Part III: The geometry of lattice congruences on posets of regions

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Main points

$\mathcal{A}$: simplicial hyperplane arrangement (e.g., Coxeter arrangement).
$\mathcal{F}$: the fan defined by $\mathcal{A}$.
The poset of regions $\mathcal{P}(\mathcal{A}, B)$ is a polygonal lattice.
A congruence Θ on $\mathcal{P}(\mathcal{A}, B)$ defines a coarsening $\mathcal{F}_\Theta$ of $\mathcal{F}$.

When $\mathcal{A}$ is a Coxeter arrangement (so $\mathcal{P}(\mathcal{A}, B)$ is the weak order) and Θ is a Cambrian congruence Θ_c, $\mathcal{F}_\Theta$ is the Cambrian fan $\mathcal{F}_c$. It is the normal fan of a realization of the W-associahedron (Hohweg-Lange-Thomas) and contains information about **g**-vectors and **c**-vectors (R.-Speyer).

A simple geometric condition cuts hyperplanes of $\mathcal{A}$ into shards, which form a geometric model for join-irreducible elements, forcing, and canonical join complex.

Section III.a: Lattice congruences and fans

The poset of regions (Edelman, 1985)

$\mathcal{A}$: a (central) hyperplane arrangement in a real vector space.
Regions: connected components of the complement of $\mathcal{A}$.
B: a distinguished "base" region.
Separating set of a region R:

$$S(R) = \{\text{hyperplanes of } \mathcal{A} \text{ separating } R \text{ from } B\}$$

Poset of regions $\mathcal{P}(\mathcal{A}, B)$ is the set of regions with

$$Q \le R \text{ if and only if } S(Q) \le S(R).$$

Alternately, take the zonotope dual to $\mathcal{A}$ and direct its 1-skeleton by a linear functional.

Proposition. If $\mathcal{A}$ is a Coxeter arrangement for W, then $w \mapsto wB$ is an isomorphism from the weak order on W to $\mathcal{P}(\mathcal{A}, B)$.

Fans

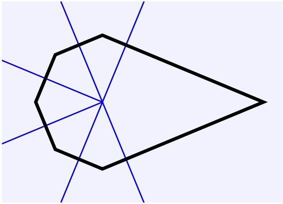
V: a real vector space.
Closed cone $C \subseteq V$: closed under nonnegative scaling, addition.
Fan $\mathcal{F}$: A collection of closed cones such that:
 If $C \in \mathcal{F}$ then all faces of C are in $\mathcal{F}$.
 If $C, D \in \mathcal{F}$ then $C \cap D$ is a face of C and of D.

$\mathcal{F}$ is complete if $\cup\mathcal{F} = V$.

Example. The normal fan of a polytope P in V.

Define an equivence relation on functionals in the dual space to V with $f \equiv f'$ if and only if f and f' are maximized on the same face of P. Cones in $\mathcal{F}$ are closures of equivalent classes.

For example, a polygon and its normal fan:



Coarsening fans by lattice congruences

Every central hyperplane arrangement defines a fan.
(Cones are the regions, together with all their faces.)
This is the normal fan of the corresponding zonotope.
Simplicial fan: all cones are simplicial.
Simplicial hyperplane arrangement: cuts space into a simplicial fan.

Theorem (Bjorner, Edelman, Ziegler, 1987). If $\mathcal{A}$ is simplicial then $\mathcal{P}(\mathcal{A}, B)$ is a lattice for any base region B.

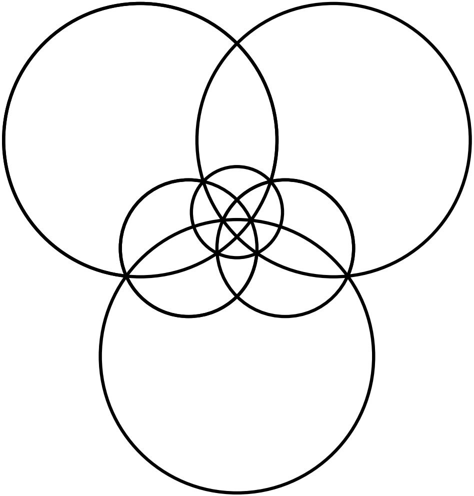
Theorem. If $\mathcal{A}$ is simplicial then $\mathcal{P}(\mathcal{A}, B)$ is a polygonal lattice.

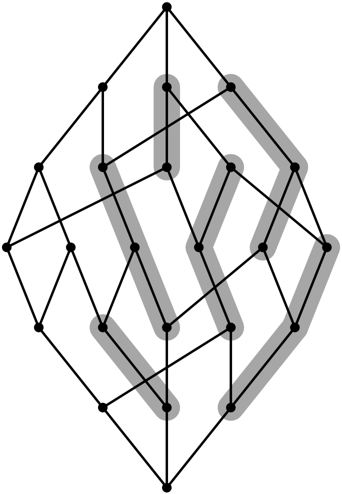
For any lattice congruence Θ on $\mathcal{P}(\mathcal{A}, B)$, define a collection $\mathcal{F}_\Theta$ of cones, closed under passing to faces.

Maximal cones of $\mathcal{F}_\Theta$ are unions, over congruence classes of Θ, of maximal cones of the fan defined by $\mathcal{A}$.

Theorem (R., 2004). $\mathcal{F}_\Theta$ is a fan.

Example: $\mathcal{F}_\Theta$ for a congruence on the weak order on S_4





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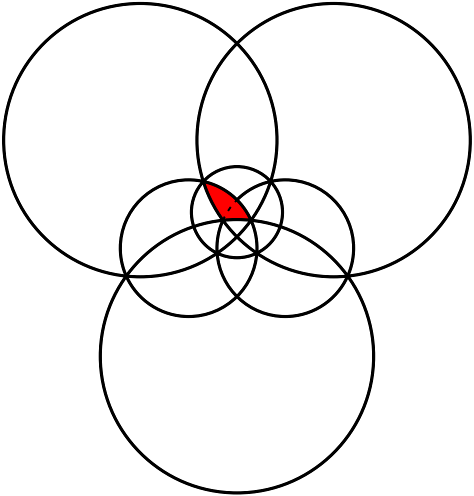

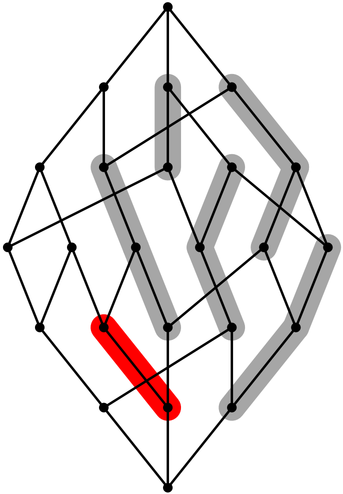

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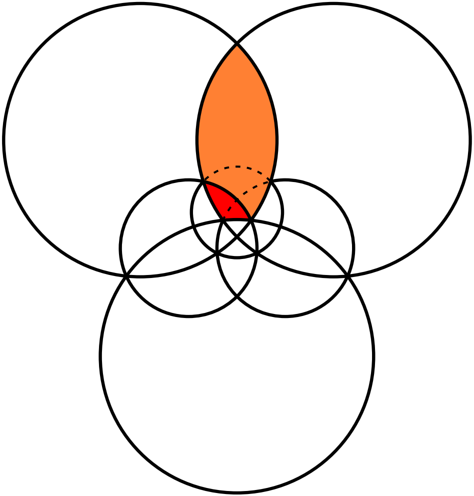

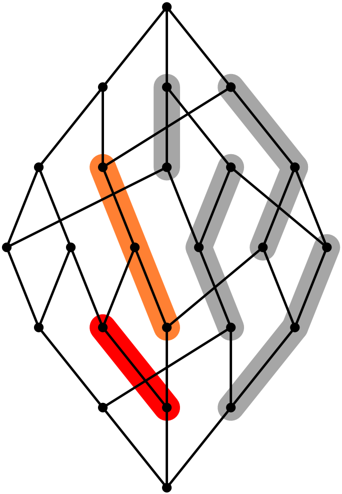

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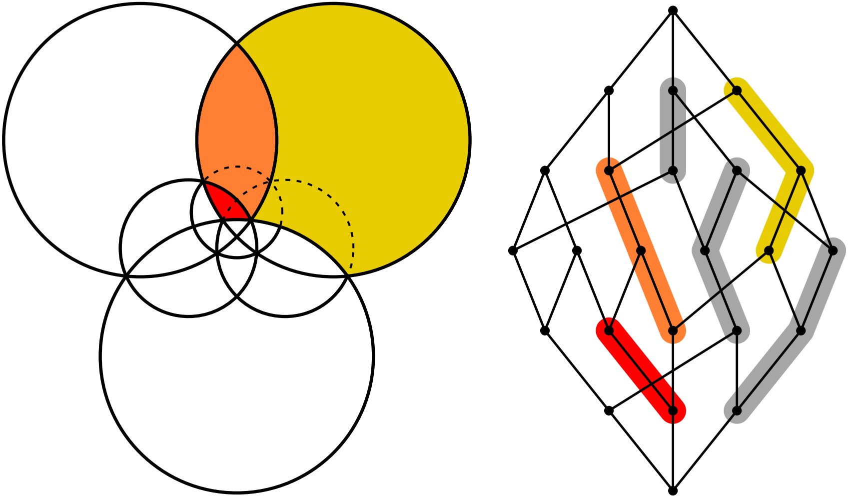

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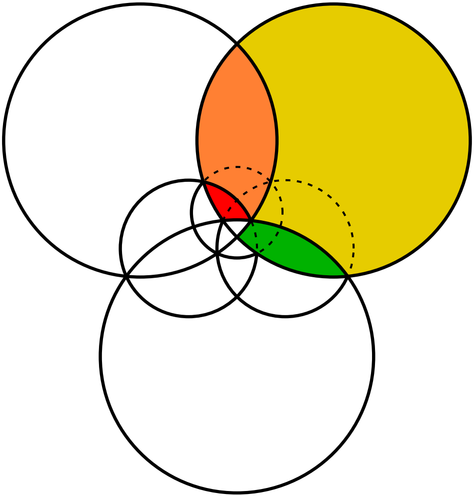

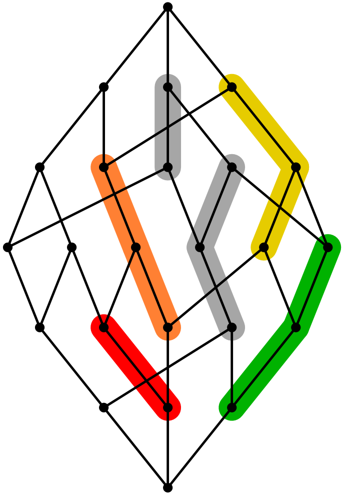

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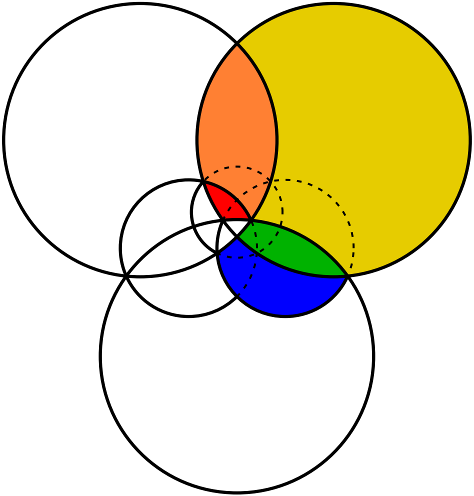

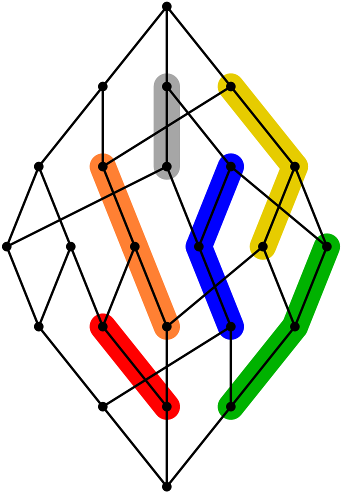

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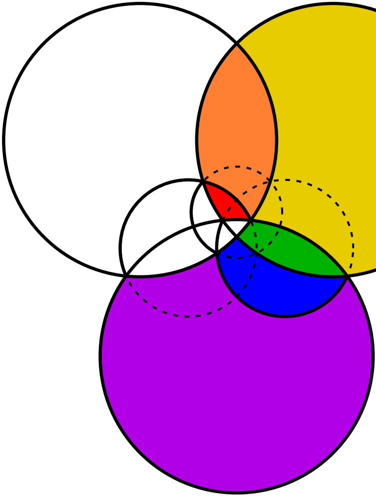

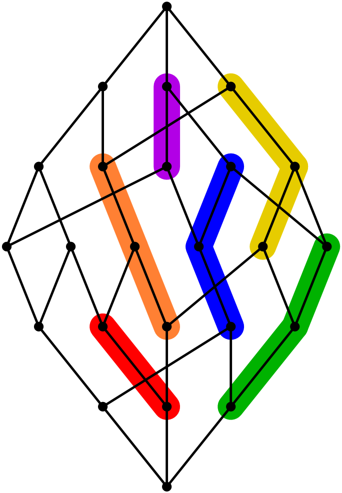

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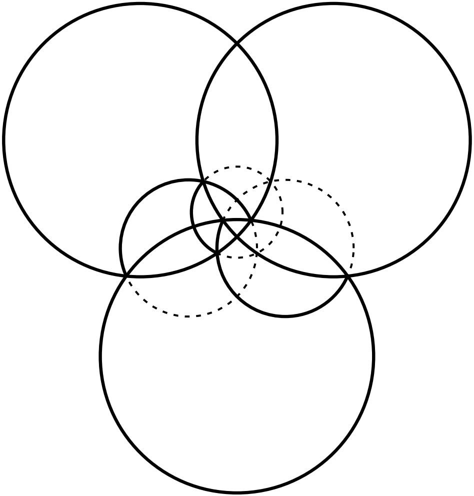
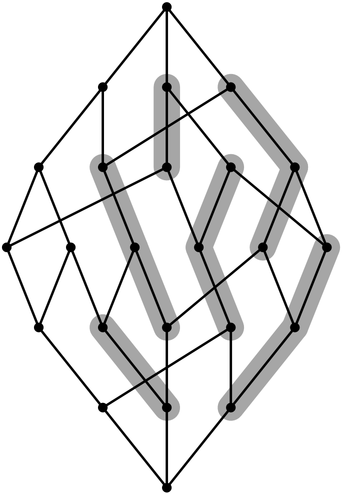
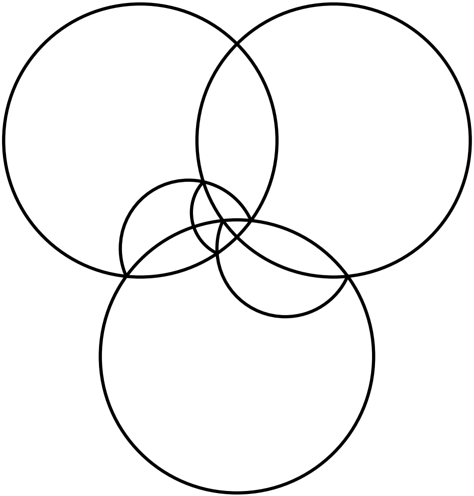
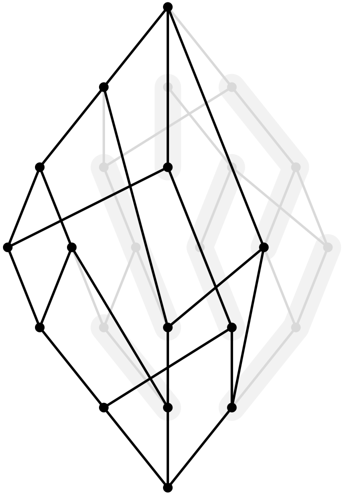




Example: $\mathcal{F}_\Theta$ for a congruence on the weak order on S_4

$\mathcal{F}_\Theta$ = normal fan of associahedron. $\mathcal{P}(\mathcal{A}, B)/\Theta$ = Tamari lattice.





