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From Christoffel words to Markoff numbers

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1 Introduction

In a short article written in Latin in 1875, Christoffel [?] introduced a family of words on a two letter alphabet, that we call *Christoffel words*. Shortly after, they were also considered by Smith [?], who did not know, as he says and regrets, Christoffel's work. These words were followed in the 20th century by the theory of Sturmian sequences, introduced in 1940 by Morse and Hedlund [?] in Symbolic Dynamics. More recently, there has been a lot of work, beginning by Jean Berstel and Aldo de Luca, on these words, from the point of view of Combinatorics on Words and also in Discrete Geometry.

Independently from Christoffel, Markoff (= Markov, famous for the Markov processes, but writing his name in the French way) wrote as young student two brilliant articles [?, ?] in 1879 and 1880, on the theory called now *Theory of Markoff*. This theory has two sides: it characterizes on one hand certain quadratic forms, and on the other, certain real numbers, defined by some extremal conditions (by their minima for quadratic forms, and by their rational approximations for real numbers). They are constructed using some special integers called *Markoff numbers*, which are among others chararacterized by a special Diophantine equation, the *Markoff equation*.

Looking at the subsequent literature, it is seen that Markoff's theory has visibly fascinated many mathematicians, which have developed, and often reproved one or the other side of the theory: Hurwitz [?], Frobenius [?], Perron [?], Remak [?], Dickson [?], Cassels [?], Cohn [?], Bombieri [?], and the list is much longer. Three books must be cited here: the unavoidable book by Cusick and Flahive [?], the book by Perrine [?] who gives among others a lot of matrix constructions, and the recent book by Aigner [?], celebrating the 100th anniversary of Frobenius' *injectivity conjecture for Markoff numbers*.

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Markoff constructs the special quadratic forms that appear in his theory by using special patterns of 1's and 2's (in the continued fraction expansion); these patterns happen to be Christoffel words. The link between Christoffel words and the theory of Markoff was explicitly noted by Frobenius in 1913 [**?**], and somewhat forgotten until recently (but it was known to Caroline Series [**?**]).

The scope of the lecture is to present Markoff's theory, from the point of view of Combinatorics on words, especially Christoffel words. We give a clear presentation, with most of the proofs, using Christoffel words, of the theory of Markoff. It will include both aspects: approximation of real numbers and minima of quadratic forms.

The interested reader may find many results and proofs in the forthcoming book [**?**]. For the theory of Christoffel and related words, see also Chapter 2 of [**?**] and the first part of the book [**?**].

2 Content of the lecture

We begin by introducing Christoffel words, geometrically by discretization of a segment, following Berstel [**?**] and Borel-Laubie [**?**]. One deduces easily the tree of Christoffel pairs of Berstel and de Luca [**?**] and the fact that Christoffel words are primitive elements in the free group F_2; the converse will be mentioned. Proofs are elementary and mostly geometric.

One parametrizes each Markoff triples (the solutions of the diophantine equation $x^2 + y^2 + z^2 = 3xyz$) by Christoffel words, using the Fricke identities, following Harvey Cohn. It allows to compute easily all solutions. The Frobenius conjecture (also called Markoff numbers injectivity conjecture) will be mentioned.

Markoff's characterization of bi-infinite words whose Markoff' supremum is less than three will be deduced from a similar result for infinite words. Christoffel words appear here as periodic patterns. The palindromic-like Markoff condition is introduced here.

The real numbers that are badly approximable, the worst being the golden ratio, are reduced to the previous infinite words.

A real binary indefinite quadratic form whose minimum m and discriminant d satisfy $3m < \sqrt{d}$ are also reduced to these bi-infinite words. One obtains in this way a family of quadratic forms, parametrized by Markoff numbers.

One sees also that the balanced (in the sense of Morse and Hedlund) bi-infinite words are exactly those whose associated Markoff constant is less or equal to 3, and that they are characterized by the combinatorial Markoff condition.

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