# Part II: Lattice congruences of the weak order

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Algebraic and Geometric Combinatorics of Reflection Groups
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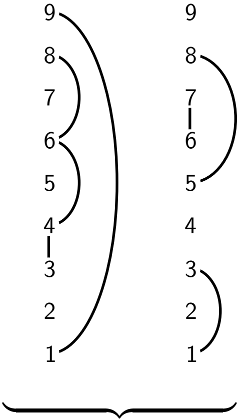
# Looking ahead: Noncrossing arc diagrams




Right nc arc diagrams

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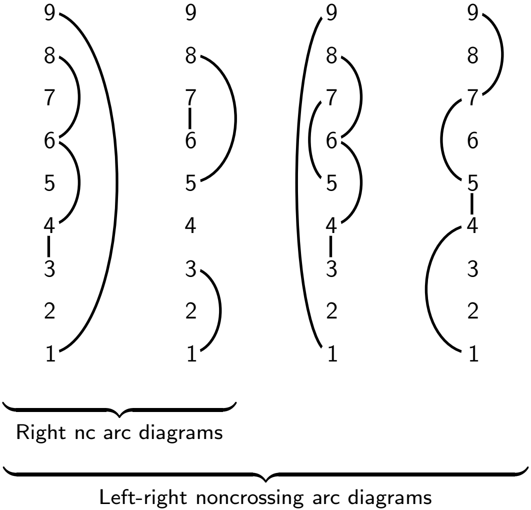

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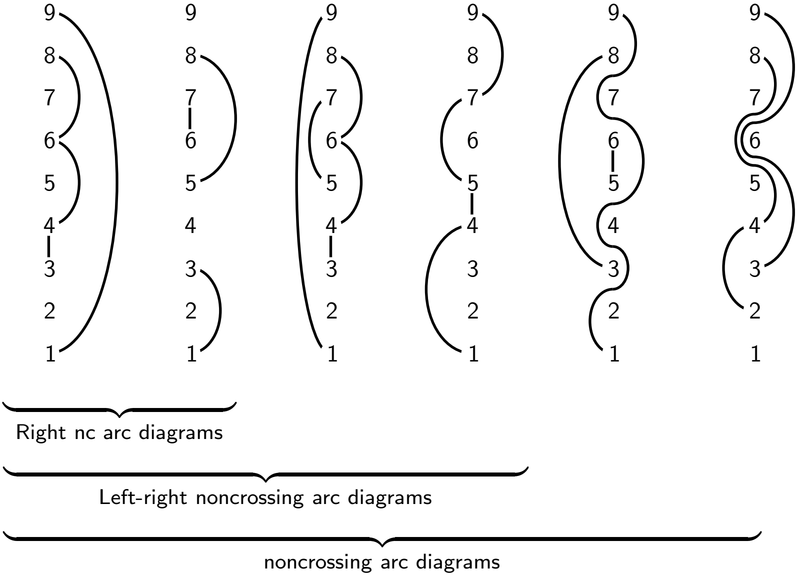

# Section II.a: The weak order on a finite Coxeter group

# Coxeter groups

$$W = \left\langle S \mid s_i^2 = 1,\ \forall\, i \ \text{ and } \ (s_i s_j)^{m(i,j)} = 1,\ \forall\, i < j \right\rangle.$$

**Examples.**

- $B_2 = \left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \right\rangle$. This is the dihedral group of order 8. Its elements are

  $$1,\ s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

- The symmetric group $S_n$ (permutations of $\{1, \ldots, n\}$) with $s_i = (i \ \ i{+}1)$ and $m(i,j) = \begin{cases} 3 & \text{if } j = i+1, \text{ or} \\ 2 & \text{if } j > i+1. \end{cases}$

Let $w \in W$. A word of minimal length, among words for $w$, is called a reduced word for $w$.

The length $\ell(w)$ of $w$ is the length of a reduced word for $w$.

# Weak order

The weak order on a Coxeter group $W$ sets $u \leq w$ if and only if a reduced word for $u$ occurs as a prefix of some reduced word for $w$.

The covers are $w \lessdot ws$ for $w \in W$ and $s \in S$ with $\ell(w) < \ell(ws)$.

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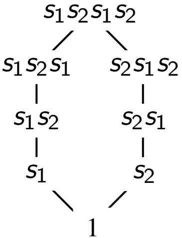


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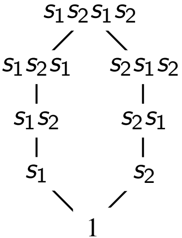



The weak order is ranked by the length function $\ell$.

It is a meet semilattice in general, and a lattice when $W$ is finite.

Alert: This is "right" weak order. There is also a "left" weak order.

# Reflections

The set $S$ is called the simple reflections. The set

$$T = \left\{wsw^{-1} : w \in W,\ s \in S\right\}$$

is called the set of reflections in $W$.

**Example.** $B_2 = \left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \right\rangle$:
$S = \{s_1, s_2\}$
$T = \{s_1, s_2, s_1 s_2 s_1, s_2 s_1 s_2\}$

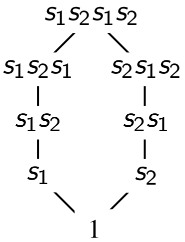
**Example.** The symmetric group:
$S = \{$adjacent transpositions $(i\ \ i{+}1)\}$
$T = \{$all transpositions $(i\ \ j)\}$

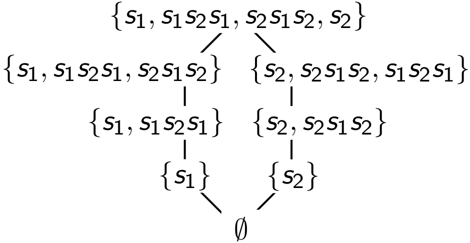
An inversion of $w \in W$ is a reflection $t \in T$ such that $\ell(tw) < \ell(w)$. The notation $\mathrm{inv}(w)$ means $\{$inversions of $w\}$.
If $a_1 \cdots a_k$ is a reduced word for $w$, then write $t_i = a_1 \cdots a_i \cdots a_1$.

$$\mathrm{inv}(w) = \{t_i : 1 \leq i \leq k\}.$$

In the weak order, $u \leq w$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$.