Why $\mathcal{F}_\Theta$ is a fan

A complete fan $\mathcal{F}'$ coarsens a complete fan $\mathcal{F}$ if each face of $\mathcal{F}'$ is a union of faces of $\mathcal{F}$.

Adjacency graph $\mathcal{G}$ of $\mathcal{F}$:

Vertices are the full-dimensional cones of $\mathcal{F}$

Edges are the pairs of adjacent full-dimensional cones

A fan $\mathcal{F}'$ coarsening $\mathcal{F}$ is determined by its edge set: the set of edges connecting adjacent full-dimensional cones of $\mathcal{F}$ that are contained in the same face of $\mathcal{F}'$.

One can show that $\mathcal{F}_\Theta$ is a fan as a special case of a characterization of which edge sets correspond to fan coarsenings.

The characterization is very general (coarsenings of polytopal complexes), but we'll phrase it for fans coming from hyperplane arrangements.

Why $\mathcal{F}_\Theta$ is a fan (continued)

When $\mathcal{F}$ comes from a hyperplane arrangement, $\mathcal{G}$ is the vertex-edge graph of the zonotope $\mathcal{Z}$ dual to F.

The polygon property of a set $\mathcal{E}$ of edges of $\mathcal{G}$:

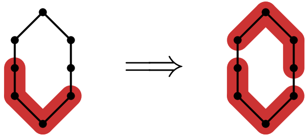
For every $2k$-gonal face P of $\mathcal{Z}$, whenever $\mathcal{E}$ contains any $k-1$ consecutive edges of P, then $\mathcal{E}$ also contains the opposite $k-1$ consecutive edges of P.

Theorem (R., 2010). Let $\mathcal{Z}$ be a zonotope and let $\mathcal{F}$ be the normal fan of $\mathcal{Z}$. Then a set $\mathcal{E}$ of edges of $\mathcal{Z}$ is the edge set of a fan coarsening $\mathcal{F}$ if and only if $\mathcal{E}$ has the polygon property.

The special case where $\mathcal{Z}$ is the permutohedron is due to Morton, Pachter, Shiu, Sturmfels, and Wienand.

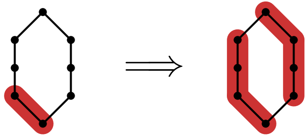
Why $\mathcal{F}_\Theta$ is a fan (continued)

Polygon property: For every $2k$-gonal face P of $\mathcal{Z}$, whenever $\mathcal{E}$ contains any $k-1$ consecutive edges of P, then $\mathcal{E}$ also contains the opposite $k-1$ consecutive edges of P.



Theorem (repeated). A set $\mathcal{E}$ of edges of $\mathcal{Z}$ is the edge set of a fan coarsening $\mathcal{F}$ if and only if $\mathcal{E}$ has the polygon property.

Forcing says:



Conclude: If $\mathcal{E}$ is chosen by edge forcing, $\mathcal{F}'$ is a fan coarsening $\mathcal{F}$. That is, $\mathcal{F}_\Theta$ is a fan coarsening $\mathcal{F}$.

Recap of Section III.a: Lattice congruences and fans

When $\mathcal{A}$ is simplicial, $\mathcal{P}(\mathcal{A}, B)$ is a polygonal lattice.

$\mathcal{F}$ is the simplicial fan defined by $\mathcal{A}$ (maximal cones are regions).

Given a congruence Θ, for each Θ-class, take the union of the corresponding regions.

These unions are the maximal cones of a complete fan $\mathcal{F}_\Theta$ that coarsens $\mathcal{F}$.

Questions?

Section III.b: Cambrian fans

Cambrian fan

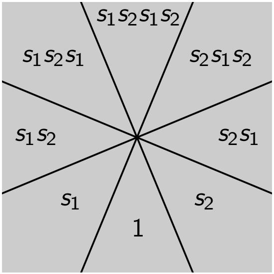
The Cambrian fan $\mathcal{F}_c$ is $\mathcal{F}_{\Theta_c}$ where Θ_c is the Cambrian congruence. That is, maximal cones are unions (over Θ_c-classes) of maximal cones of the Coxeter fan.

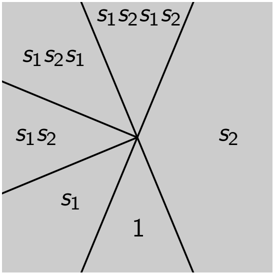
Theorem (R., Speyer, 2006).
The bijection $\mathrm{cl}_c : \{c\text{-sortables}\} \to \{\text{clusters}\}$ induces a combinatorial isomorphism between the Cambrian fan $\mathcal{F}_c$ and the normal fan of the generalized associahedron.

Theorem (Hohlweg, Lange, Thomas, 2010). The Cambrian fan $\mathcal{F}_c$ is the normal fan of a realization of the generalized associahedron. (They gave an explicit construction.)

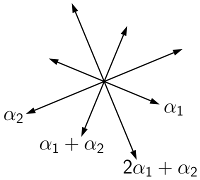
Example ($W = B_2$, $c = s_1 s_2$)

$\mathcal{F}$

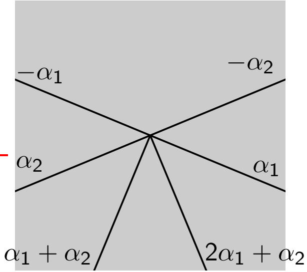


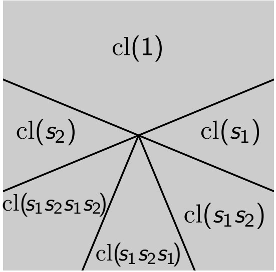


$\mathcal{F}_c$

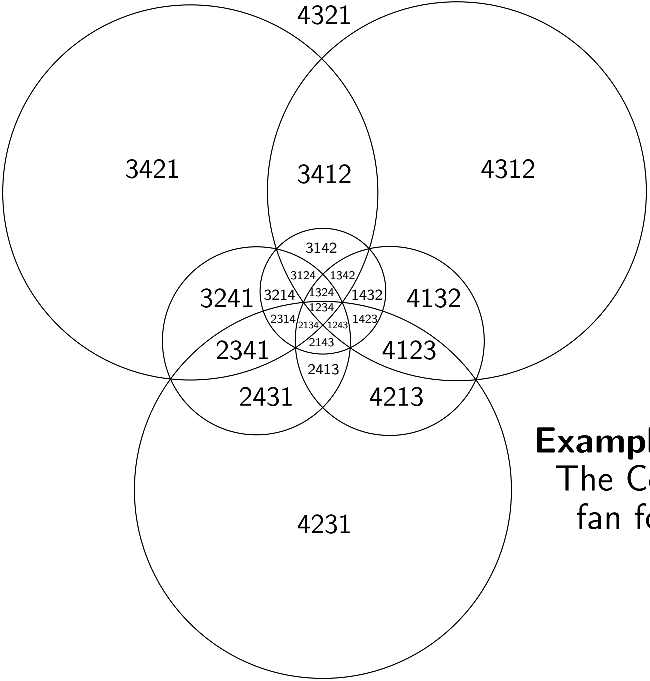


Normal fan of associa-hedron

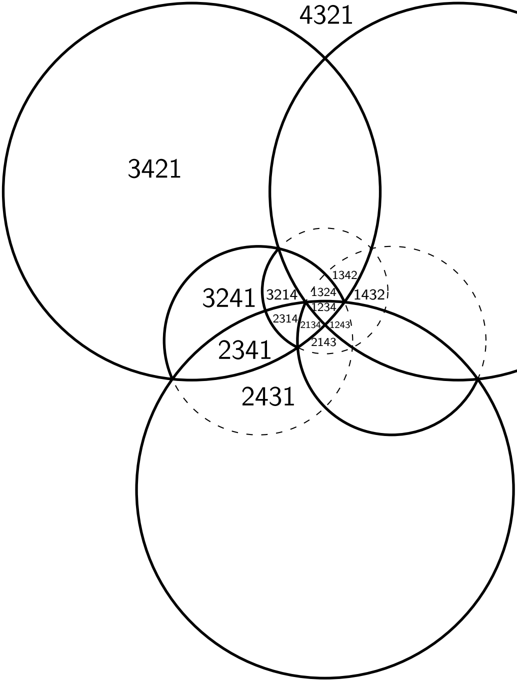




The bijection

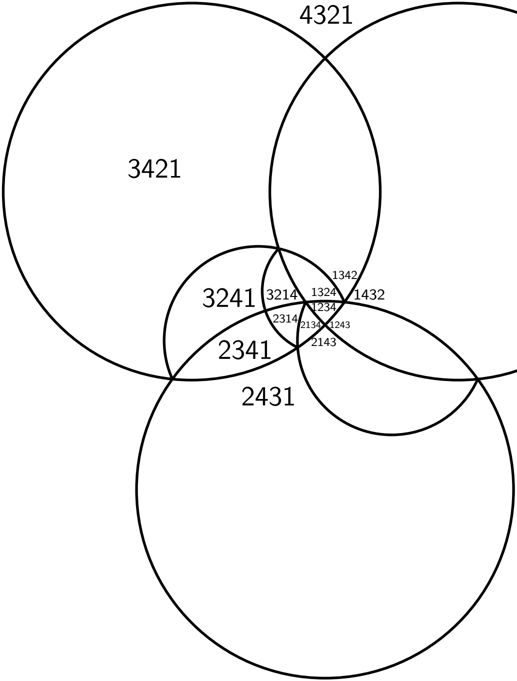


Example:
The Coxeter fan for S_4



A Cambrian fan for S_4

In this case, sortable means 312-avoiding.



A Cambrian fan for S_4

In this case, sortable means 312-avoiding.

Cambrian fans from the combinatorics of sortable elements

v: a c-sortable element of W with c-sorting word $a_1 \cdots a_k$.

Recall: $c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$

For each $s_i \in S$, there is a leftmost instance of s_i in c^∞ which is not in the subword of c^∞ corresponding to $a_1 \cdots a_k$.

Let $a_1 \cdots a_j$ be the initial segment of $a_1 \cdots a_k$ consisting of those letters that occur in c^∞ before the omission of s_i. Say $a_1 \cdots a_k$ skips s_i after $a_1 \cdots a_j$.

Define: $C_c^{s_i}(v) = a_1 \cdots a_j \cdot \alpha_i$.

$C_c(v) = \{C_c^{s_i} : s_i \in S\}$.

$\mathrm{Cone}_c(v) = \{\mathbf{x} \in (\mathbb{R}^n)^* : \langle \mathbf{x}, C_c^{s_i}(v) \rangle \geq 0 \, \forall s_i \in S\}$.

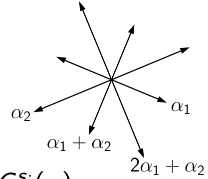
Theorem (R., Speyer, 2014). The maximal cones of the Cambrian fan $\mathcal{F}_c$ are $\mathrm{Cone}_c(v)$ as v runs over all c-sortable elements.

Skips example: $W = B_2, \quad c = s_1 s_2$

$c^\infty = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \cdots$

c-sortable: $1, \ s_1, \ s_1 s_2, \ s_1 s_2 | s_1, \ s_1 s_2 | s_1 s_2, \ s_2$

not c-sortable: $s_2 | s_1, \ s_2 | s_1 s_2$



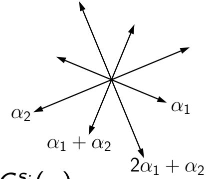
v	s_i	skip	$C_c^{s_i}(v)$
1	s_1		
	s_2		
s_1	s_1		
	s_2		
$s_1 s_2$	s_1		
	s_2		
$s_1 s_2 s_1$	s_1		
	s_2		
$s_1 s_2 s_1 s_2$	s_1		
	s_2		
s_2	s_1		
	s_2		

Skips example: $W = B_2, \quad c = s_1 s_2$

$c^\infty = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \cdots$

c-sortable: $1,\ s_1,\ s_1 s_2,\ s_1 s_2 | s_1,\ s_1 s_2 | s_1 s_2,\ s_2$

not c-sortable: $s_2 | s_1,\ s_2 | s_1 s_2$



v	s_i	skip	$C_c^{s_i}(v)$
1	s_1	unforced (s_1 reduced)	α_1
	s_2		
s_1	s_1		
	s_2		
$s_1 s_2$	s_1		
	s_2		
$s_1 s_2 s_1$	s_1		
	s_2		
$s_1 s_2 s_1 s_2$	s_1		
	s_2		
s_2	s_1		
	s_2		

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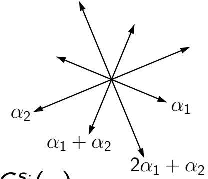

v	s_i	skip	$C_c^{s_i}(v)$
1	s_1	unforced (s_1 reduced)	α_1
	s_2	unforced (s_2 reduced)	α_2
s_1	s_1		
	s_2		
$s_1 s_2$	s_1		
	s_2		
$s_1 s_2 s_1$	s_1		
	s_2		
$s_1 s_2 s_1 s_2$	s_1		
	s_2		
s_2	s_1		
	s_2		

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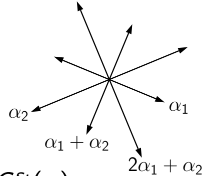

v	s_i	skip		$C_c^{s_i}(v)$
1	s_1	unforced	(s_1 reduced)	α_1
	s_2	unforced	(s_2 reduced)	α_2
s_1	s_1	forced	(s_1s_1 not reduced)	$-\alpha_1$
	s_2			
s_1s_2	s_1			
	s_2			
$s_1s_2s_1$	s_1			
	s_2			
$s_1s_2s_1s_2$	s_1			
	s_2			
s_2	s_1			
	s_2			

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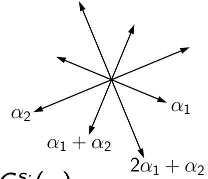
not c-sortable: $s_2|s_1,\ s_2|s_1s_2$



