}^N} \otimes \bar{a} \in \mathcal{M}' \tag{48}$$

In this system, we construct the density matrix

$$\left|\widetilde{e}_i^{(\alpha)}(u)\right\rangle = \phi\Big(J\pi_\omega(u)J\left|e_i^{(\alpha)}\right\rangle\Big) = |i\rangle \otimes \bar{u}\,|\alpha\rangle \tag{49}$$

where $\bar{u}\,|\alpha\rangle = \sum_k \bar{u}_{k\alpha}\,|k\rangle$

$$\widetilde{P}_{(\alpha)}(u) = \mathbb{1}_{\mathbb{C}^N} \otimes (\bar{u}\,|\alpha\rangle\langle\alpha|\,\bar{u}^*) \in \mathcal{M}' \tag{50}$$

$$\widetilde{\rho}_\omega(u) = \sum_{\alpha,i,j} \sqrt{\lambda_i \lambda_j}\, u_{i\alpha} \bar{u}_{j\alpha} [(|i\rangle\,\langle j|) \otimes (\bar{u}\,|\alpha\rangle\,\langle\alpha|\,\bar{u}^*)] \tag{51}$$

$$\widetilde{\mu}_\alpha(u) = \sum_{i=1}^{N} \lambda_i |u_{i\alpha}|^2 \tag{52}$$

Entropy

$\mathbb{C}^N \otimes \mathbb{C}^N = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$\begin{aligned} \omega(a) &= \mathrm{tr}_{\mathbb{C}^N \otimes \mathbb{C}^N} \left(\widetilde{\rho}_\omega(u) \pi_\omega(a) \right) \\ &= \mathrm{tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \left(\widetilde{\rho}_\omega(u)(a \otimes \mathbb{1}_{\mathcal{H}_2}) \right) \\ &= \mathrm{tr}_{\mathcal{H}_1} \left(\widetilde{\rho}_1(u) a \right) \end{aligned} \tag{53}$$

Reduced density matrix

$$\widetilde{\rho}_1(u) = \mathrm{tr}_{\mathcal{H}_2} \widetilde{\rho}_\omega(u) = \sum_i \lambda_\alpha \left|\alpha\right\rangle \left\langle\alpha\right| \tag{54}$$

with spectrum $\widetilde{\mu}_\alpha^{(1)}(u) = \lambda_\alpha$.

On the other hand

$$\widetilde{\rho}_2(u) = \mathrm{tr}_{\mathcal{H}_1}\, \widetilde{\rho}_\omega(u) = \sum_{\alpha,i} \lambda_i |u_{i\alpha}|^2 \, |\Phi_\alpha(u)\rangle \langle \Phi_\alpha(u)| \tag{55}$$

where $|\Phi_\alpha(u)\rangle := \bar{u}\,|\alpha\rangle$ is an orthonormal basis.
With spectrum

$$\widetilde{\mu}_\alpha^{(2)}(u) \;=\; \sum_i \lambda_i |u_{i\alpha}|^2 \tag{56}$$

In summary, we obtained

$$S_{\mathrm{vN}}(\widetilde{\rho}_1(u)) \;=\; S_{\mathrm{vN}}(\widetilde{\rho}_\omega(\mathbb{1})) \tag{57}$$

$$S_{\mathrm{vN}}(\widetilde{\rho}_2(u)) \;=\; S_{\mathrm{vN}}(\widetilde{\rho}_\omega(u)) \tag{58}$$

Ambiguity = Anomaly

- Anomaly $\rightarrow$ gauge (commutant) algebra action which preserves the state but changes the entropy.
- Haag, Hugenholtz and Winnink (HHW) approach; unitary maps for invertible density matrix

$$|[\mathbb{1}]\rangle \rightarrow |\sqrt{\rho}\rangle\,, \qquad |[A]\rangle \rightarrow A\,|\sqrt{\rho}\rangle \tag{59}$$