**Example.** $B_2 = \left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \right\rangle$



# Inversions and weak order on the symmetric group

Write a permutation $\pi \in S_n$ in one-line notation $\pi_1 \cdots \pi_n$.

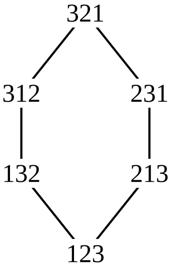
Inversions are

$$\mathrm{inv}(\pi) = \{\text{transpositions } (i\ \ j) : \ i \text{ comes before } j \text{ in } \pi\},$$

and this is the origin of the term "inversion."

Cover relations in the weak order are transpositions of adjacent entries. Going "up" means putting the entries out of numerical order.

**Example.** The weak order on $S_3$:

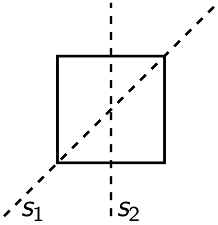


# Coxeter groups are reflection groups

A finite reflection group is a finite group generated by reflections.

**Theorem.** A finite group is a Coxeter group if and only if it is a finite reflection group.

**Example**: $B_2 = \{\text{symmetries of a square}\}$.
All symmetries of the square are compositions of the reflections $s_1$ and $s_2$.
This is a finite reflection group.




Since $s_1 s_2$ is a $90^\circ$ rotation, $(s_1 s_2)^4 = 1$.
Abstractly, this group is

$$B_2 = \left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 \right\rangle.$$

# Coxeter groups are reflection groups (continued)

The point is to find hyperplanes $H_i$ at appropriate angles so that, if $s_i$ is the orthogonal reflection in $H_i$, the order of $s_i s_j$ is $m(i,j)$.

Besides the generators $S$, other elements act as reflections, namely the set $T = \left\{wsw^{-1} : w \in W,\, s \in S\right\}$.

The collection of reflecting hyperplanes for all these reflections (the Coxeter arrangement $\mathcal{A}$) cuts space into "regions."

The generators $S$ are the reflections in the walls of some region $B$. Identify $B$ with the identity element 1. The map $w \mapsto wB$ is a bijection from $W$ to $\mathcal{A}$-regions.

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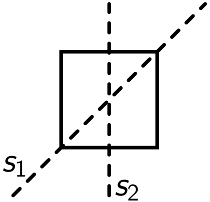

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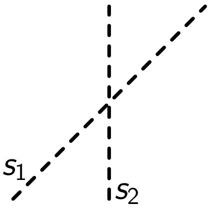

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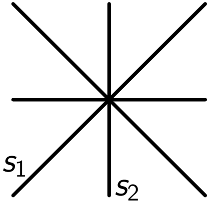

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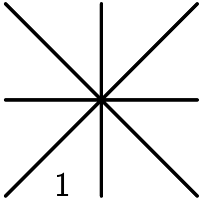

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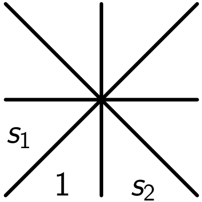

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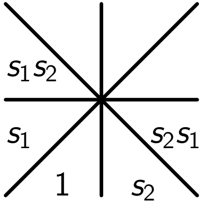

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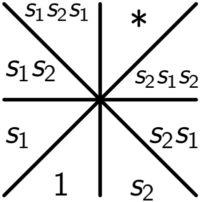
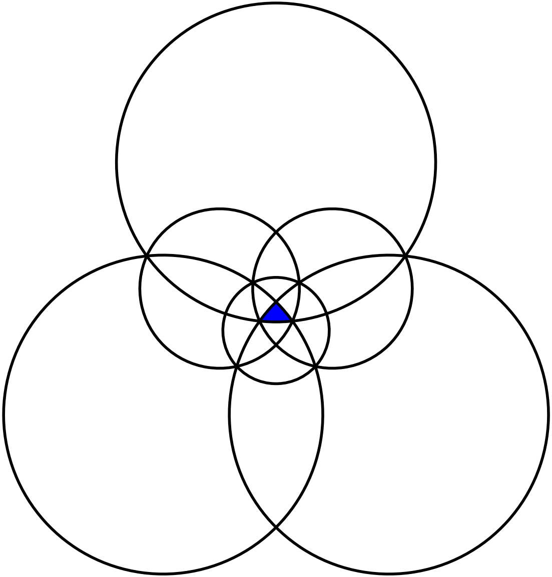



$$* = s_1s_2s_1s_2$$
$$= s_2s_1s_2s_1$$

# Example: $S_4 = \{\text{symmetries of regular tetrahedron}\}$



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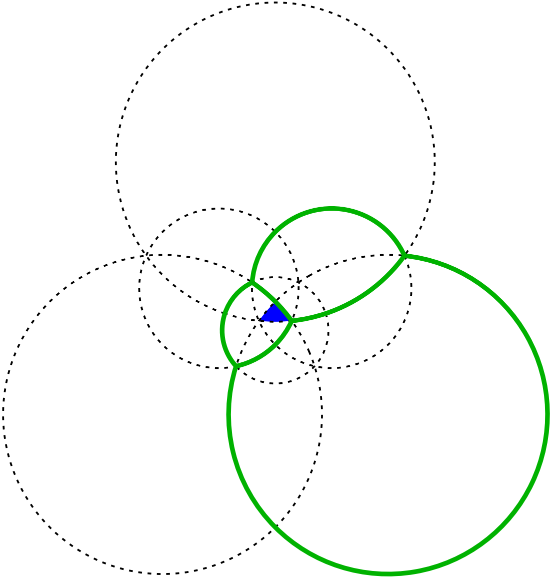