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	s_2	unforced	(s_2 reduced)	α_2
s_1	s_1	forced	(s_1s_1 not reduced)	$-\alpha_1$
	s_2	unforced	(s_1s_2 reduced)	$2\alpha_1 + \alpha_2$
s_1s_2	s_1			
	s_2			
$s_1s_2s_1$	s_1			
	s_2			
$s_1s_2s_1s_2$	s_1			
	s_2			
s_2	s_1			
	s_2			

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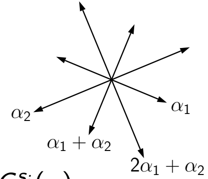

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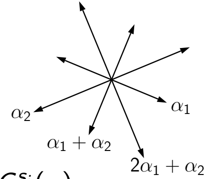
	s_2	unforced	(s_2 reduced)	α_2
s_1	s_1	forced	($s_1 s_1$ not reduced)	$-\alpha_1$
	s_2	unforced	($s_1 s_2$ reduced)	$2\alpha_1 + \alpha_2$
$s_1 s_2$	s_1	unforced	($s_1 s_2 s_1$ reduced)	$\alpha_1 + \alpha_2$
	s_2			
$s_1 s_2 s_1$	s_1			
	s_2			
$s_1 s_2 s_1 s_2$	s_1			
	s_2			
s_2	s_1			
	s_2			

Skips example: $W = B_2, \quad c = s_1 s_2$

$c^\infty = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \cdots$

c-sortable: $1,\ s_1,\ s_1s_2,\ s_1s_2|s_1,\ s_1s_2|s_1s_2,\ s_2$

not c-sortable: $s_2|s_1,\ s_2|s_1s_2$



v	s_i	skip		$C_c^{s_i}(v)$
1	s_1	unforced	(s_1 reduced)	α_1
	s_2	unforced	(s_2 reduced)	α_2
s_1	s_1	forced	(s_1s_1 not reduced)	$-\alpha_1$
	s_2	unforced	(s_1s_2 reduced)	$2\alpha_1 + \alpha_2$
s_1s_2	s_1	unforced	($s_1s_2s_1$ reduced)	$\alpha_1 + \alpha_2$
	s_2	forced	($s_1s_2s_2$ not reduced)	$-2\alpha_1 - \alpha_2$
$s_1s_2s_1$	s_1			
	s_2			
$s_1s_2s_1s_2$	s_1			
	s_2			
s_2	s_1			
	s_2			

Skips example: $W = B_2, \quad c = s_1 s_2$

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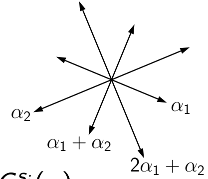

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	s_2	unforced	($s_1 s_2$ reduced)	$2\alpha_1 + \alpha_2$
$s_1 s_2$	s_1	unforced	($s_1 s_2 s_1$ reduced)	$\alpha_1 + \alpha_2$
	s_2	forced	($s_1 s_2 s_2$ not reduced)	$-2\alpha_1 - \alpha_2$
$s_1 s_2 s_1$	s_1	forced	($s_1 s_2 s_1 s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2			
$s_1 s_2 s_1 s_2$	s_1			
	s_2			
s_2	s_1			
	s_2			

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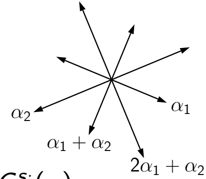

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$s_1 s_2$	s_1	unforced	($s_1 s_2 s_1$ reduced)	$\alpha_1 + \alpha_2$
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$s_1 s_2 s_1$	s_1	forced	($s_1 s_2 s_1 s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2	unforced	($s_1 s_2 s_1 s_2$ reduced)	α_2
$s_1 s_2 s_1 s_2$	s_1			
	s_2			
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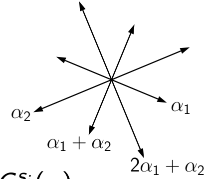

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	s_2	unforced	($s_1 s_2$ reduced)	$2\alpha_1 + \alpha_2$
$s_1 s_2$	s_1	unforced	($s_1 s_2 s_1$ reduced)	$\alpha_1 + \alpha_2$
	s_2	forced	($s_1 s_2 s_2$ not reduced)	$-2\alpha_1 - \alpha_2$
$s_1 s_2 s_1$	s_1	forced	($s_1 s_2 s_1 s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2	unforced	($s_1 s_2 s_1 s_2$ reduced)	α_2
$s_1 s_2 s_1 s_2$	s_1	forced	($s_1 s_2 s_1 s_2 s_1$ not reduced)	$-\alpha_1$
	s_2			
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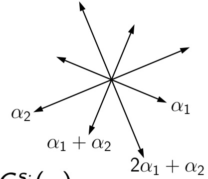

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$s_1s_2s_1$	s_1	forced	($s_1s_2s_1s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2	unforced	($s_1s_2s_1s_2$ reduced)	α_2
$s_1s_2s_1s_2$	s_1	forced	($s_1s_2s_1s_2s_1$ not reduced)	$-\alpha_1$
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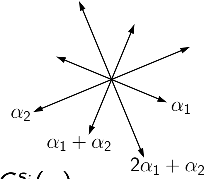

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$s_1 s_2 s_1$	s_1	forced	($s_1 s_2 s_1 s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2	unforced	($s_1 s_2 s_1 s_2$ reduced)	α_2
$s_1 s_2 s_1 s_2$	s_1	forced	($s_1 s_2 s_1 s_2 s_1$ not reduced)	$-\alpha_1$
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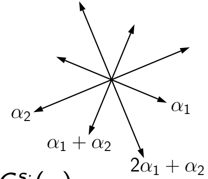

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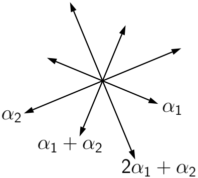
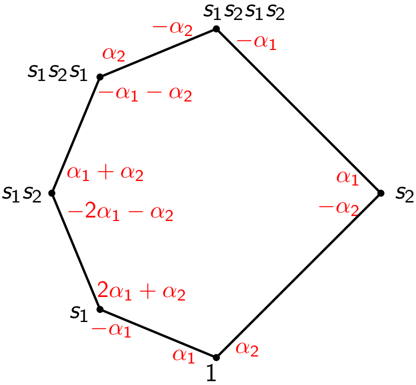
s_2	s_1	unforced	(s_1 reduced)	α_1
	s_2	forced	($s_2 s_2$ not reduced)	$-\alpha_2$

The Cambrian fan by skips

$W = B_2$, $c = s_1 s_2$

$C_c(v)$ shown in red.





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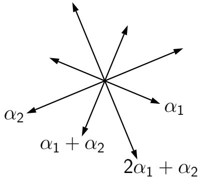

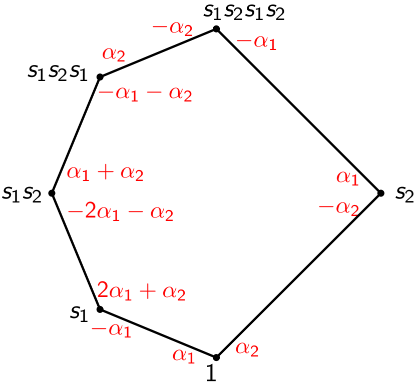

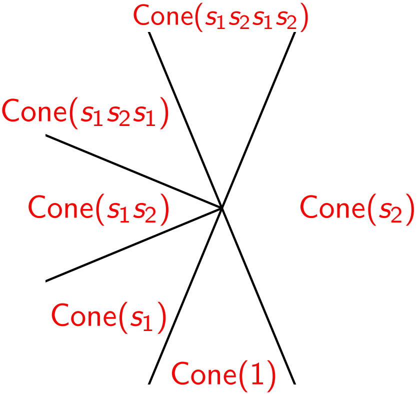

Recap of Section III.b: Cambrian fans

The Cambrian fan is $\mathcal{F}_c = \mathcal{F}_{\Theta_c}$ for Θ_c the c-Cambrian congruence.

It is the normal fan of a generalized associahedron.

Its geometry can be read off from the combinatorics of c-sortable elements (skips in c-sorting words).

Questions?

Section III.c: Shards

Recall: Combinatorial models

When we talked about noncrossing arc diagrams, we said we wanted a combinatorial model for congruences. Specifically, we wanted a set of objects

- in bijection with join-irreducible elements of W.
- with a compatibility relation modeling edges of the CJC (so pairwise compatible sets of objects are in bijection with W).
- with forcing among j.i. elements read off combinatorially.

Recall: Combinatorial or geometric models

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- with a compatibility relation modeling edges of the CJC (so pairwise compatible sets of objects are in bijection with W).
- with forcing among j.i. elements read off combinatorially.

Now we'll consider a general geometric model based on shards.

(It's not so un-combinatorial... In some sense noncrossing diagrams are shards in type A.)

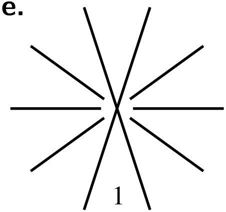
What shards are

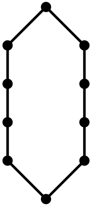
To make the fan $\mathcal{F}_\Theta$ for a congruence Θ on the weak order, we glue cones of the Coxeter fan together according to congruence classes.

So: contracting an edge means removing the wall between two adjacent cones.

A shard is (the union of) a maximal collections of walls which must always be removed together in a lattice congruence. Each shard turns out to consist of walls all in the same hyperplane.

Example.





We describe a congruence by specifying which shards are removed. Edge-forcing also implies some forcing relations among shards.

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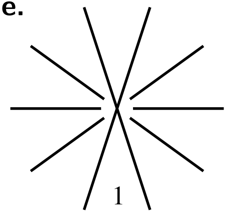
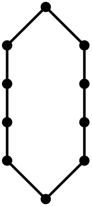
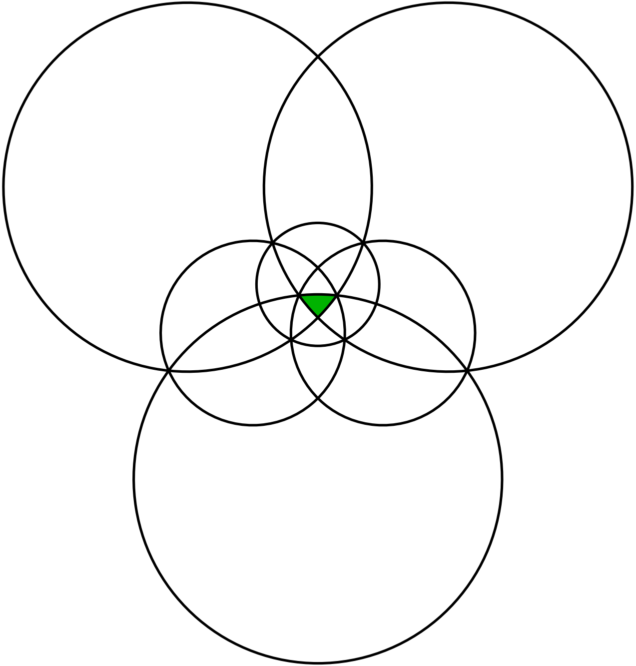
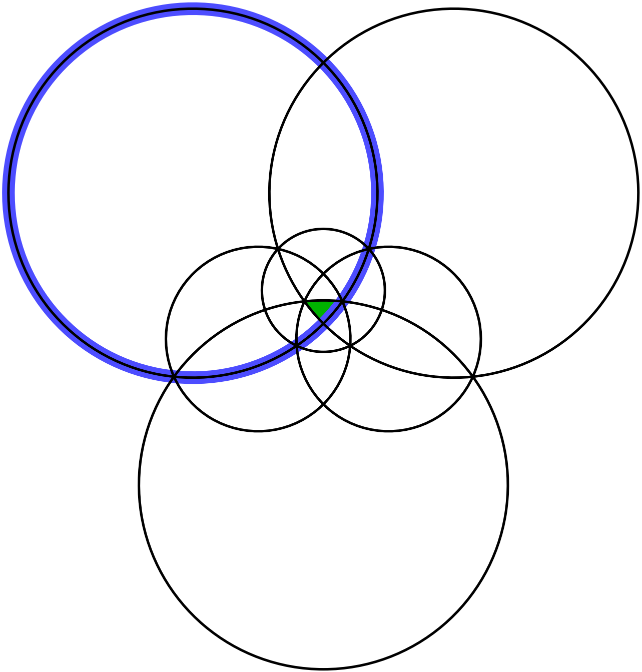


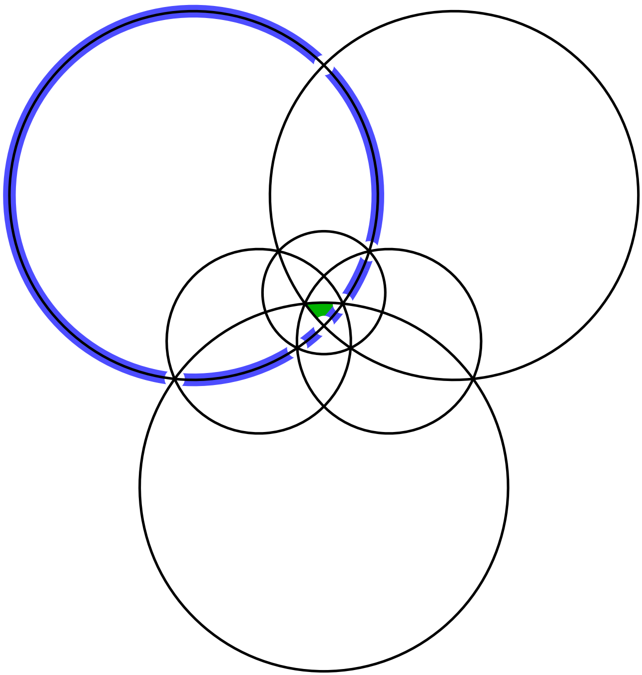


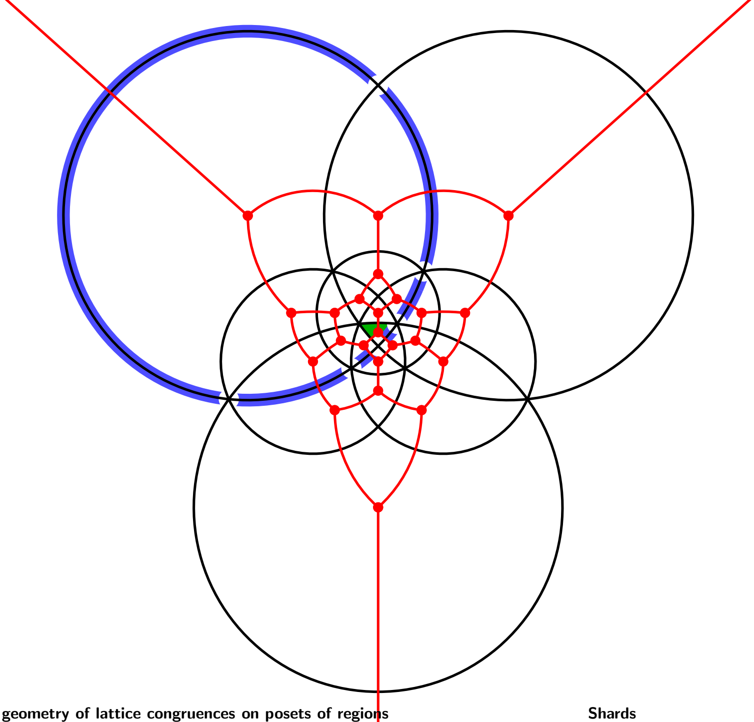
*Hypotheses: Weak order on a finite Coxeter group or congruence uniform poset of regions of a simplicial arrangement.

