Asymptotic behavior of regular sequences

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joint work with

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Let $k \geq 2$ be an integer.

$(u_n)_n \in \mathbb{Z}^{\mathbb{N}}$ is *k-automatic* if its *k-kernel*

$$\left\{(u_{k^a n+b})_n \mid a, b \in \mathbb{N}, 0 \leq b < k^a\right\}$$

is finite.

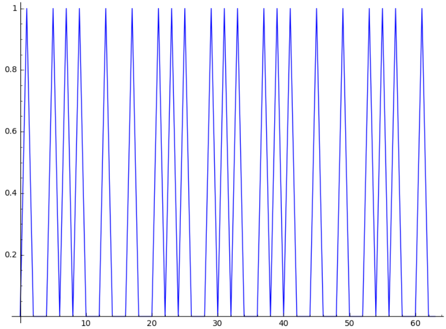
Example

The *period-doubling* sequence $(u_n)_n$ is 2-automatic.

$$\begin{aligned}
(u_n)_n &= 01000101010001000100010101000101\cdots \\
(u_{2n})_n &= 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \cdots \\
(u_{2n+1})_n &= 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\cdots \\
(u_{4n+1})_n &= 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad \cdots \\
(u_{4n+3})_n &= 0\quad 1\quad 0\quad 0\quad 0\quad 1\quad 0\quad 1\cdots
\end{aligned}$$

k-automatic sequences are bounded

If $(u_n)_n$ is k-automatic, there is N such that $|u_n| \leq N$ for all n.



Period-doubling sequence.

k-regular sequences generalize k-automatic ones

$(u_n)_n$ is k-regular if the $\mathbb{Z}$-module generated by its k-kernel is finitely generated.

Thus we have recurrence relations of the form

$$u_{k^a n+b} = \sum_{\substack{0\le c<N\\ 0\le d<k^c}} e_{c,d}\, u_{k^c n+d}, \qquad a\ge N, 0\le b<k^a$$

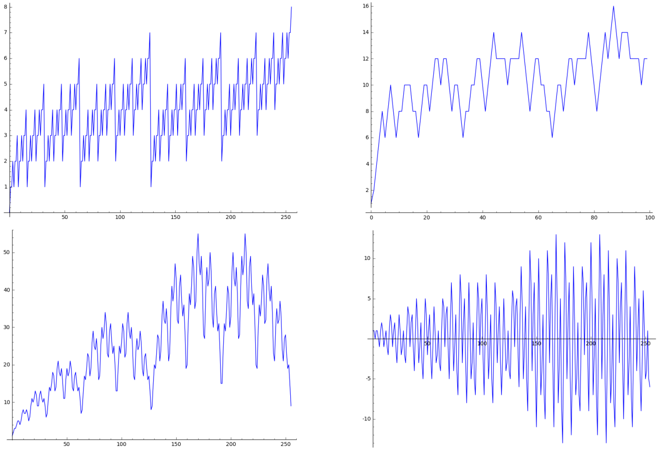
Example (sum-of-digits)

$$n=\sum_{j=0}^{\ell} d_j 2^j \quad\mapsto\quad s_2(n)=\sum_{j=0}^{\ell} d_j, \qquad d_j\in\{0,1\},$$

is 2-regular: for all n, we have

$$s_2(2n)=s_2(n) \quad\text{and}\quad s_2(2n+1)=s_2(n)+1.$$

k-regular sequences are much more chaotic...

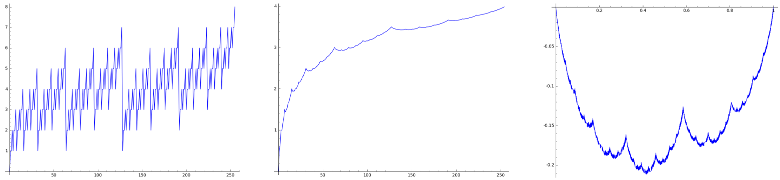


...but their summatory function have asymptotic behaviors

Theorem (Delange 1975)

There is a continuous nowhere differentiable periodic function φ *of period 1 such that*

$$\frac{1}{N}\sum_{n=0}^{N-1} s_2(n) = \frac{1}{2}\log_2(N) + \varphi(\log_2(N))$$



Delange's result holds for any k-regular sequence

Theorem (Dumas 2013)

If $(u_n)_n$ is k-regular, then its summatory function admits an asymptotic expansion which is a sum of terms of the form

$$N^{\log_k \rho} \binom{\log_k N}{m} e^{i\theta \log_k N} \varphi(\log_k N),$$

where φ is a continuous periodic function of period 1.