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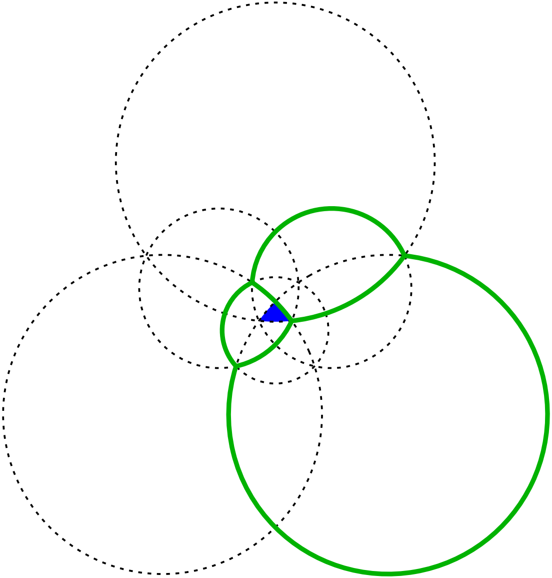

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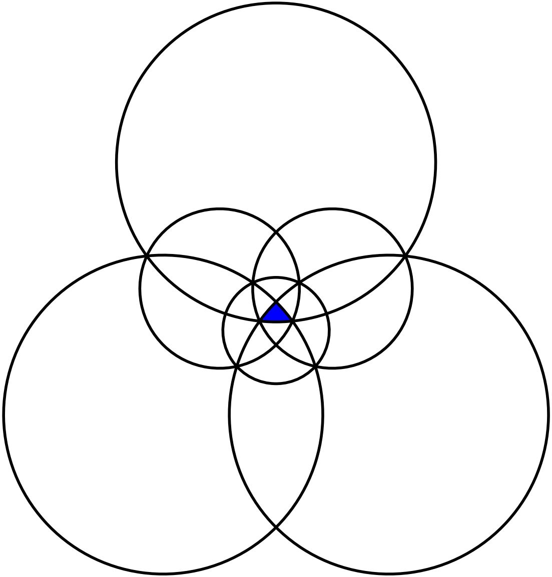

Blue region is 1.

Regions $\leftrightarrow$ elements.

Largest circles: hyperplanes for $s_1$, $s_2$, and $s_3$. ($s_2$ on top.)

$m(s_1, s_2) = 3.$
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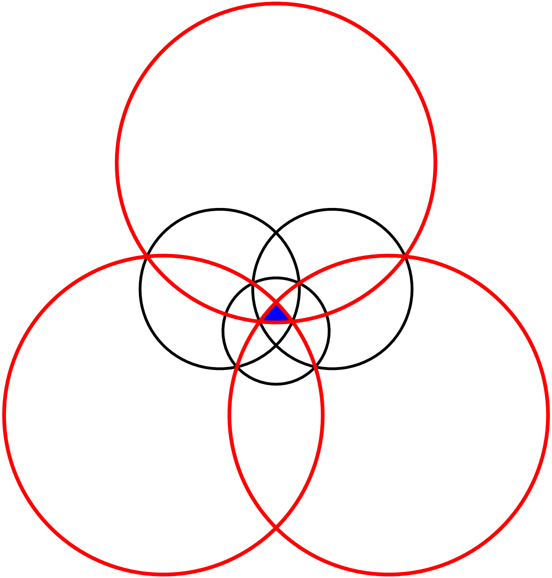

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# The weak order is a poset of regions

Let $\mathcal{A}$ be any arrangement of hyperplanes. Choose any region $B$.

Separating set of a region $R$:

$$S(R) = \{\text{hyperplanes of } \mathcal{A} \text{ separating } R \text{ from } B\}$$

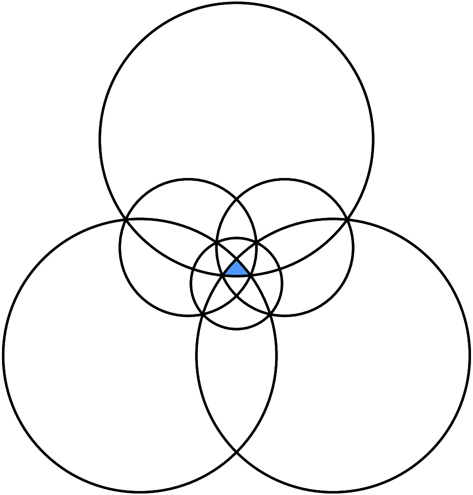
Poset of regions $\mathcal{P}(\mathcal{A}, B)$ is the set of regions with

$$Q \leq R \text{ if and only if } S(Q) = S(R).$$

**Proposition.** If $\mathcal{A}$ is a Coxeter arrangement for $W$, then $w \mapsto wB$ is an isomorphism from the weak order on $W$ to $\mathcal{P}(\mathcal{A}, B)$.

Proof: Each $H \in \mathcal{A}$ is $H_t$ for a unique reflection $t \in T$. Check that $\text{inv}(w) \leftrightarrow S(wB)$.

# Example: The weak order on $S_4$



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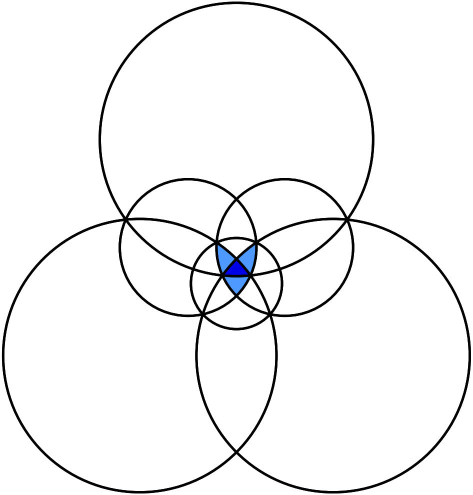