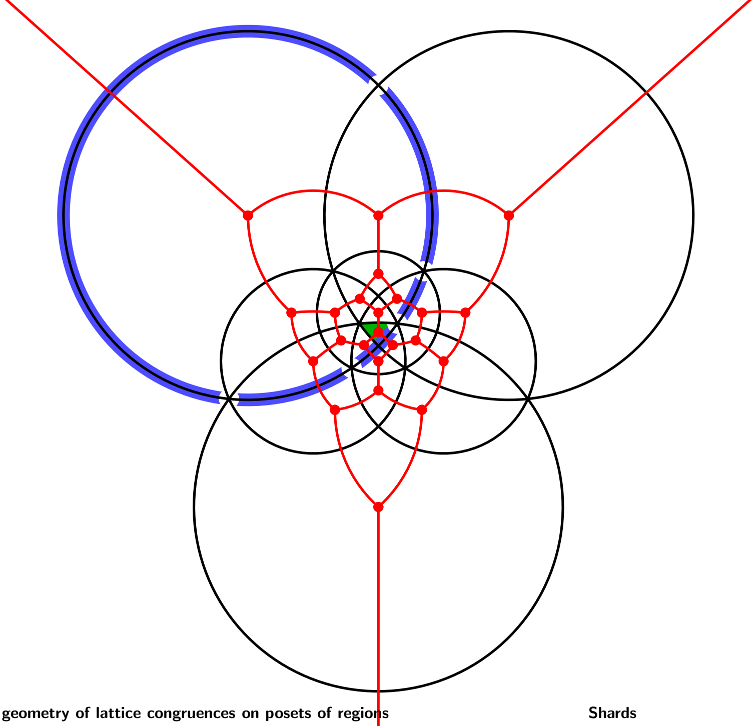
We describe a congruence by specifying which shards are removed. Edge-forcing also implies some forcing relations among shards.