The proof is based on the linear representation of $(u_n)_n$: there are matrices $A_0, A_1, \ldots A_{k-1}$, a row vector r and a column vector c such that

$$u_n = rA_{d_0}A_{d_1} \cdots A_{d_l}c \qquad \text{if } n = \sum_{i=1}^{l} d_i k^i.$$

There is an error term in $\mathcal{O}(N^{\log_k r})$ for every $r > \rho^*$.

Two remarks about Dumas's result

1. The result holds for the sequence itself:
   if $(u_n)_n$ is k-regular, then so are
   $$\left(\sum_{j=0}^{n-1} u_j\right)_n \quad \text{and} \quad (u_{n+1} - u_n)_n$$
2. The error term in the asymptotic behavior seems to be inherent to the proof, not to the sequence itself.

We present a method based on numeration systems

Settings:

$$(u_n)_n \text{ is } k\text{-regular} \qquad S_n := \sum_{j=0}^{n-1} u_j$$

Mains ideas:

1. Find a linear recurrent sequence $(V_n)_n$ such that

$$S_{ak^n} = c(a)V_n \qquad \text{for all } n;$$

2. Deduce recurrence relations for $(S_n)_n$ with $(V_n)_n$ occurring;

3. Express φ using a numeration system based on $(V_n)_n$.

Observation:

- Seems to provide exact formulas
- Also works for not k-regular sequences

Binomial coefficients of words generalize usual ones

$v, w \in A^*$

The binomial coefficient $\binom{v}{w}$ is the number of times that w occurs as a *scattered* subword of v.

Example

$\binom{10101}{101} = 4$ because, writing

$$v_1 v_2 v_3 v_4 v_5 = 10101,$$

we have

$$v_1 v_2 v_3 = v_1 v_2 v_5 = v_1 v_4 v_5 = v_3 v_4 v_5 = 101.$$

Usual coefficients are obtained by considering unary alphabets:

$$\binom{a^p}{a^q} = \binom{p}{q}.$$

This leads to generalized Pascal triangles...

$m, n \in \mathbb{N}$

$\binom{m}{n}$		k				
		0	1	2	3	4
	0	1				
	1	1	1			
	2	1	2	1		
m	3	1	3	3	1	
	4	1	4	6	4	1

$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$$

local rule

$v, w \in \text{rep}_2(\mathbb{N})$

$\binom{v}{w}$		w				
		ε	1	10	11	100
	ε	1				
	1	1	1			
	10	1	1	1		
v	11	1	2	0	1	
	100	1	1	2	0	1

$$\binom{va}{wb} = \binom{v}{wb} + \delta_{a,b}\binom{v}{w}$$

not so local rule

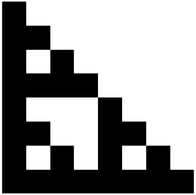
...and generalized Sierpińsky gaskets

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

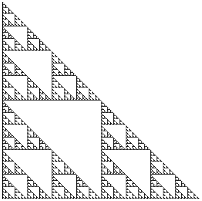
$\xrightarrow{\text{mod } 2}$

1							
1	1						
1	0	1					
1	1	1	1				
1	0	0	0	1			
1	1	0	0	1	1		
1	0	1	0	1	0	1	
1	1	1	1	1	1	1	1

$$\begin{cases} 0 \mapsto \square \\ 1 \mapsto \blacksquare \end{cases} \longrightarrow$$



$\longrightarrow$



...and generalized Sierpińsky gaskets

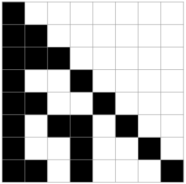
Theorem (L., Rigo, Stipulanti 2016)

1							
1	1						
1	1	1					
1	2	0	1				
1	1	2	0	1			
1	2	1	1	0	1		
1	2	2	1	0	0	1	
1	3	0	3	0	0	0	1

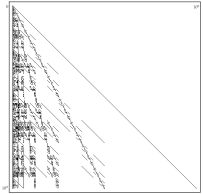
$\xrightarrow{\text{mod } 2}$

1							
1	1						
1	1	1					
1	0	0	1				
1	1	0	0	1			
1	0	1	1	0	1		
1	0	0	1	0	0	1	
1	1	0	1	0	0	0	1