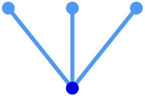

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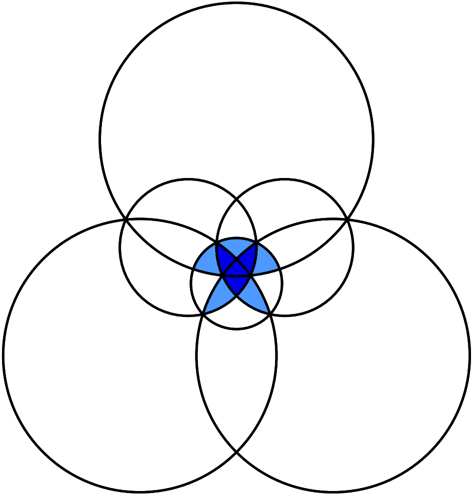

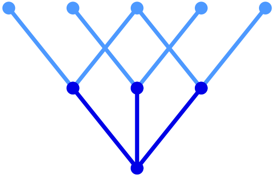

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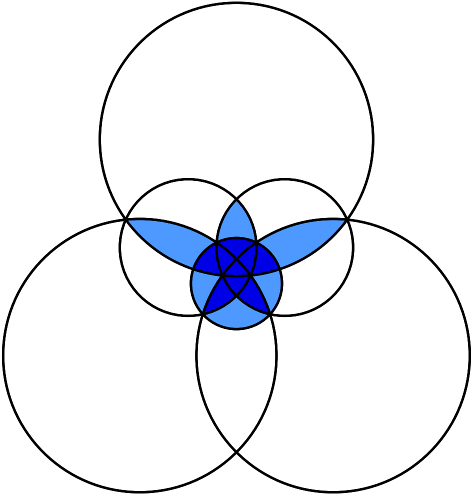

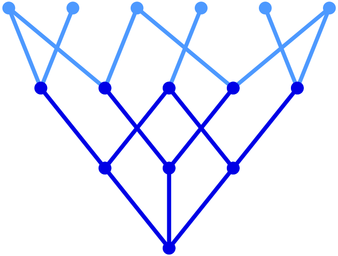

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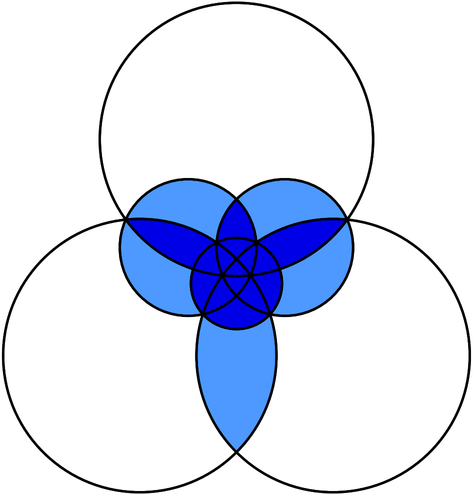

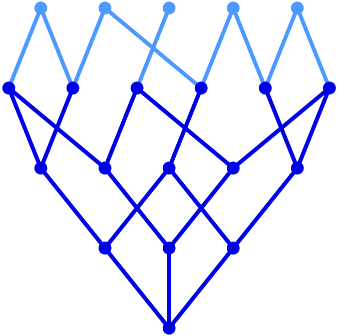

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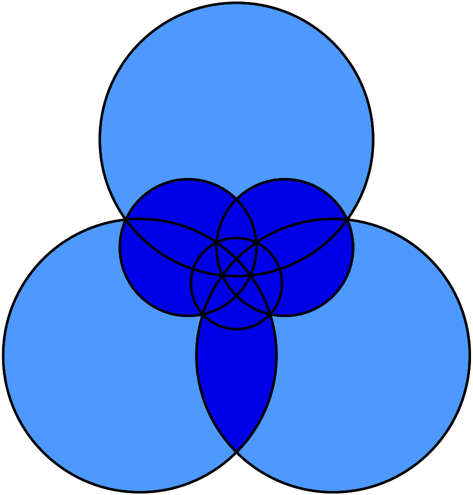

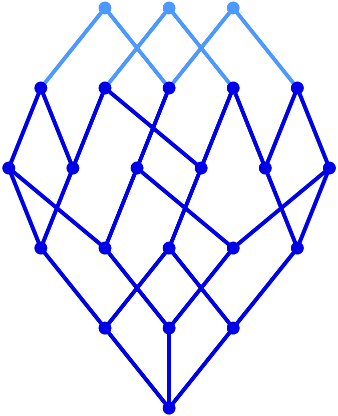

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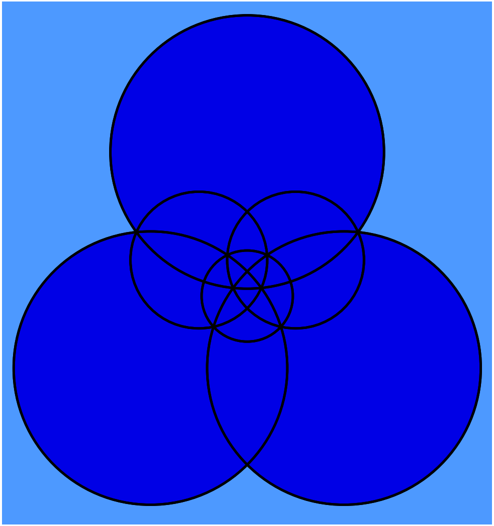