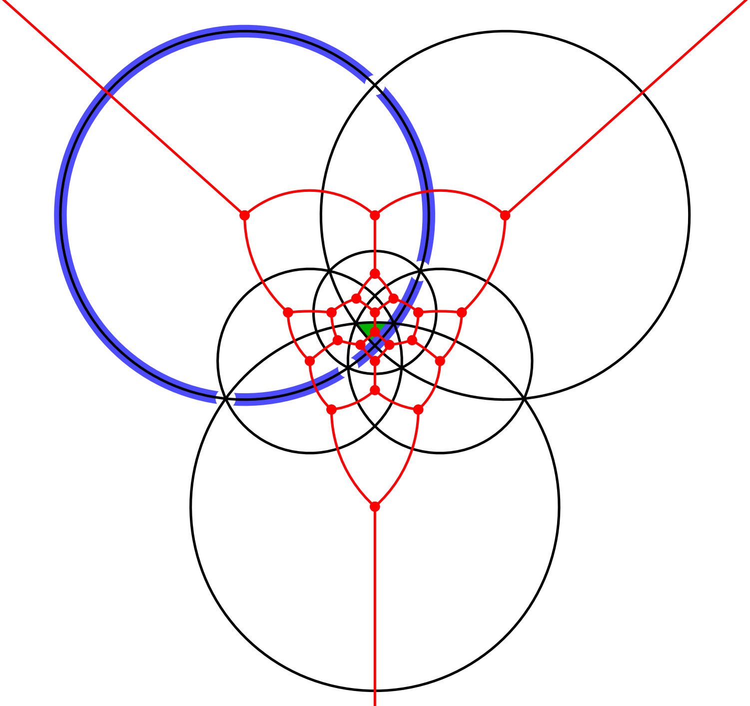


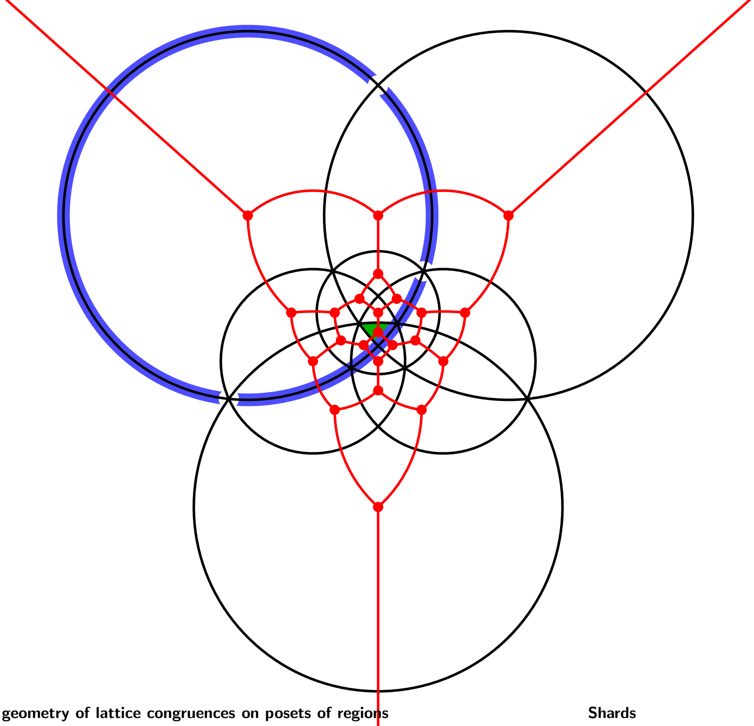


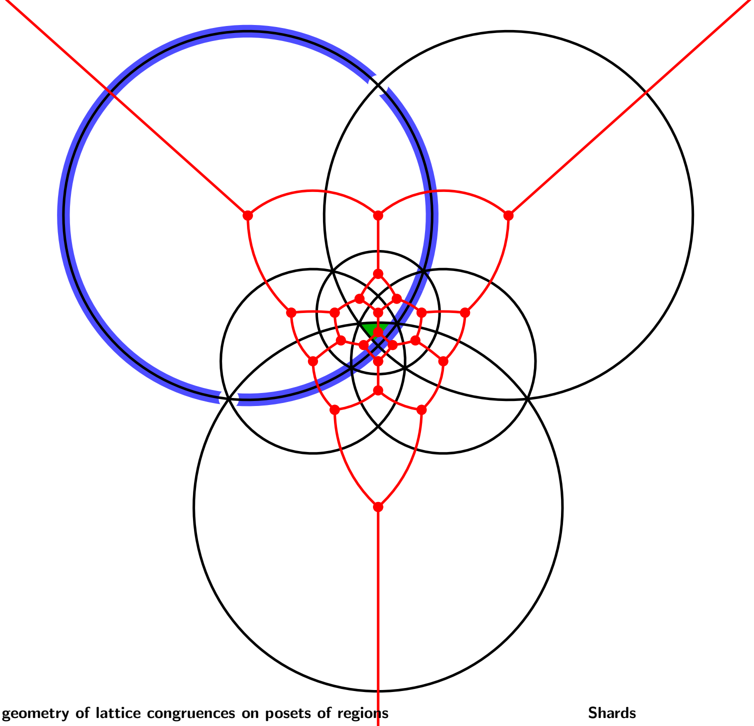


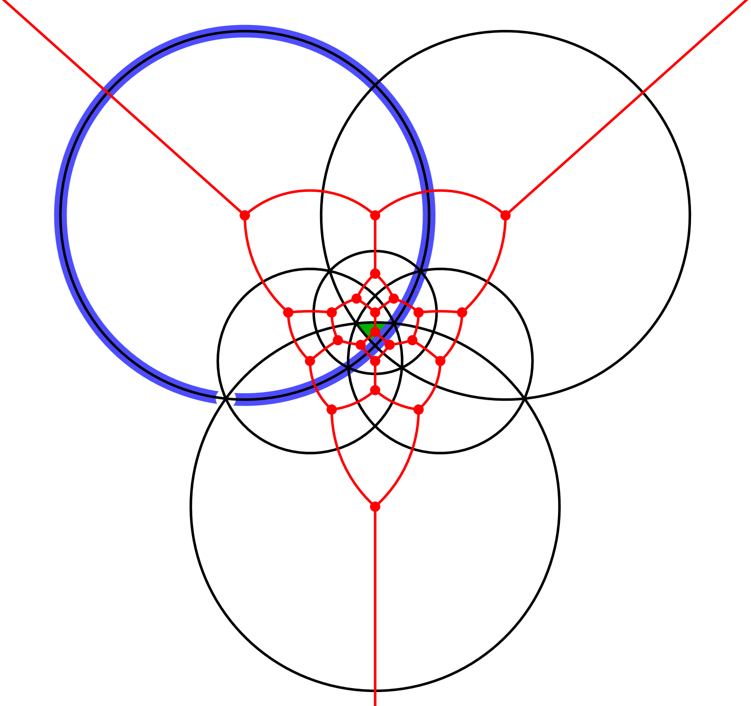


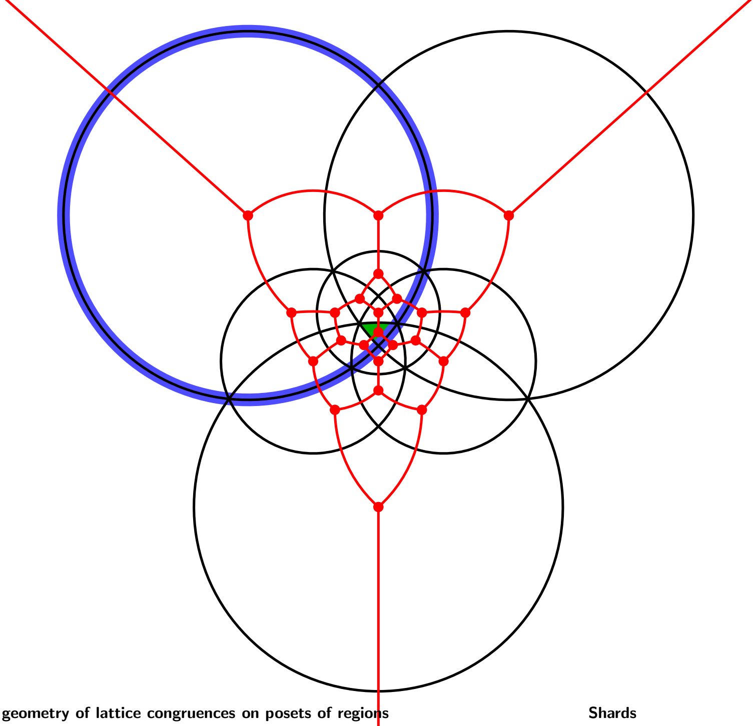


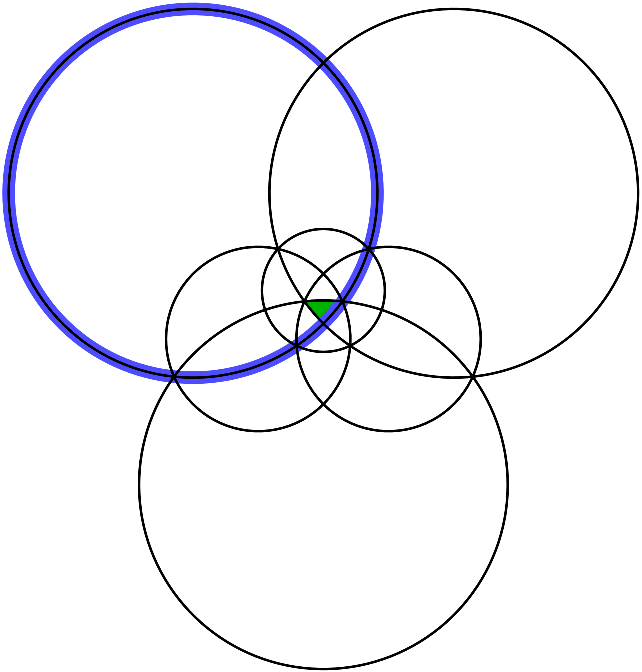


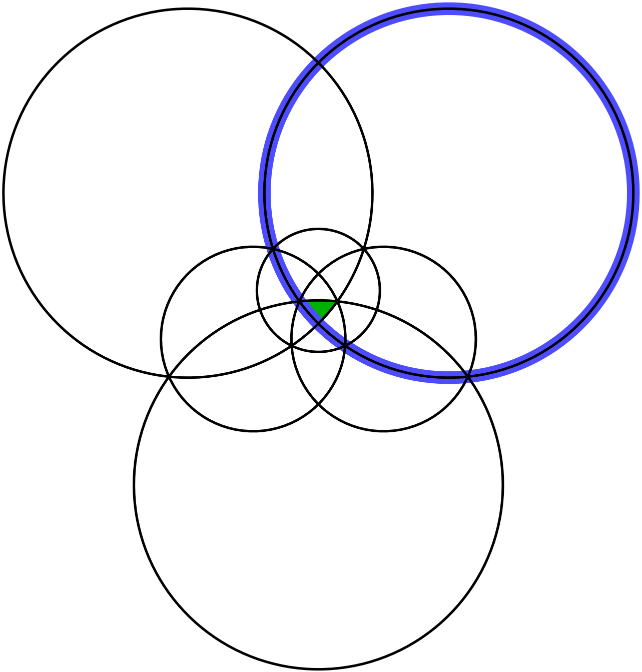


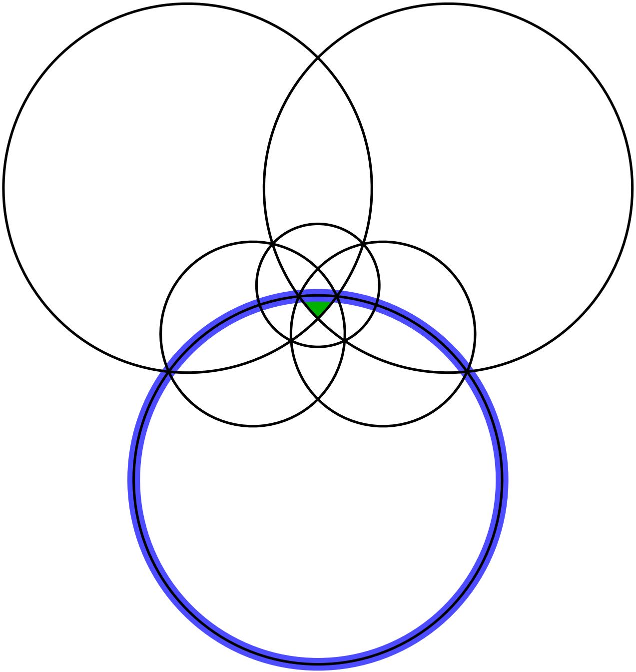


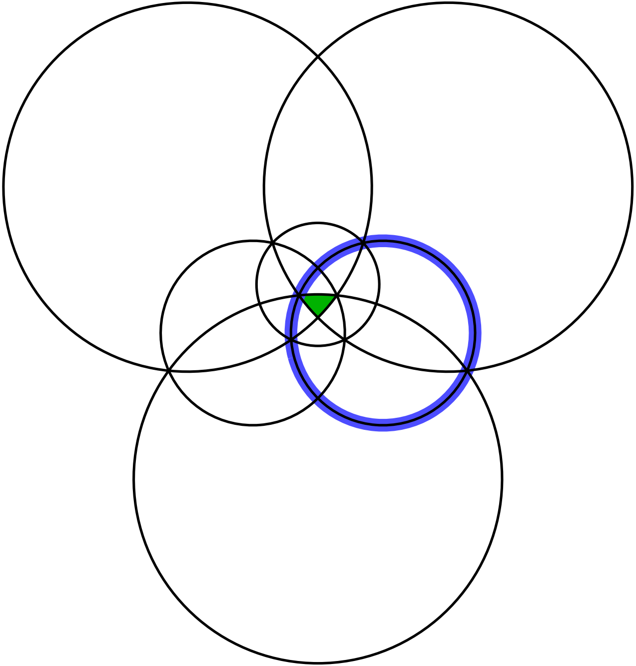


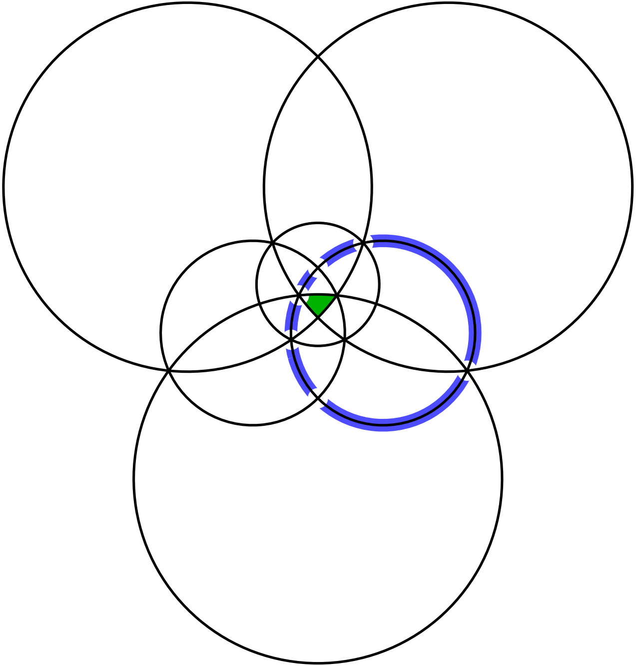


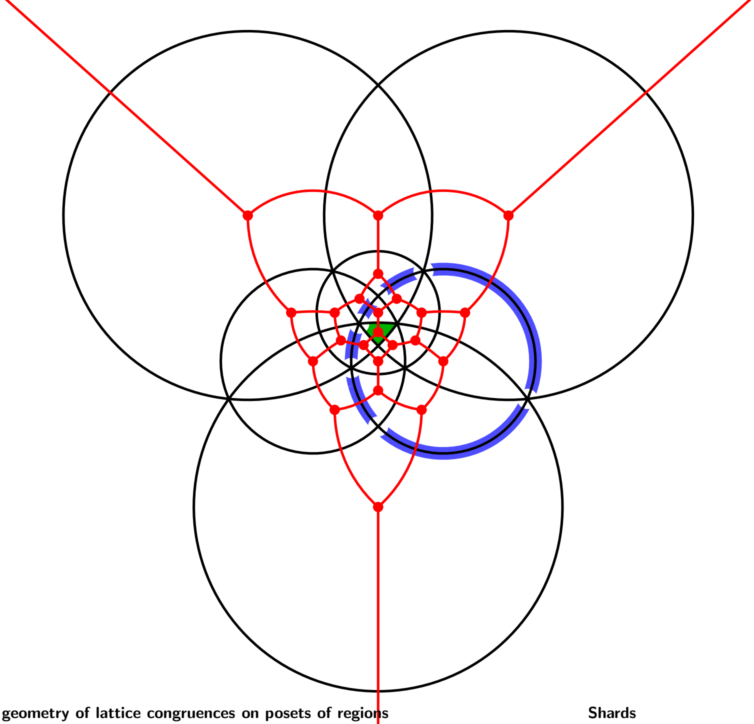


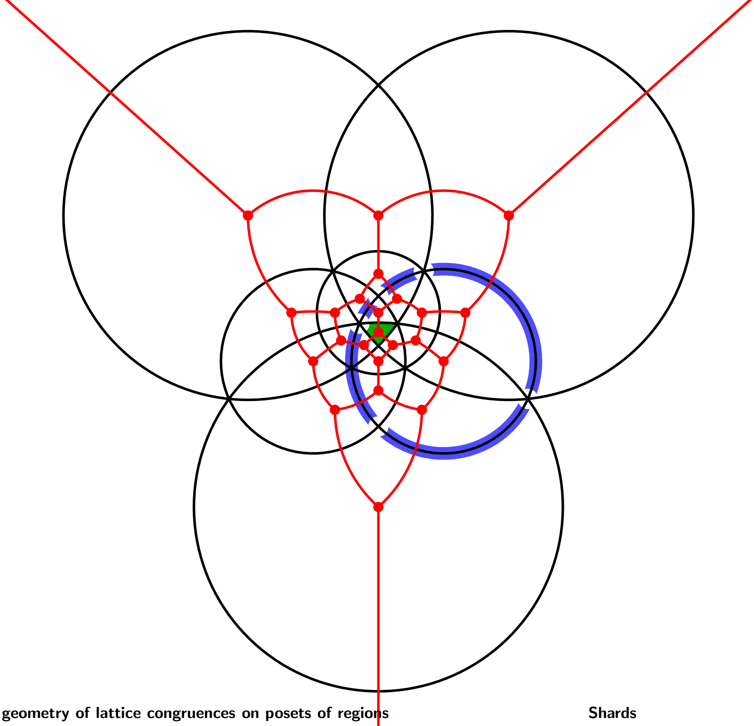


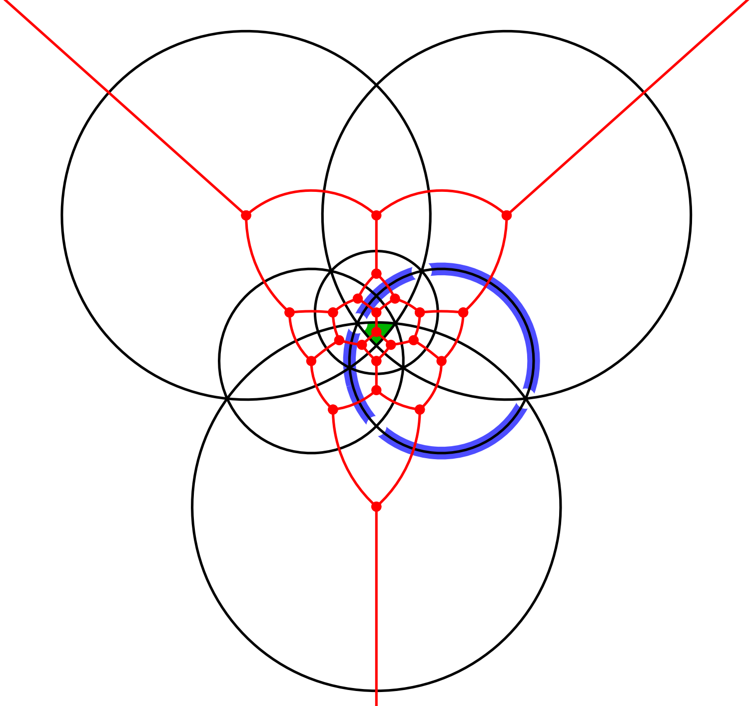


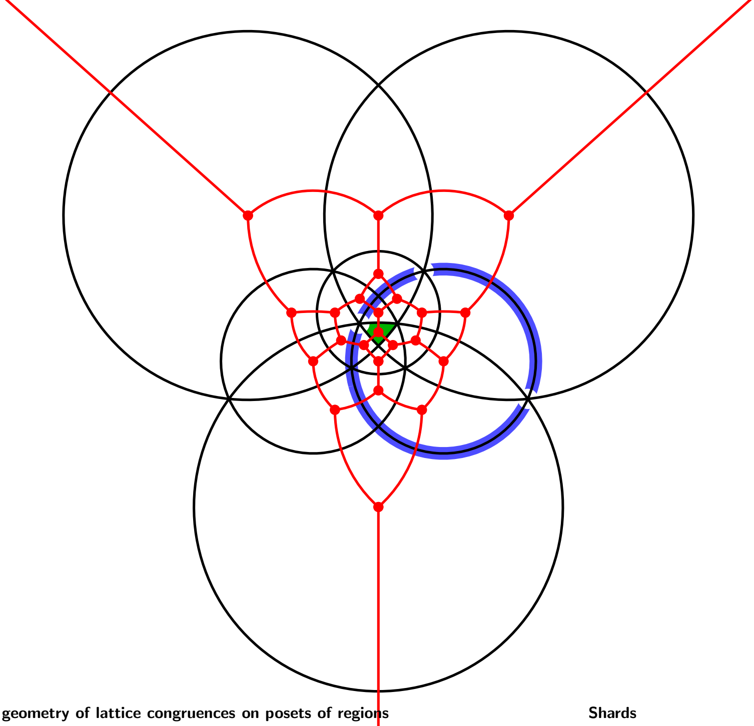


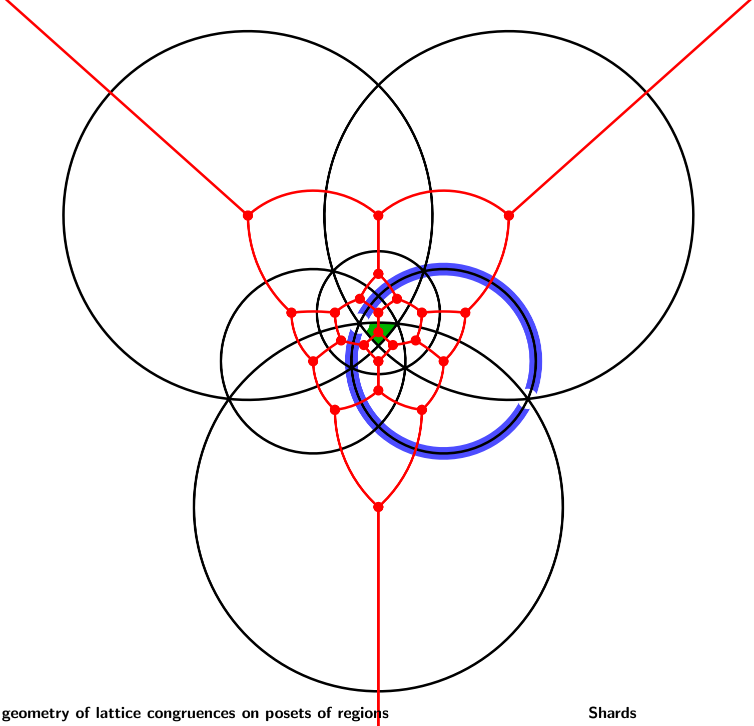


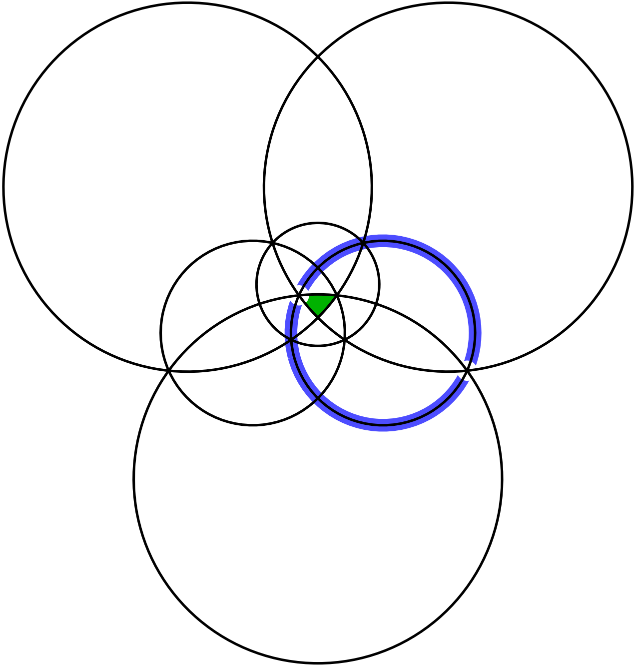


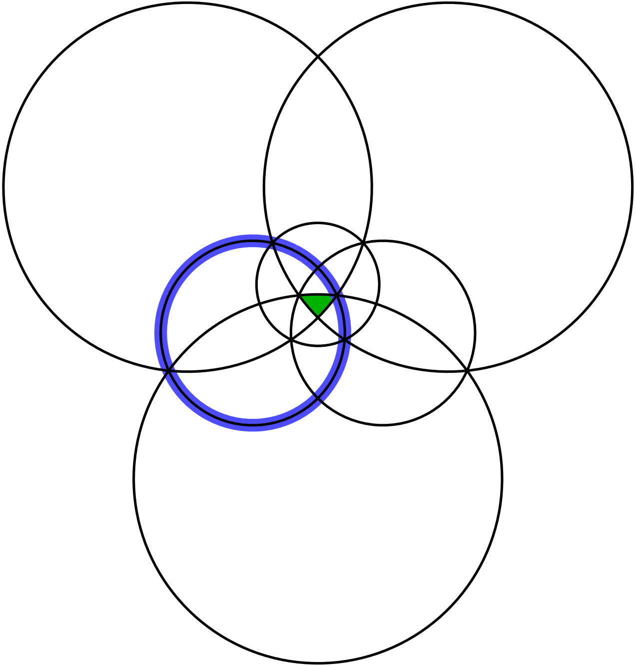


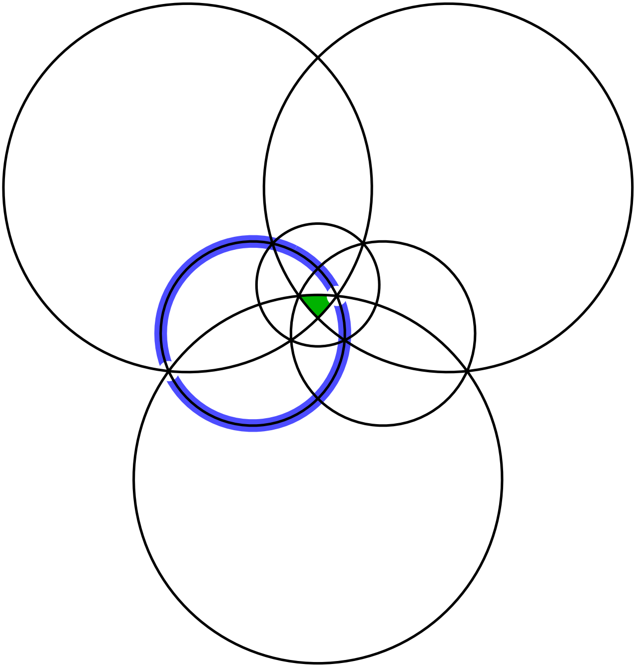


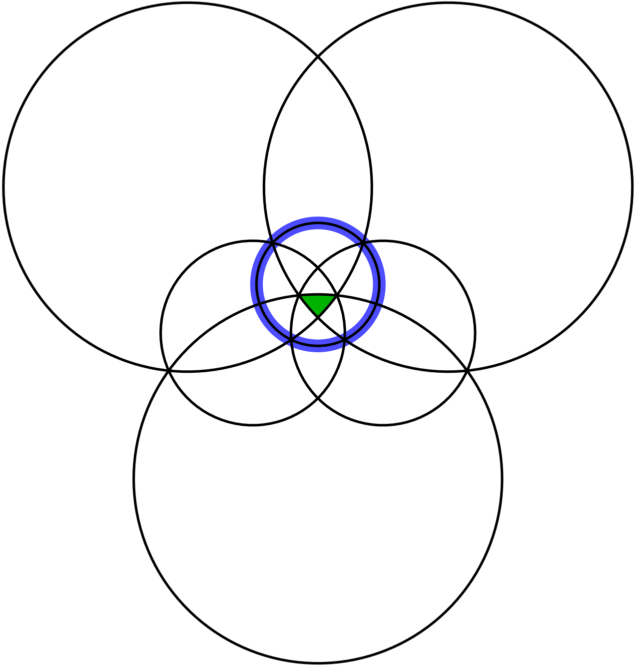


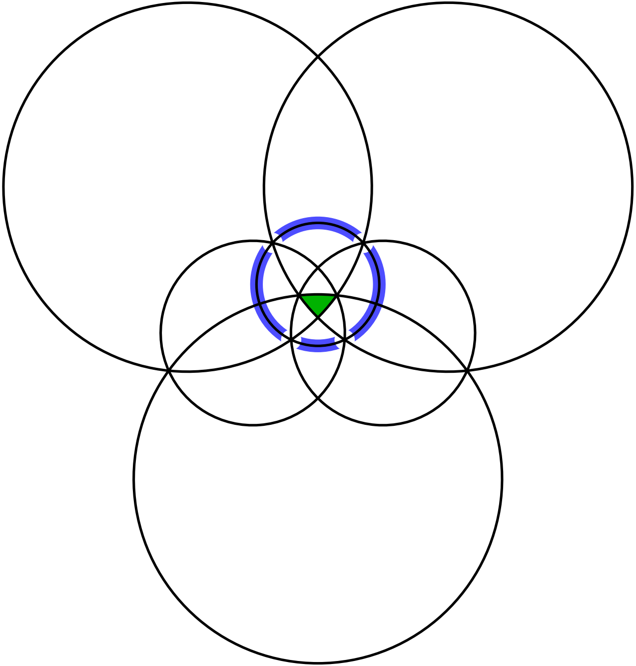


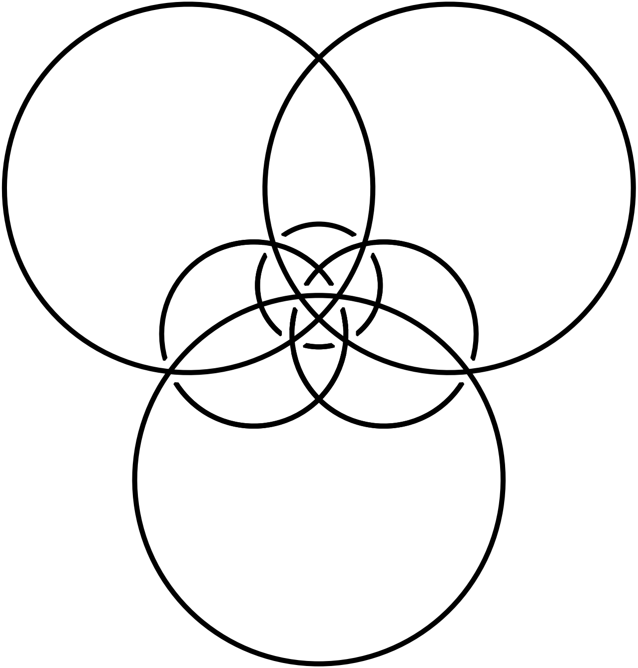






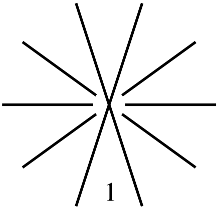






Shards, defined purely geometrically

Shards in a dihedral (or "rank 2") Coxeter group: The two hyperplanes bounding the "identity region" are not cut. The remaining hyperplanes are cut in half.



Important technical point: all of the shards contain the origin. We "cut" along the intersection of the hyperplanes, then take closures of the pieces.

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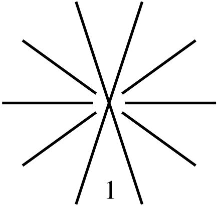

Important technical point: all of the shards contain the origin. We "cut" along the intersection of the hyperplanes, then take closures of the pieces. Technical point about to disappear...

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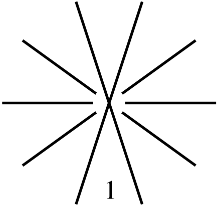

In higher ranks, we do this cutting in every rank-2 subarrangement.

Why is this the same as the definition by "maximal collections of walls which must always be removed together?" Because the lattice is polygonal!

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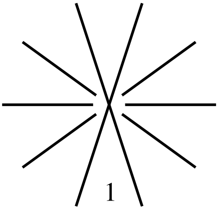
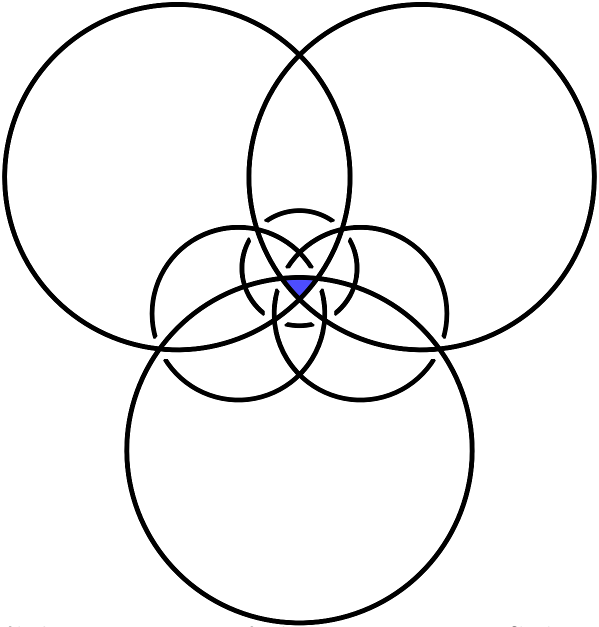


In higher ranks, we do this cutting in every rank-2 subarrangement.

Why is this the same as the definition by "maximal collections of walls which must always be removed together?" Because the lattice is polygonal!

Again, need weak order or the congruence uniform simplicial case for these two definitions to coincide.

Shards in S_4

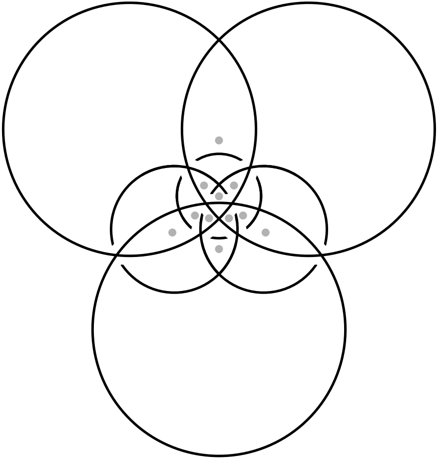