- Since $\left|\sqrt{\rho}\right\rangle$ is cyclic and separator, we can apply the modular theory

$$SA\,|\sqrt{\rho}\rangle = A^*\,|\sqrt{\rho}\rangle\,. \tag{60}$$

- Since the action of the commutant changes ρ to ρ' and hence $\left|\sqrt{\rho}\right\rangle$ to $\left|\sqrt{\rho'}\right\rangle$, we can define a relative modular objects

$$S_{\rho'\rho}A\left|\sqrt{\rho}\right\rangle = A^* \left|\sqrt{\rho'}\right\rangle , \quad S_{\rho'\rho} = J_{\rho'\rho}\Delta_{\rho'\rho}^{1/2} \tag{61}$$

- If $U_{\rho'\rho}(t) := \Delta_{\rho'\rho}^{it}$, the operator

$$u_{\rho'\rho}(t) := U_{\rho'\rho''}(t)U_{\rho\rho''}^*(t) \in \mathcal{A} \tag{62}$$

fulfills the cocycle identity and define an algebraic analogue to Radon-Nikodym derivative $\rightarrow$ anomaly.

Anomalies

Anomalies: whenever a symmetry of a classical theory is not preserved in the corresponding quantum theory:

- Central charges $\leftrightarrow$ obstructions
- Renormalization $\rightarrow$ Path integral
- Generators of the symmetry do not leave invariant the domain of definition of the Hamiltonian [4]

[4] Esteve (1986)

Example: S^1

- Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\phi^2} \tag{63}$$

- Domain

$$D_\eta \equiv \mathrm{Dom}(H) = \{\psi \in L^2([0, 2\pi]) \,|\, \psi(2\pi) = \eta\psi(0)\}, \quad \eta = e^{i\theta} \tag{64}$$

- Parity $P : \phi \mapsto 2\pi - \phi$

$$\psi \in D_\eta \to \big(P\psi\big)(\phi) = \psi(2\pi - \phi) \tag{65}$$

$$\big(P\psi\big)(2\pi) = \psi(0) = \overline{\eta}\psi(2\pi) = \big(P\psi\big)(0) \tag{66}$$

- Anomaly

$$\psi \in D_\eta \Rightarrow P\psi \in D_{\overline{\eta}} \tag{67}$$

C*-algebra for $\mathbb{R}^2 = \mathcal{W}_{\mathbb{R}^2}$

- $\mathbb{R}^2 \to T^*\mathbb{R}, \qquad (\hat{q}, \hat{p}) \to$ Weyl algebra
- Unitaries $U(a) = e^{ia\hat{p}}$, $V(b) = e^{ib\hat{q}}$, $[\hat{q}, \hat{p}] = i$, $a, b \in \mathbb{R}$

$$\Big(U(a)\psi\Big)(x) = \psi(x+a), \quad \Big(V(b)\psi\Big)(x) = e^{ibx}\psi(x) \tag{68}$$

$$U(a)V(b) = e^{iab}V(b)U(a) \tag{69}$$

- Weyl *-algebra $\mathcal{W}$
  - Generators

$$W(a,b) = e^{iab/2}V(b)U(a) = e^{i(a\hat{p}+b\hat{q})} \tag{70}$$

  - Relations

$$W(a_1,b_1)W(a_2,b_2) = e^{-i(a_1b_2-a_2b_1)/2}W(a_1+a_2, b_1+b_2) \tag{71}$$

$$W(a,b)^* = W(-a,-b) \tag{72}$$

- In general, (V, σ): real symplectic vector space, $u, v \in V$

$$W(u)W(v) = e^{-i\sigma(u,v)/2}W(u+v), \quad W(u)^* = W(-u) \tag{73}$$

C*-algebra for S^1

- $S^1 = \mathbb{R}/\mathbb{Z}$
- C*-algebra for S^1

$$\mathcal{W}_{S^1} = \text{span}\{W(a,n) \in \mathcal{W}_{\mathbb{R}^2} \mid n \in \mathbb{Z}, a \in \mathbb{R}\} \subset \mathcal{W}_{\mathbb{R}^2} \tag{74}$$