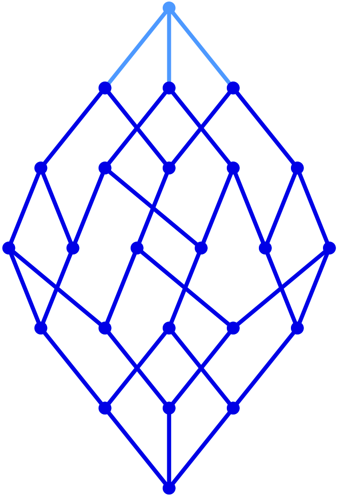

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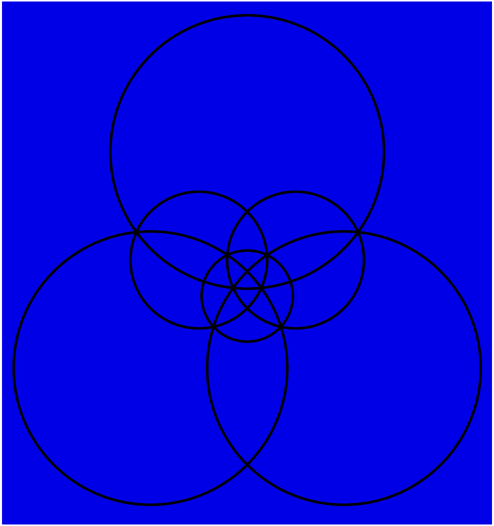
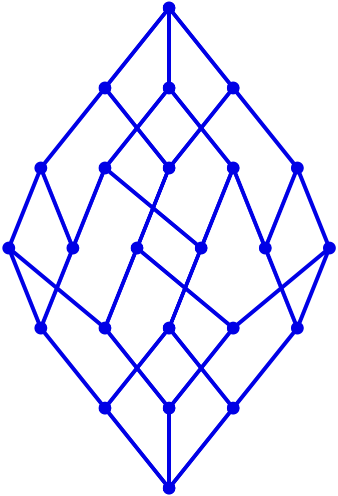
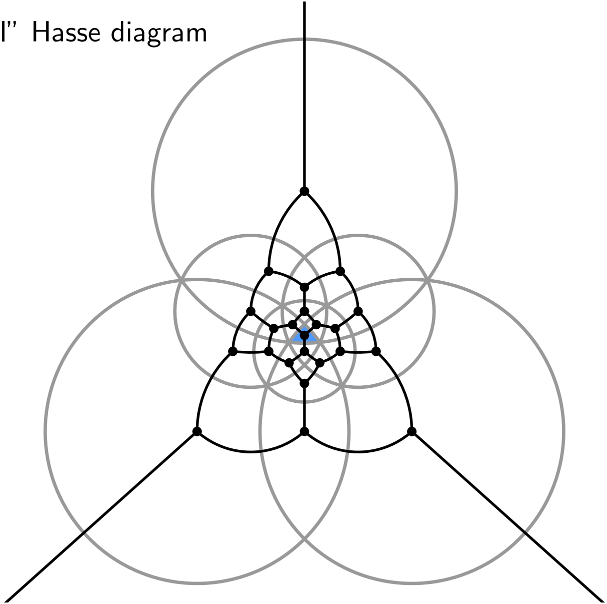




# Another view of the weak order on $S_4$

A "radial" Hasse diagram



# Recap of Section II.a: The weak order on a finite Coxeter group

Coxeter group:

- Special presentation
- Length and reduced words
- Reflection group

Weak order:

- Prefix order
- Containment of inversion sets
- Poset of regions

Questions?

# Section II.b: Lattice properties of the weak order

# The weak order is a lattice

**Lemma** (BEZ lemma, A. Björner, P. Edelman, G. Ziegler, 1987). Suppose $P$ is a finite poset with a unique minimal element. Suppose also that, for all $x$ and $y$ in $P$ such that $x$ and $y$ cover a common element $z$, the join $x \vee y$ exists. Then $P$ is a lattice.

**Theorem** (A. Björner, 1984). The weak order is a lattice.

Proof: If $x$ and $y$ cover a common element $z$ then $x = zs_i$ and $y = zs_j$ for $s_i, s_j \in S$. Convince yourself that $x \vee y = z(s_i s_j s_i \cdots)$, where "$s_i s_j s_i \cdots$" alternates and contains $m(i,j)$ letters.

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**Lemma** (BEZ lemma, A. Björner, P. Edelman, G. Ziegler, 1987). Suppose $P$ is a finite poset with a unique minimal element. Suppose also that, for all $x$ and $y$ in $P$ such that $x$ and $y$ cover a common element $z$, the join $x \vee y$ exists. Then $P$ is a lattice.

**Theorem** (A. Björner, 1984). The weak order is a lattice.

Proof: If $x$ and $y$ cover a common element $z$ then $x = zs_i$ and $y = zs_j$ for $s_i, s_j \in S$. Convince yourself that $x \vee y = z(s_i s_j s_i \cdots)$, where "$s_i s_j s_i \cdots$" alternates and contains $m(i,j)$ letters.

In fact, it is impossible to convince yourself of this without either using the geometry of $\mathcal{A}$ directly or using some property of reduced words that ultimately depends on the geometry. A direct geometric proof is straightforward: Reason about separating sets.

# The weak order is a polygonal lattice

Again:
If $x$ and $y$ cover a common element $z$ then $x = zs_i$ and $y = zs_j$.

$x \vee y = z(s_i s_j s_i \cdots)$, where "$s_i s_j s_i \cdots$" alternates and contains $m(i,j)$ letters.

The interval $[z, z(s_i s_j s_i \cdots)]$ is isomorphic to $[1, s_i s_j s_i \cdots]$, which is a polygon. For example, if $m(i,j) = 4$:

$$\begin{tikzcd}[column sep=small]
 & zs_is_js_is_j & \\
zs_is_js_i \arrow[ur, dash] & & zs_js_is_j \arrow[ul, dash] \\
zs_is_j \arrow[u, dash] & & zs_js_i \arrow[u, dash] \\
zs_i \arrow[u, dash] & & zs_j \arrow[u, dash] \\
 & z \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

This is one half of the polygonal property. The other half follows because the weak order is self-dual. The map is $w \mapsto ww_0$, where $w_0$ is the top element.
(On posets of regions, it is $R \mapsto -R$.)

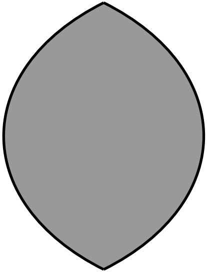
# The weak order is a congruence uniform lattice

We'll understand this by the doubling construction...

# Doubling and congruence uniformity

Doubling an interval $I$ in a lattice $L$:

Replace $I$ its product with a 2-element chain $\mathbf{2}$. Define order relations between the doubled interval and the rest of the lattice in a natural way.