Shards are in bijection with join-irreducible congruences. (Maximal collections of walls which must always be removed together are maximal collections of edges that must be contracted together!)

Again, this is for weak order or congruence uniform simplicial posets of regions. So also shards are in bijection with j.i. elements.

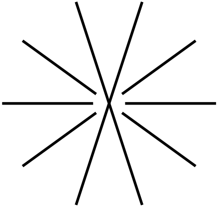
In the non congruence uniform case, the bijection from shards to j.i. elements still works, taking the geometric definition of shards.

In any case, the bijection sends a shard to the lowest region above that shard.



Compatibility of shards

Two shards are compatible (i.e. form an edge in the canonical join complex) if and only if their relative interiors intersect.



We know that the CJC is flag. Therefore, faces of the CJC are sets of shards that pairwise intersect in their relative interiors.

Forcing among shards

(i.e. removing one shard forces removal of others)

Facets of shards are maximal proper faces of the shards (codimension 2 in the ambient space).

The shard digraph: $\Sigma_1 \to \Sigma_2$ iff Σ_1 has a codimension-2 (in the ambient space) intersection with a facet of Σ_2. This can only happen if Σ_1 is in a hyperplane that "cuts" Σ_2 to create that facet.

A shard Σ forces another shard Σ' if and only if there is a directed path from Σ to Σ' in the shard digraph. (Again, because the lattice is polygonal!)

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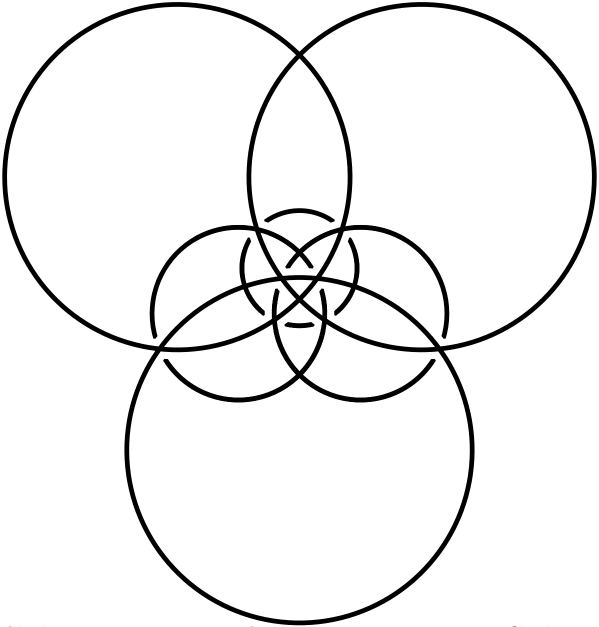
A shard Σ forces another shard Σ' if and only if there is a directed path from Σ to Σ' in the shard digraph. (Again, because the lattice is polygonal!)

Some simplicial hyperplane arrangements have non congruence uniform posets of regions: These are the cases where the shard digraph has oriented cycles.

The proof that the weak order is congruence uniform consists of showing that its shard digraph has no directed cycles.

Shard removal, forcing and fans in S_4



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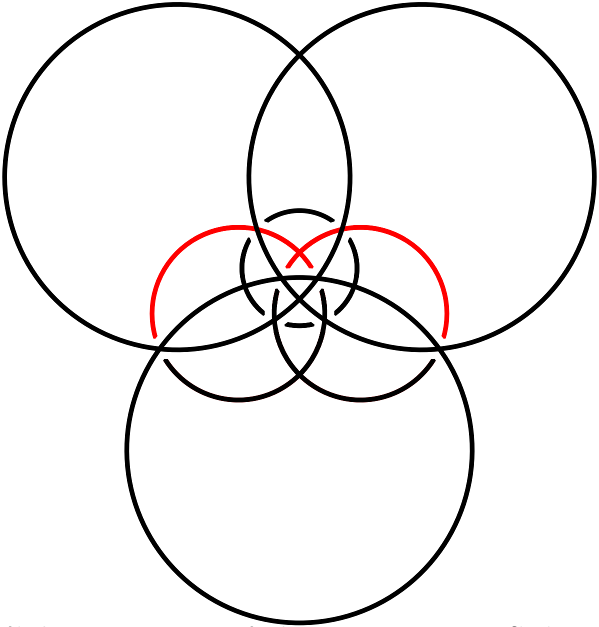

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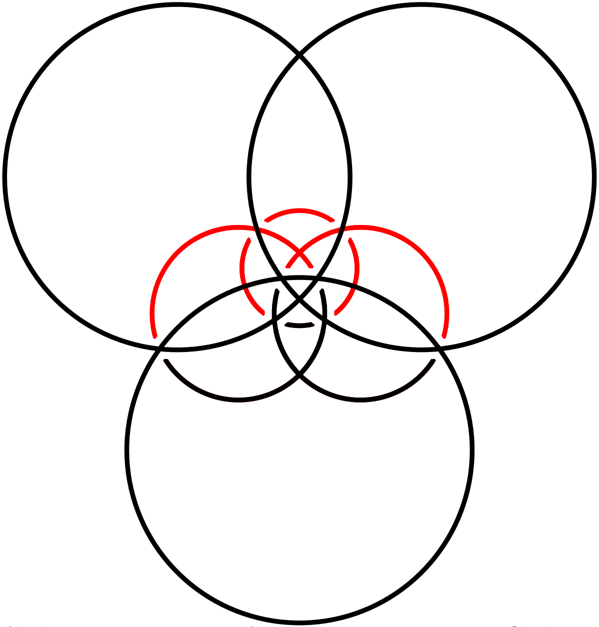

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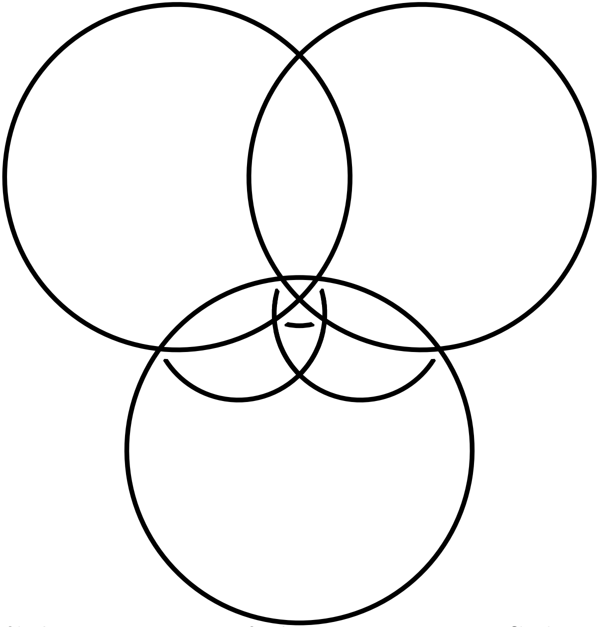
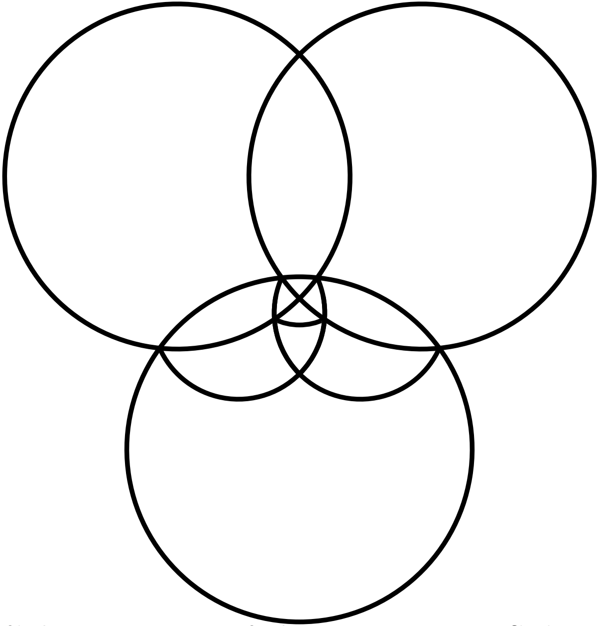


Shard removal, forcing and fans in S_4 (A Cambrian fan)



Recap of Section III.c: Shards

Shards are pieces of hyperplanes that constitute a model for join-irreducible elements and forcing in simplicial hyperplane arrangements.