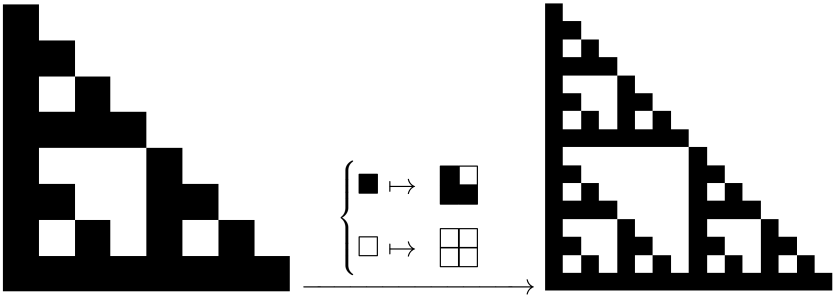
$$\xrightarrow{\begin{cases} 0 \mapsto \square \\ 1 \mapsto \blacksquare \end{cases}}$$



$\longrightarrow$



There is a substitution rule for the Sierpińsky gasket

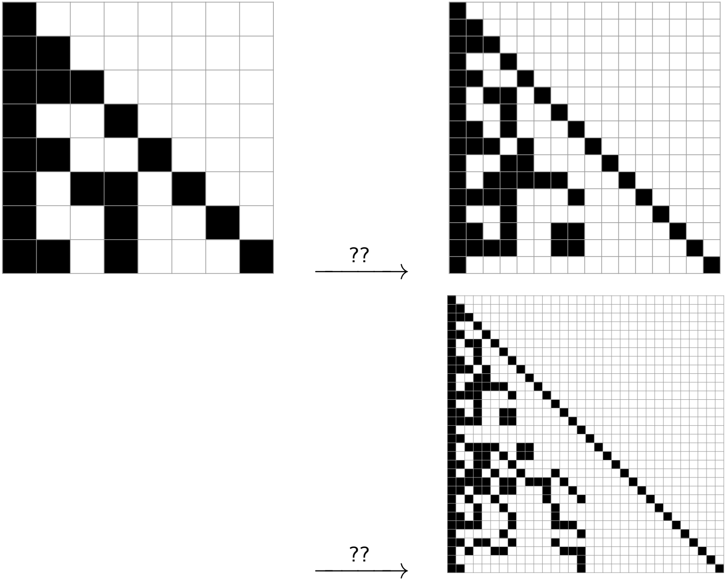


Theorem (Lucas 1878)

$$\binom{m}{n} \equiv \prod_i \binom{m_i}{n_i} \mod 2,$$

where $\mathrm{rep}_2(m) = (m_i)_i$ *and* $\mathrm{rep}_2(m) = (n_i)_i$.

It does not look so clear for the generalized one

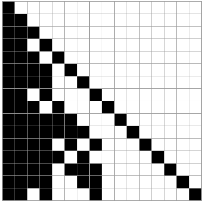


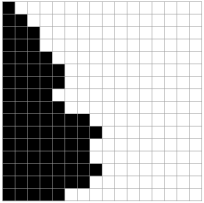
Another difference is the number of positive elements in a row

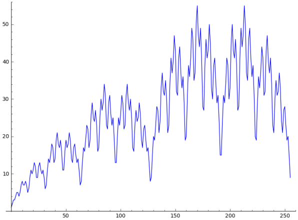
In the classical Pascal triangle, all elements of a row are positive, not in the generalized one:

1								1							
1	1							1	1						
1	2	1						1	1	1					
1	3	3	1					1	2	0	1				
1	4	6	4	1				1	1	2	0	1			
1	5	10	10	5	1			1	2	1	1	0	1		
1	6	15	20	15	6	1		1	2	2	1	0	0	1	
1	7	21	35	35	21	7	1	1	3	0	3	0	0	0	1

Positive coefficients draws a familiar graph







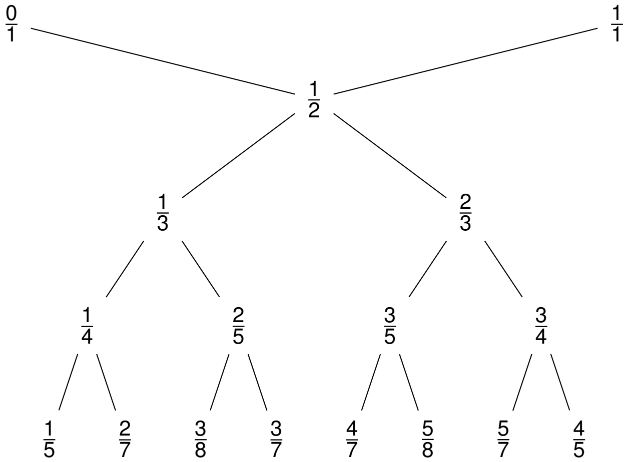
$$u_n = \# \left\{ m \mid \binom{\mathrm{rep}_2(n)}{\mathrm{rep}_2(m)} > 0 \right\}$$

$$(u_n)_n = 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, \ldots.$$

$(u_n)_n$ is 2-regular

Theorem (L. Rigo, Stipulanti 2017)

$(u_n)_n = (SB_{2n+1})_n$, where $(SB_n)_n$ is the 2-regular Stern-Brocot sequence.