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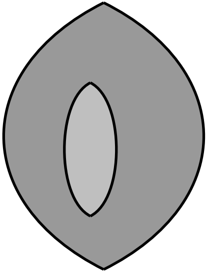

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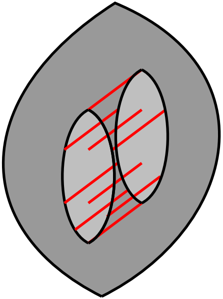


# Doubling and congruence uniformity

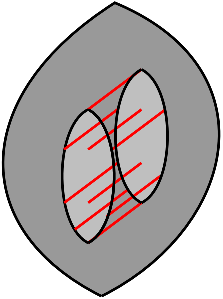
Doubling an interval $I$ in a lattice $L$:

Replace $I$ its product with a 2-element chain $\mathbf{2}$. Define order relations between the doubled interval and the rest of the lattice in a natural way.

Specifically, the doubling $L[I]$ is a subset of the product $L \times \mathbf{2}$:

$$L[I] := [((L - X) \cup I) \times \{1\}] \cup (X \times \{2\})$$

where $X = \{x \in L : x \geq c \text{ for some } c \in I\}$.



Can check: $L[I]$ is a lattice. (Write down meet and join explicitly in terms of meet and join in $L$.)

# Doubling and congruence uniformity (continued)

**Theorem** (A. Day 1994, W. Geyer 1994). A finite lattice $L$ is congruence uniform if and only if it can be obtained from a 1-element lattice by a sequence of doublings of intervals.

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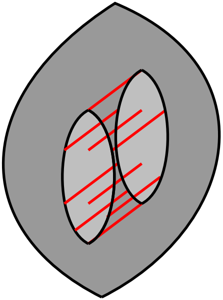
How to understand the direction we need (doublings $\Longrightarrow$ congruence uniform):

1. Doubling creates exactly one new join-irreducible element $j$.
(The lowest red edge is $j_* \lessdot j$.)

2. Doubling creates exactly one new join-irreducible congruence.
(All red edges give the same congruence. All other edges give "old" congruences.)



# The weak order is a congruence uniform lattice

We'll understand this by the doubling construction.

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The depth of a hyperplane $H \in \mathcal{A}$ is the length of a shortest walk from $B$ to a region $R$ with $H \in S(R)$.

Construct the weak order on $W$ as a sequence of posets of regions, adding hyperplanes one at a time in order of depth.

**Theorem** (R., 2003). This constructs the weak order by a sequence of doublings of intervals.

**Corollary** (N. Caspard, C. Le Conte de Poly-Barbut, and M. Morvan, 2002). The weak order is congruence uniform.

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This implies that $\mathrm{Irr}(\mathrm{Con}(L))$ is a partial order on j.i. elements. To describe $\mathrm{Irr}(\mathrm{Con}(L))$ in general, we will need the geometry of shards. We will hopefully discuss this in Part III.

# Example: Weak order on $S_4$ by doubling

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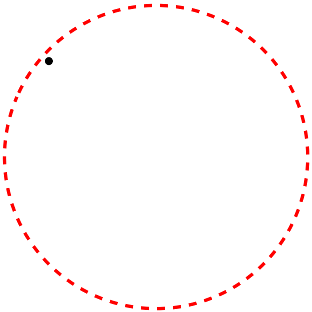

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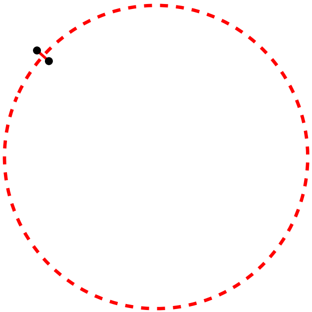

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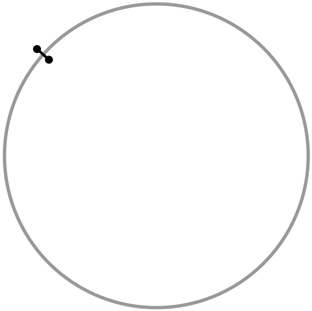

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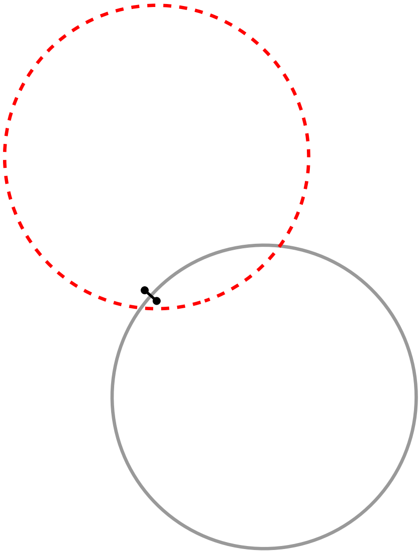

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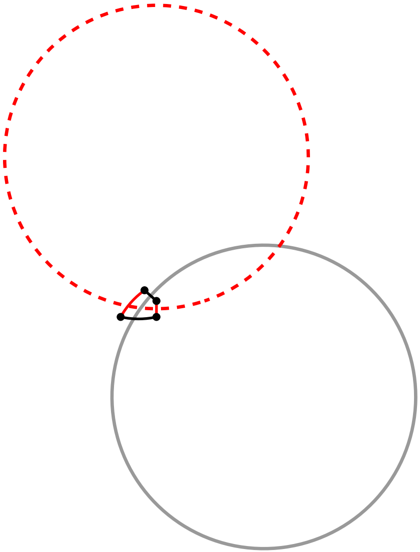

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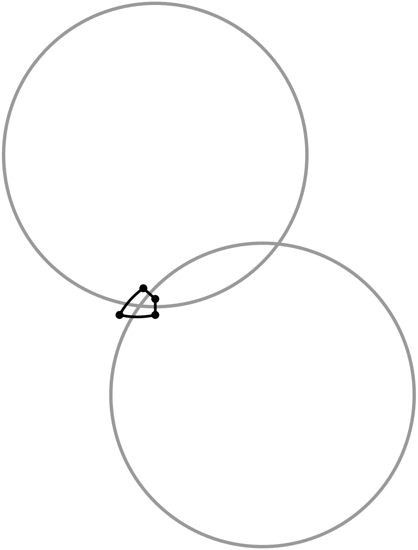