"quantization of T^*S^1"

- Ground states $\omega_\theta : \mathcal{W}_{S^1} \to \mathbb{C}$

$$\omega_\theta\Big(W(a,n)\Big) = \begin{cases} \delta_{n,0} e^{ia\frac{\theta}{2\pi}}, & \theta \in [0,\pi] \\ \delta_{n,0} e^{ia\left(\frac{\theta}{2\pi}-1\right)}, & \theta \in [\pi, 2\pi) \end{cases} \tag{75}$$

w.r.t. the canonical dynamic

- Parity is restored as a symmetry for the mixed state (also a GS)

$$\frac{1}{2}(\omega_\theta + \omega_{\overline{\theta}}) \tag{76}$$

GNS

- $\theta \in [0, \pi]$
- $\Omega_\theta = |[1]_\theta\rangle$
- $\mathcal{N}_\theta \neq \{0\}$, in fact, $a \neq b \in \mathbb{R}, n \in \mathbb{Z}$

$$\left(W(a,n) - e^{-i(b-a)\left(\frac{\theta}{2\pi} + \frac{n}{2}\right)} W(b,n) \right) \in \mathcal{N}_\theta \tag{77}$$

- $\mathcal{H}_\theta = \overline{\mathcal{W}_{S^1}/\mathcal{N}_\theta}, \qquad \omega_\theta(u^* v) = \langle [u]_\theta, [v]_\theta \rangle$

- Translation $U(2\pi) \equiv W(2\pi, 0) \in \mathcal{N}_\theta = e^{i\theta}$ $(a = 2\pi, b = 0 = m)$

$$\pi_\theta\Big(U(2\pi)\Big)\,|\Omega_\theta\rangle = e^{i\theta}\,|\Omega_\theta\rangle \tag{78}$$

- As $U(2\pi)$ is in the center of the algebra

$$\pi_\theta\Big(U(2\pi)\Big)\,|[W(a,n)]_\theta\rangle = e^{i\theta}\,|[W(a,n)]_\theta\rangle \tag{79}$$

- $$D_\eta \leftrightsquigarrow \mathcal{W}_{S^1}/\mathcal{N}_\theta \tag{80}$$

Anomalies

- Principal bundle, G: compact Lie group, H: non-abelian finite subgroup

$$\begin{array}{ccc} H & \hookrightarrow & G \\ & & \downarrow \\ & & G/H \end{array} \tag{81}$$

- Transition function $\tilde{\tau}_{\alpha\beta}(h)$, $h \in H$
- Representation $\rho : H \to Gl(\mathbb{C}^k)$
- Associated bundle

$$\begin{array}{c} \xi = G \times_\rho \mathbb{C}^k \\ \downarrow \\ G/H \end{array} \tag{82}$$

- Wave function $\psi \in \Gamma(\xi)$
- Transition function $\tau_{\alpha\beta}(h) = \rho(\tilde{\tau}_{\alpha\beta}(h))$, $\quad \psi_\alpha = \tau_{\alpha\beta}\psi_\beta$
- (discrete) symmetry transformation: gauge $\to \tau_{\alpha\beta} \neq h\tau_{\alpha\beta}h^{-1}$ in general, so h cannot implement the H-action.

Example: Ethylene molecule C_2H4

- $$\mathcal{Q} = G/H \tag{83}$$

  G: compact Lie group, H: non- abelian finite subgroup
- The configuration space associated to the ethylene molecule [5]

  $$\mathcal{Q} = SO(3)/H \sim SU(2)/H^* \tag{84}$$

  H dihedral group: gauge group
- Quantization on $T^*\mathcal{Q} \to \mathcal{A}$: generated by $C^0(\mathcal{Q})$ and $\mathbb{C}SU(2)$
- GNS construction $\to u \in \mathcal{A} \to$ entropy ambiguity

[5] Balachandran, de Queiroz, Vaidya (2013)

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