Dyck paths and Diophantine equations

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work-in-progress

The polynomials $P_n(q)$

Kassel and Reutenauer[1] obtained the following result.

Theorem

For each $n \geq 1$, there is a polynomial $P_n(q)$, having nonnegative coefficients, such that

$$C_n(q) = (1-q)^2 \; P_n(q).$$

[1] Kassel, C., and Reutenauer, C. (2016). *Complete determination of the zeta function of the Hilbert scheme of n points on a two-dimensional torus*. arXiv preprint arXiv:1610.07793.

Dyck paths

Symmetric Dyck paths

Theorem

The coefficients of the polynomial

$$(1-q)\ P_n(q)$$

belong to the set $\{-1,0,1\}$. *Furthermore, when reading in degree-increasing way, the nonzero coefficients of* $P_n(q)$ *determine a symmetric Dyck path.*

$$\begin{aligned}(1-q)\,P_{126}(q) &= +q^{0}+q^{85}-q^{96}+q^{111}+q^{116}-q^{121}\\ &\quad +q^{130}-q^{135}-q^{140}+q^{155}-q^{166}-q^{251}\\ \text{Dyck word} &= (\,(\,)\,(\,(\,)\,(\,)\,)\,(\,)\,)\end{aligned}$$

Dyck path of $P_{126}(q)$

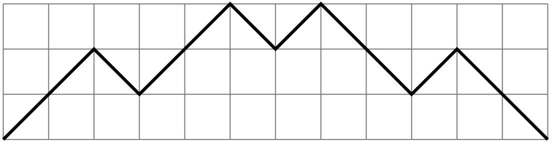

Figure: Dyck path of $(()(()())())$.

Yitang Zhang, Terence Tao and the Twin Prime Conjecture

In 2013, Yitang Zhang established the first finite bound on gaps between prime numbers[2]. In order to refine Zhang's result, with a view towards the twins prime conjecture, the *polymath8* project, led by Terence Tao[3], introduced the so-called λ-*densely divisible* numbers as the positive integers n for which there is at least a divisor on the interval $\left[\lambda^{-1} R, R\right]$, for all real numbers $1 \leq R \leq n$.

[2] Reference: Zhang, Y. (2014). Bounded gaps between primes. Annals of Mathematics, 179(3), 1121-1174.

[3] Reference: Castryck, W., Fouvry, É., Harcos, G., Kowalski, E., Michel, P., Nelson, P., ... and Xie, X. F. (2014). New equidistribution estimates of Zhang type. Algebra & Number Theory, 8(9), 2067-2199.

For example, 12 is 2-densely divisible because the quotient of two consecutive divisors

$$1, 2, 3, 4, 6, 12$$

is ≤ 2.

Theorem

An integer $n \geq 1$ is 2-densely divisible if and only if for each $0 \leq k \leq 2\,n - 2$, the term q^k appears with a non-zero coefficients in the polynomial $P_n(q)$.

Caballero, J. M. R., Irreducible Dyck words and densely divisible numbers. To appear.

The number 12 is 2-densely divisible, because each q^k, with $0 \leq k \leq 2 \cdot 12 - 2$, appears with a nonzero coefficient in

$$\begin{aligned} P_{12}(q) &= q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} \\ &\quad + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^{9} + q^{8} + q^{7} + q^{6} + q^{5} \\ &\quad + q^{4} + q^{3} + q^{2} + q + 1. \end{aligned}$$

The number 11 is not 2-densely divisible, because the coefficient of q^6 is zero.

$$P_{11}(q) = q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^5 + q^4 + q^3 + q^2 + q + 1$$

Erdös, Nicolas and the Pythagorean triangles

Erdös and Nicolas defined the following function[4]

$$F(n) := \max\{q_t(n) : \quad t \in \mathbb{R} \quad \text{and} \quad t > 0\},$$

where $q_t(n) := \#\left\{d : \quad d|n \quad \text{and} \quad \frac{1}{2}\,t < d \leq t\right\}$ is the number of divisors d of n satisfying $\frac{1}{2}\,t < d \leq t$.

Theorem

The largest coefficient of $P_n(q)$ is $F(n)$.

Caballero, J. M. R., On a function introduced by Erdős and Nicolas. To appear.