Compatibility of shards means intersecting in their relative interiors.

Forcing is described in terms of incidence relations among shards.

Questions?

Section III.d: The shard intersection order

The shard intersection order

Initial motivation: The lattice property for the noncrossing partition lattice was first proved uniformly by Brady and Watt (2005), and differently (for W crystallographic) Ingalls and Thomas (2006).

Shard intersections give a new proof that $\mathrm{NC}(W)$ is a lattice: Construct a lattice $(W, \preceq)$ on the elements of W, and identify a sublattice of $(W, \preceq)$ isomorphic to $\mathrm{NC}(W)$.

Beyond the initial motivation: $(W, \preceq)$ turns out to have very interesting properties, analogous to the properties of $\mathrm{NC}(W)$.

Proofs are simple and natural in the Coxeter context. (More broadly: in the context of simplicial hyperplane arrangements.)

This approach brings to light how $\mathrm{NC}(W)$ arises naturally in the context of semi-invariants of quivers.

There are intriguing connections to certain "pulling" triangulations of associahedra and permutohedra.

The shard intersection order

Let $\Psi(W)$ be the set of arbitrary intersections of shards.
We partially order this set by reverse containment.

Immediate: $(\Psi(W), \supseteq)$ is a join semilattice. (Join is intersection.) It also has a unique minimal element (the empty intersection, i.e. the ambient vector space), so it is a lattice. Also immediate: $(\Psi(W), \supseteq)$ is atomic.

Less obvious: $(\Psi(W), \supseteq)$ is graded (ranked by codimension) and coatomic.

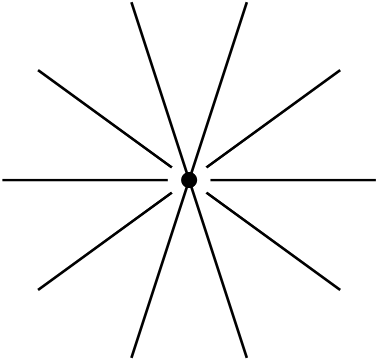
Surprising: The elements of $\Psi(W)$ are in bijection with the elements of W.

$$w \in W \longleftrightarrow \text{ a region } R \longleftrightarrow \bigcap\{\text{shards below } R\}$$

In particular, $(\Psi(W), \supseteq)$ induces a partial order $\preceq$ on W.

Also surprising: Every lower interval in $(\Psi(W), \supseteq)$ is isomorphic to $(\Psi(W_J), \supseteq)$ for some standard parabolic subgroup W_J.

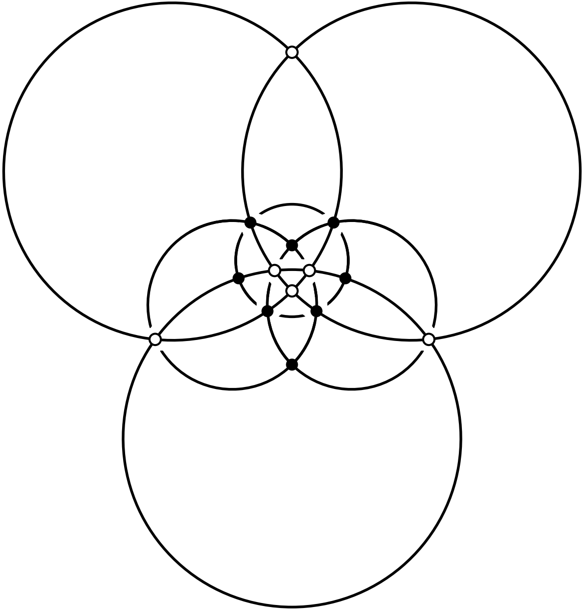
Shard intersections in $I_2(5)$



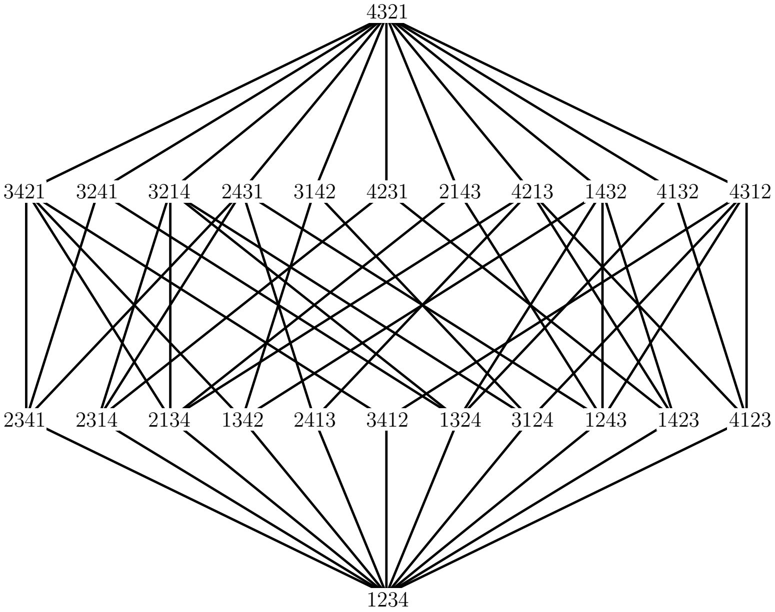
The poset $(\Psi(I_2(5)), \supseteq)$ has $\mathbb{R}^2$ as its unique minimal element and the origin as its unique maximal element. The 8 (1-dimensional) shards are pairwise incomparable under containment, and live at rank 1 (i.e. codimension 1).

The poset $(I_2(5), \preceq)$ has 1 as its unique minimal element and w_0 as its unique maximal element. The other 8 elements of W are pairwise incomparable and live at at rank 1.

Shard intersections in S_4



The shard intersection lattice on S_4



Shard intersections and lattice congruences

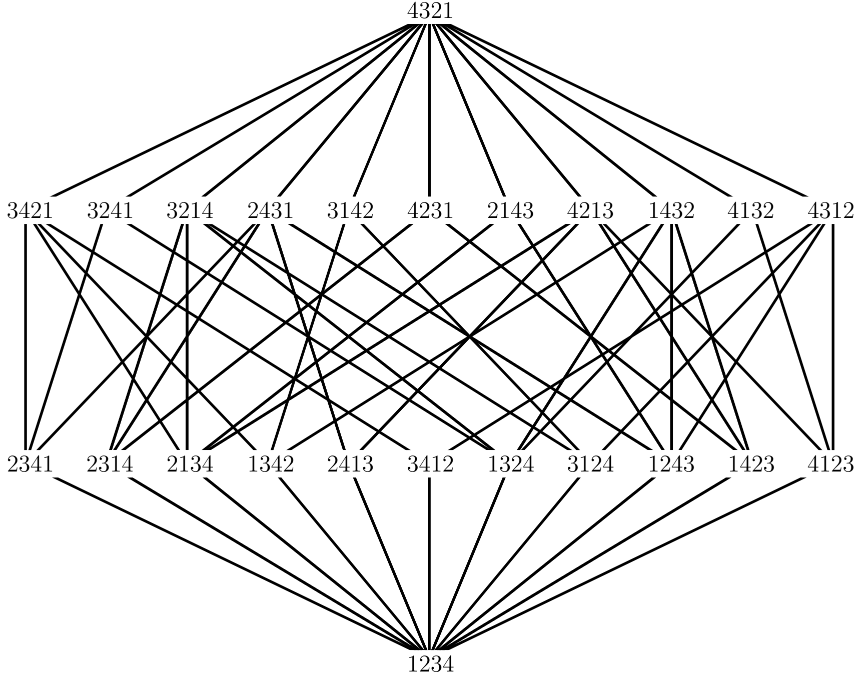
Since shards are so central to lattice congruences on the weak order, it is perhaps not surprising that lattice congruences "play nicely" with the shard intersection order.

Specifically, let $\pi_\downarrow^\Theta(W)$ be the collection of "bottom elements" of congruences classes of a congruence Θ. Then the restriction $(\pi_\downarrow^\Theta(W), \preceq)$ is a lattice and a join-sublattice of $(W, \preceq)$).

For Θ_c the c-Cambrian congruence, the lattice $(\pi_\downarrow^{\Theta_c}, \preceq)$—the restriction to c-sortable elements—is isomorphic to $\mathrm{NC}_c(W)$.

As a consequence, $\mathrm{NC}_c(W)$ is a lattice. (In fact, $\mathrm{NC}_c(W)$ is a sublattice of $(W, \preceq)$.)

The earlier proof (by Brady and Watt) that $\mathrm{NC}(W)$ is a lattice also used the polyhedral geometry of cones. Their proof is "dual" to the new proof (in the broadest outlines but not in any of the details).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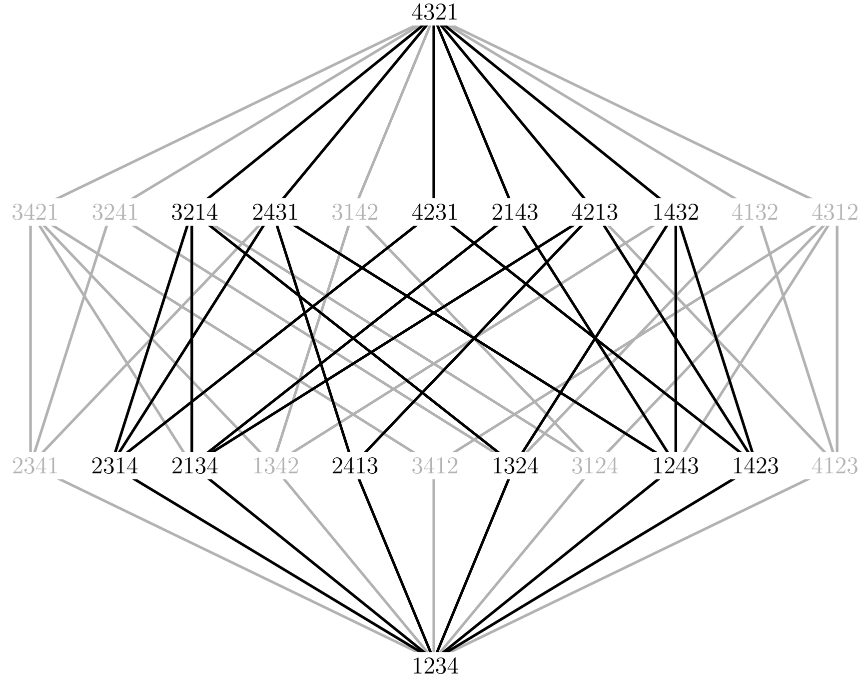
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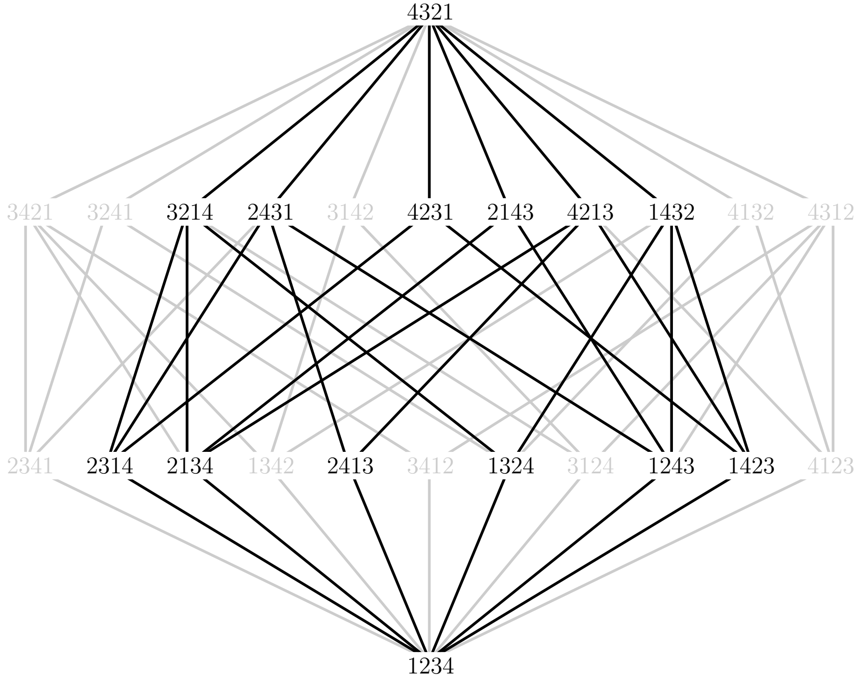
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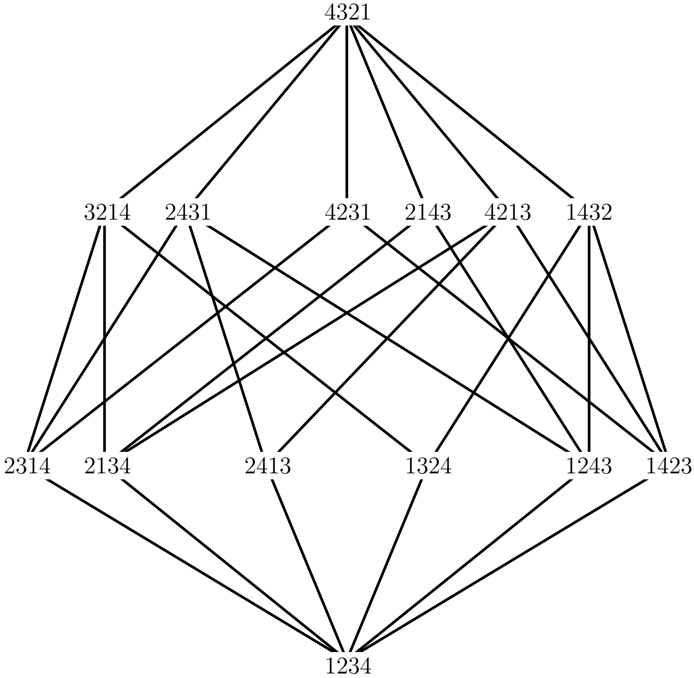
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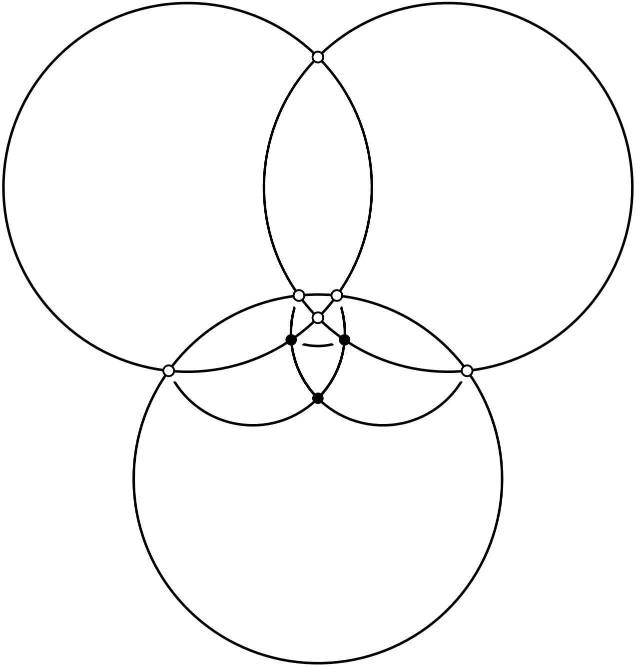


Properties of $(W, \preceq)$ and $\mathrm{NC}(W)$

$(W, \preceq)$	$\mathrm{NC}(W)$
Lattice	Lattice (sublattice of $(W, \preceq)$)
Weaker than weak order	Weaker than Cambrian lattice
Atomic and coatomic	Atomic and coatomic
Graded (W-Eulerian numbers)	Graded (W-Narayana)
Not self-dual	Self-dual
Lower intervals $\cong (W_J, \preceq)$	Lower intervals $\cong \mathrm{NC}(W_J)$
Möbius number: $\pm$number of "positive" elements of W.	Möbius number: $\pm$number of "positive" elements of $\mathrm{NC}(W)$.