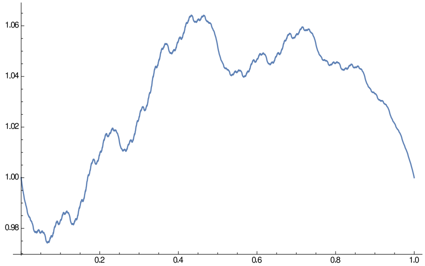


We got an exact formula for the summatory function

Theorem (L. Rigo, Stipulanti 2017)

There is a continuous and periodic function φ of period 1 such that for all n,

$$\sum_{j=0}^{n-1} u_n = 3^{\log_2 n}\varphi(\log_2(n)).$$



Our proof uses a particular numeration system

$S(n) = \sum_{j=0}^{n-1} u_n$

1. for all n, $S(2^n) = 3^n$
2. For all $\ell \geq 1$,

$$S(2^\ell + r) = \begin{cases} 2 \cdot 3^{\ell-1} + S(2^{\ell-1} + r) + S(r), \\ \qquad \text{if } 0 \leq r \leq 2^{\ell-1}; \\ 4 \cdot 3^\ell - 2 \cdot 3^{\ell-1} - S(2^{\ell-1} + r') - S(r') \\ \qquad \text{if } 2^{\ell-1} < r < 2^\ell, \text{ where } r' = 2^\ell - r. \end{cases}$$

3. This defines a numeration system based on $(3^n)_n$:

$$S(n) = \sum_{i=0}^{\ell} a_i(n) 3^{\ell-i} = 3^\ell \sum_{i=0}^{\ell} \frac{a_i(n)}{3^i}$$

4. The function φ is expressed using these 3-representations.

A closer look at the function φ

If $n \in [2^\ell, 2^{\ell+1})$ and $m \in [2^{\ell'}, 2^{\ell'+1})$ are such that

$\mathrm{rep}_2(n)$	$=$	v	w
$\mathrm{rep}_2(m)$	$=$	v	x

with v “long”,

then

- m, n have the same “position” in $[2^\ell, 2^{\ell+1})$ and $[2^{\ell'}, 2^{\ell'+1})$
- $a_i(n) = a_i(m)$ for the first i’s.

For $x \in [0, 1]$, we set

$$e_n(x) = 2^{n+1} + 2\lfloor x2^n \rfloor + 1 \in [2^{n+1}, 2^{n+2})$$

and $(a_i(x))_i = \lim_n (a_i(e_n(x)))_i$ We get

$$\varphi(x) = \frac{1}{3^{\log_2(x+1)}} \sum_{i \geq 0} \frac{a_i(x)}{3^i}.$$

What about other sequences?

The Fibonacci sequence $(F_n)_n$

$$F_0 = 1, \quad F_1 = 2, \quad F_{n+2} = F_{n+1} + F_n,$$

defines the Zeckendorf numeration system:

$$n \mapsto \mathrm{rep}_F(n) = d_\ell d_{\ell-1} \cdots d_0 \in \{0,1\}^*,$$

where

$$n = \sum_{i=0}^{\ell} d_i F_i, d_\ell = 1, d_i d_{i+1} \neq 11.$$

The sequence $(v_n)_n$

$$v_n = \# \left\{ m \mid \binom{\mathrm{rep}_F(n)}{\mathrm{rep}_F(m)} > 0 \right\}$$

is not known to be k-regular for any k.

We can more or less do the same work

1. There is a Fibonacci-Pascal triangle and a conjectured associated Fibonacci-Sierpińsky gasket.
2. $(v_n)_n$ is Fibonacci-regular
3. There exists a continuous and periodic function ψ of period 1 such that

$$\sum_{j=0}^{n-1} v_j = c\,\beta^{\log_F n}\psi(\log_F n) + o(\beta^{\lfloor \log_F n \rfloor}),$$

where β is the dominant root of $X^3 - 2X^2 - X + 1$.

Many things need to be done

1. Does the method work for any k-regular sequence and does it always give an exact formula?
2. What about other numeration systems like Tribonacci?
3. Pascal triangle and Sierpińsky gasket are well studied objects. Say something clever about their generalized versions.

$\vdots$

Questions?