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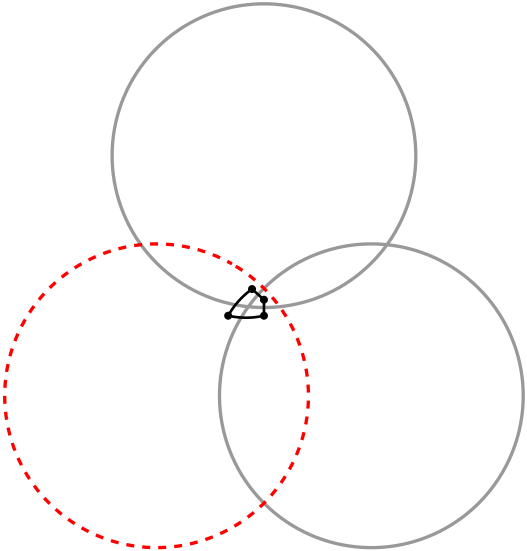

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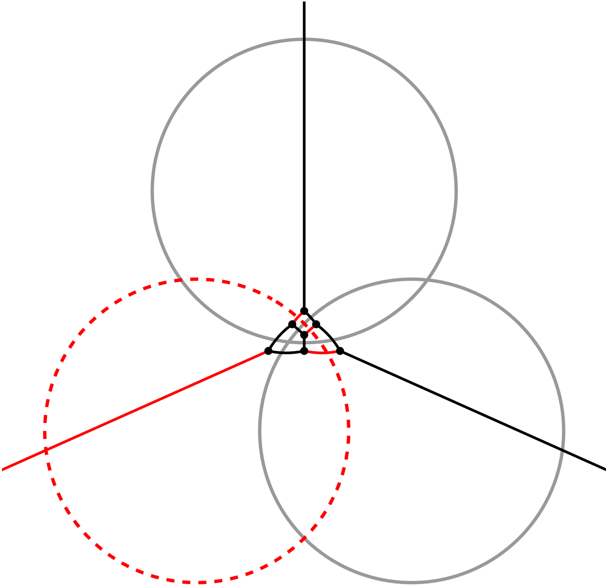

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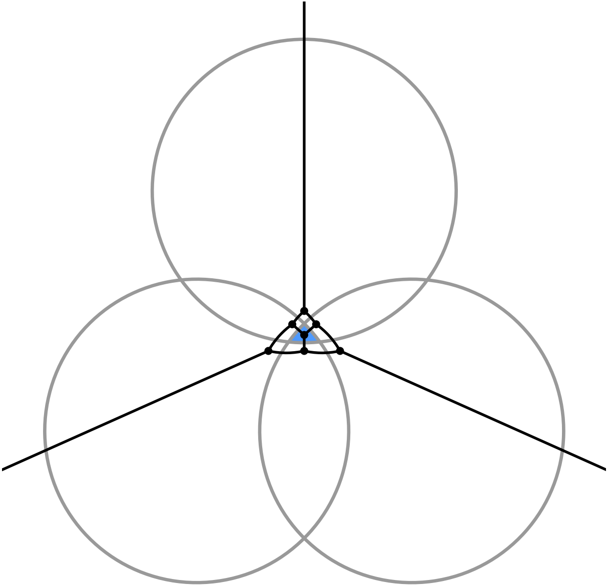

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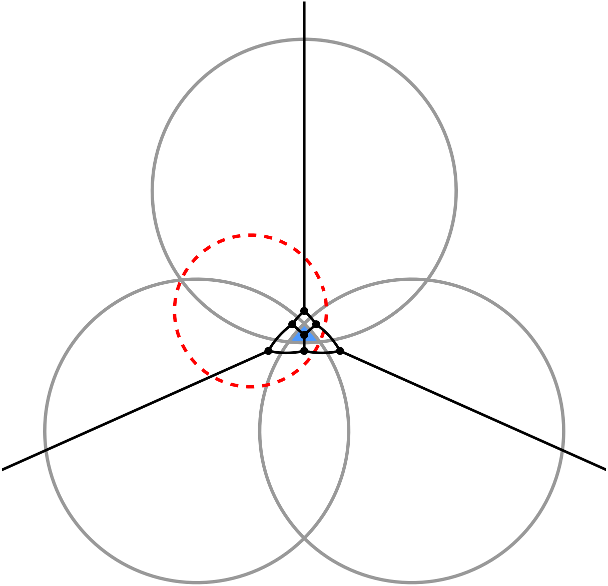

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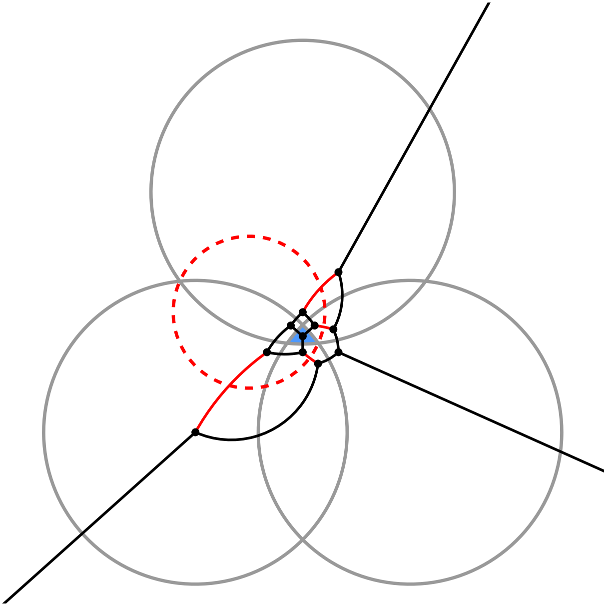