[4]Reference: Erdös, P., and Nicolas, J. L. (1976). Méthodes probabilistes et combinatoires en théorie des nombres. Bull. SC. Math, 2, 301-320.

A *Pythagorean triangle* is a right triangle with integer side lengths.

Theorem

The polynomial $P_n(q)$ has a coefficient larger than 1 if and only if $2\,n$ is the perimeter of a Pythagorean triangle

Caballero, J. M. R., On a function introduced by Erdős and Nicolas. To appear.

Diophantine equations

Kassel and Reutenauer proved the following results[5].

curve	# of integer points
$x^2 + y^2 = n$	$4\,P_n(-1)$
$x^2 + 2\,y^2 = n$	$2\,\left\|P_n\left(\sqrt{-1}\right)\right\|$
$x^2 + x\,y + y^2 = n$	$6\,\mathrm{Re}\;P_n\left(\frac{-1+\sqrt{-3}}{2}\right)$

[5] Kassel, C., and Reutenauer, C. (2016). *Complete determination of the zeta function of the Hilbert scheme of n points on a two-dimensional torus*. arXiv preprint arXiv:1610.07793.

Explicit formula for the coefficients of $P_n(q)$

Theorem

$$P_n(q) = a_{n,0}\, q^{n-1} + \sum_{k=1}^{n-1} a_{n,k} \left(q^{n-1+k} + q^{n-1-k} \right),$$

$$a_{n,k} = \# \left\{ d|n : \quad \frac{k + \sqrt{2\,n + k^2}}{2} < d \leq k + \sqrt{2\,n + k^2} \right\}.$$

Kassel, C., and Reutenauer, C. (2016). *Complete determination of the zeta function of the Hilbert scheme of n points on a two-dimensional torus*. arXiv preprint arXiv:1610.07793.

New polynomials

For the equation $x^2 + 3\,y^2 = n$ we need the following polynomials.

$$
\begin{aligned}
L_n(q) &:= b_{n,0}\, q^{n-1} + \sum_{k=1}^{n-1} b_{n,k} \left(q^{n-1+k} + q^{n-1-k} \right), \\
b_{n,k} &:= \#\left\{ d|n : \frac{3\,k + \sqrt{12\,n + 9\,k^2}}{6} < d \leq \frac{3\,k + \sqrt{12\,n + 9\,k^2}}{2} \right\}.
\end{aligned}
$$

Caballero, J. M. R., *On the number of points from the hexagonal lattice on a given circle*. To appear.

Theorem

The number of integer points on the ellipse

$$x^2 + 3\,y^2 = n$$

is $2\,L_n(-1)$.

Reference: Caballero, J. M. R., *On the number of integer solutions of* $x^2 + 3\,y^2 = n$. To appear.

The hexagonal lattice

The regular lattice with hexagonal symmetry identified with $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is called the *hexagonal lattice*. This lattice arises in Lattice Gas Cellular Automata via the Lattice Boltzmann Equation

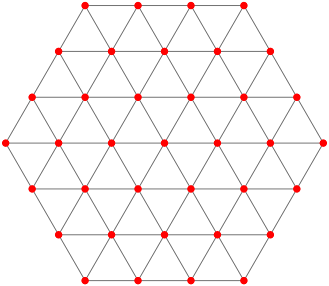

Figure: Hexagonal lattice.

The function $\mathrm{HEX}(n)$ gives the number of points belonging to the hexagonal lattice which are on the circle centered at the origin and a radius of $\sqrt{n}$.

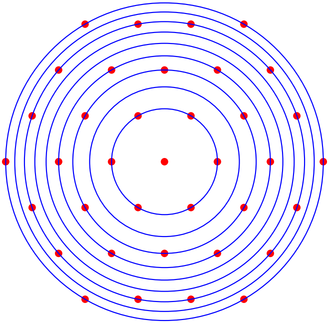

Figure: Hexagonal lattice and circles of radii $\sqrt{n}$, with $1 \leq n \leq 9$.

A relationship between $L_n(1)$ and $P_n(1)$

Theorem

For all $n \geq 1$,

$$HEX(n) + 18\, L_n(1) = 24\, P_n(1).$$

Reference: Caballero, J. M. R., *On the number of points from the hexagonal lattice on a given circle*. To appear.