Details on the Möbius number

Theorem. The Möbius function of $(W, \preceq)$ satisfies

$$\mu(1, w_0) = \sum_{J \subseteq S} (-1)^{|J|} \, |W_J| \, .$$

Proof. Since lower intervals $[1, w]$ are isomorphic to $(W_{\mathrm{Des}(w)}, \preceq)$, checking the defining recursion for μ becomes

$$\sum_{w \in W} \sum_{J \subseteq \mathrm{Des}(w)} (-1)^{|J|} \, |W_J| = \sum_{J \subseteq S} (-1)^{|J|} \, |W_J| \sum_{\substack{w \in W \text{ s.t.} \\ J \subseteq \mathrm{Des}(w)}} 1.$$

The inner sum is $|W| / |W_J|$, the number of maximal-length representatives of cosets of W_J in W. Thus the double sum reduces to zero.

Properties of $(W, \preceq)$ and $\mathrm{NC}(W)$ (continued)

$(W, \preceq)$	$\mathrm{NC}(W)$
Recursion counting maximal chains: sum over max'l proper standard parabolic subgroups. $\mathrm{MC}(W) = \sum_{s \in S} \left(\frac{\lvert W \rvert}{\lvert W_{\langle s \rangle} \rvert} - 1 \right) \mathrm{MC}(W_{\langle s \rangle})$	Recursion counting maximal chains: sum over max'l proper standard parabolic subgroups. (R., 2007.) $\mathrm{MC}(W) = \frac{h}{2} \sum_{s \in S} \mathrm{MC}(W_{\langle s \rangle})$.

These types of recursions are very natural in the context of Coxeter groups/root systems. For example:

1. Recursions for the W-Catalan number.
2. Volume of W-permutohedron (weight polytope). This follows from Postnikov's formula in terms of Φ-trees.

Properties of $(W, \preceq)$ and NC(W) (concluded)

$(W, \preceq)$	NC(W)
Maximal chains $\longleftrightarrow$ maximal simplices in a pulling triangulation of the W-permutohedron. (S_n case: Loday described the triangulation, 2005.)	Maximal chains $\longleftrightarrow$ maximal simplices in a pulling triangulation of the W-associahedron. (S_n case: Loday, 2005.) (General case: R., 2008.)
k-Chains $\longleftrightarrow$ k-simplices in the same triangulation of the W-permutohedron.	k-Chains $\longleftrightarrow$ k-simplices in the same triangulation of the W-associahedron. (R., 2008.)

Loday: Noticed that maximal simplices in a certain pulling triangulation of the S_n-associahedron biject with parking functions. Constructed the analogous triangulation of the S_n-permutohedron and asked what played the role of parking functions.

Details on the triangulations

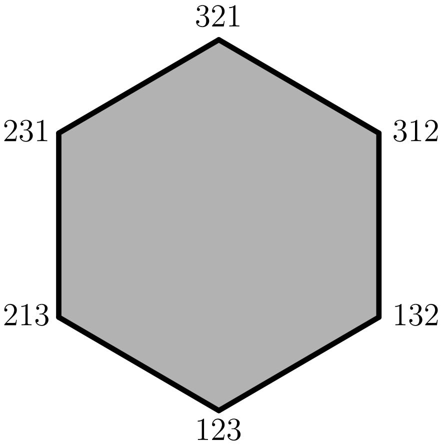
The bijection between intersections of shards and elements of W extends to a bijection between k-chains in $(W, \preceq)$ and k-simplices in a pulling triangulation of the W-permutohedron.

In particular: The order complex of $(W, \preceq)$ has f-vector equal to the f-vector of a pulling triangulation of the W-permutohedron.

Key point: For any $w \in W$, the lower interval $[1, w]$ in $(W, \preceq)$ is isomorphic to $(W_J, \preceq)$ for some W_J. The elements of W_J are in bijection with vertices of the face below w in the permutohedron.

All of this works for $\mathrm{NC}(W)$ and the W-associahedron as well. Maximal chains in $\mathrm{NC}(S_n)$ are in bijection with parking functions, so we recover the Loday result as a special case.

S_3 Permutohedron example



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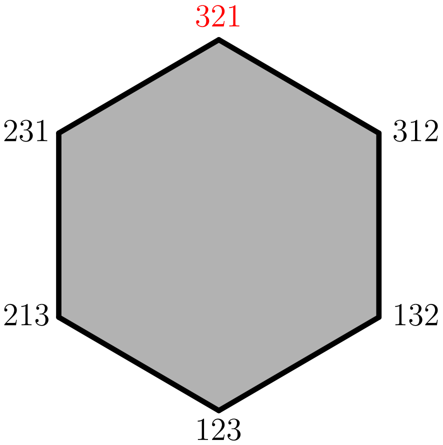

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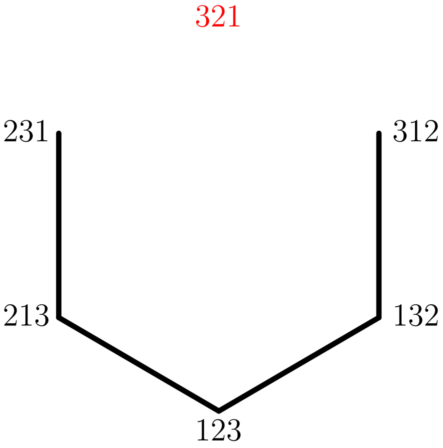

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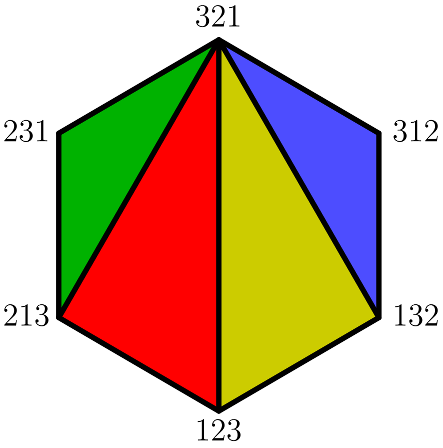

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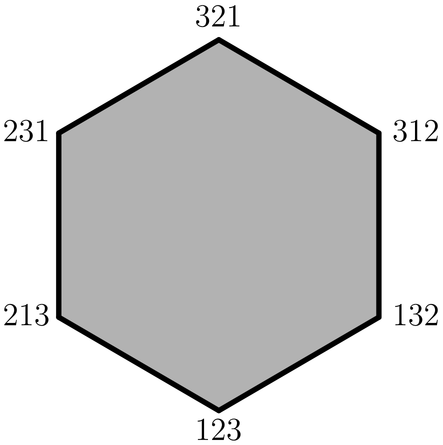

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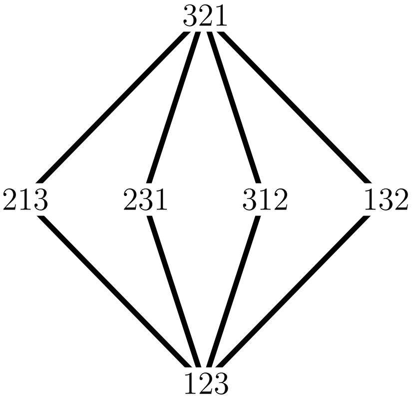

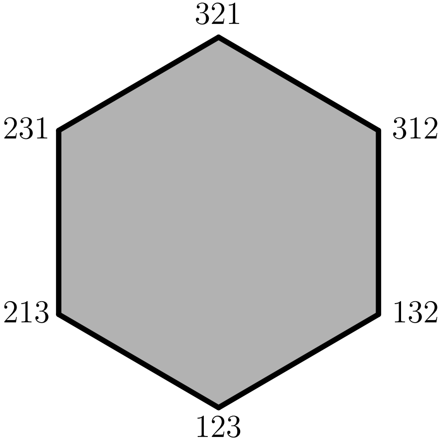

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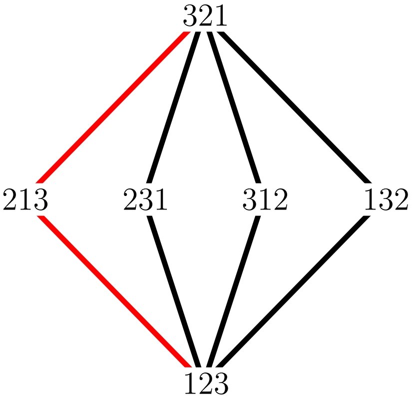

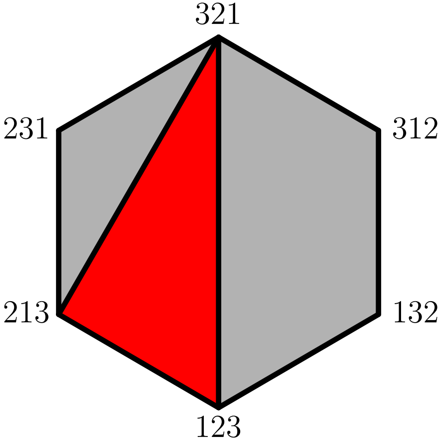

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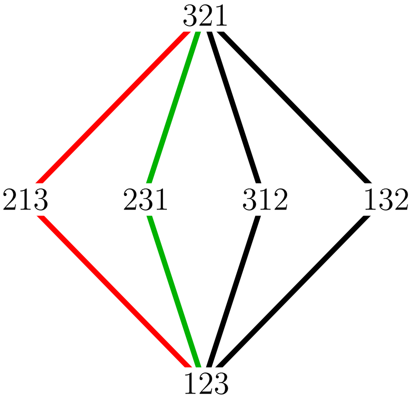

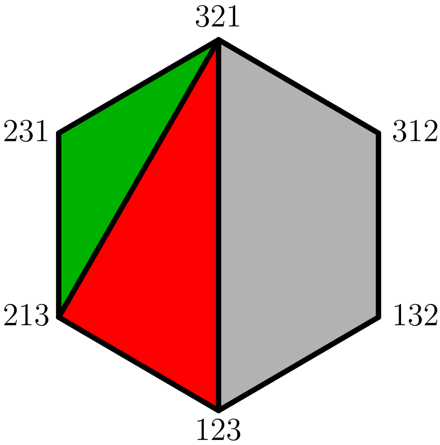

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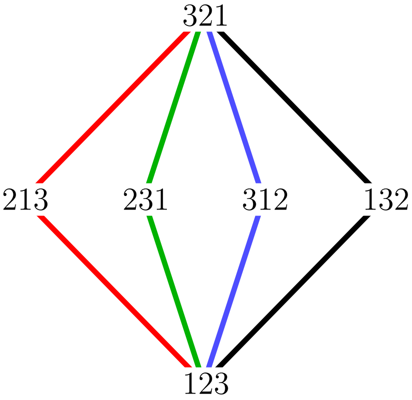
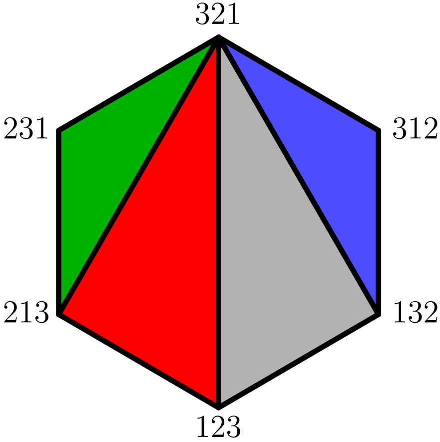
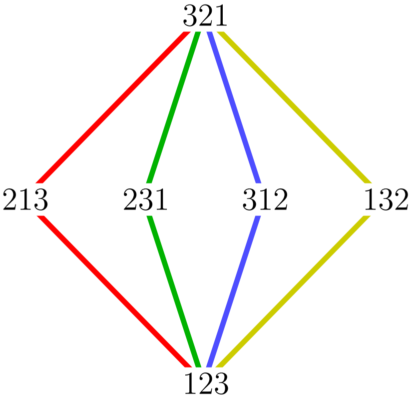
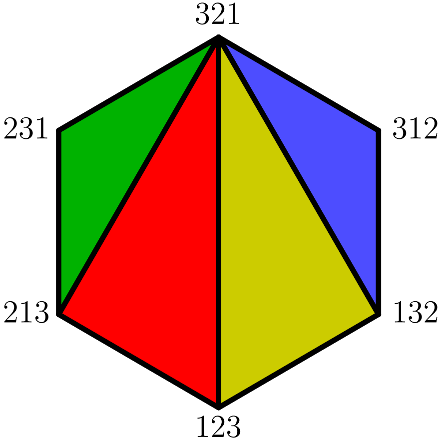




S_3 Permutohedron example





Recap of Section III.d: The shard intersection order

Elements of W are in bijection with intersections of shards.

Intersections of shards form a lattice, which can be interpreted as a new lattice structure on W.

c-Sortable elements induce a sublattice isomorphic to the lattice of noncrossing partitions.

NC partitions and the shard intersection order have analogous structural and enumerative properties.

Questions?

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