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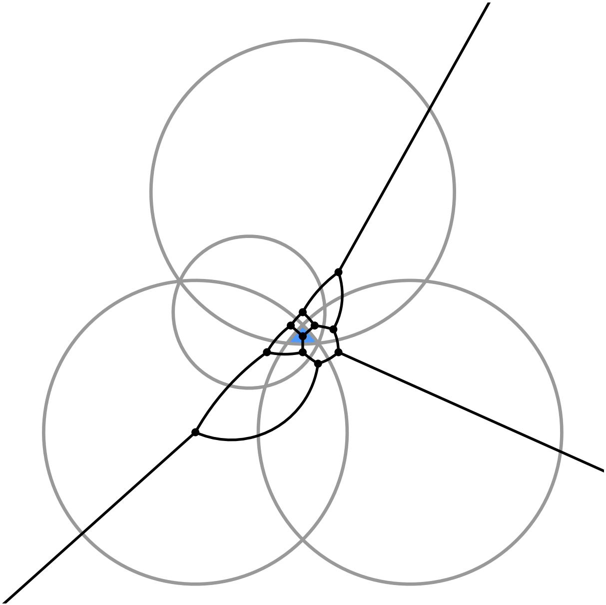

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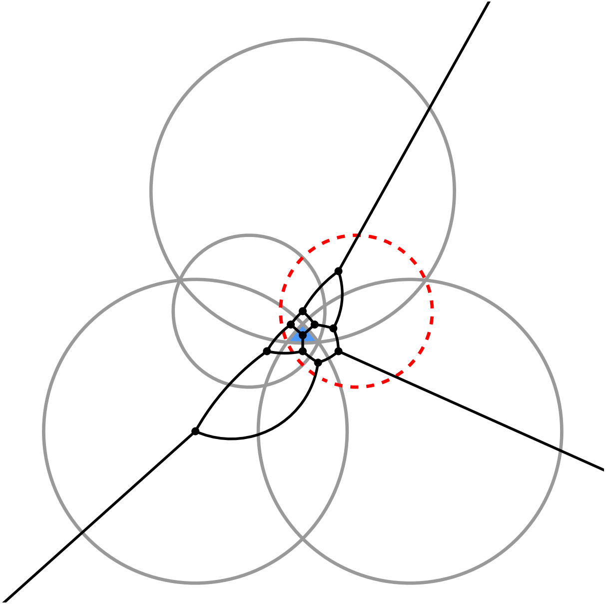

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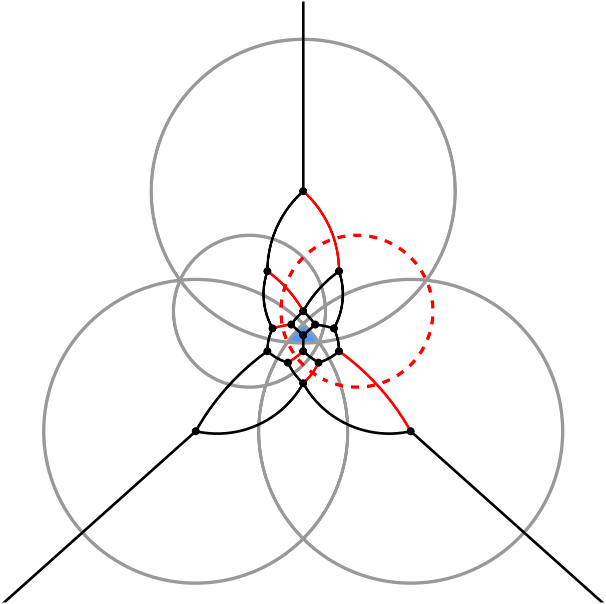

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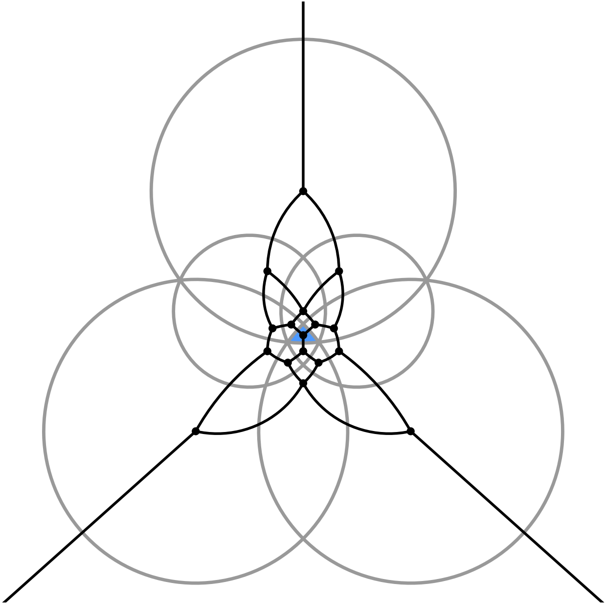

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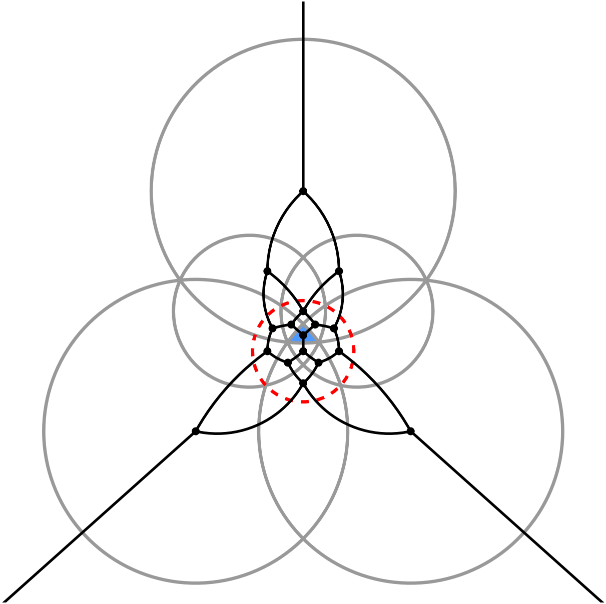

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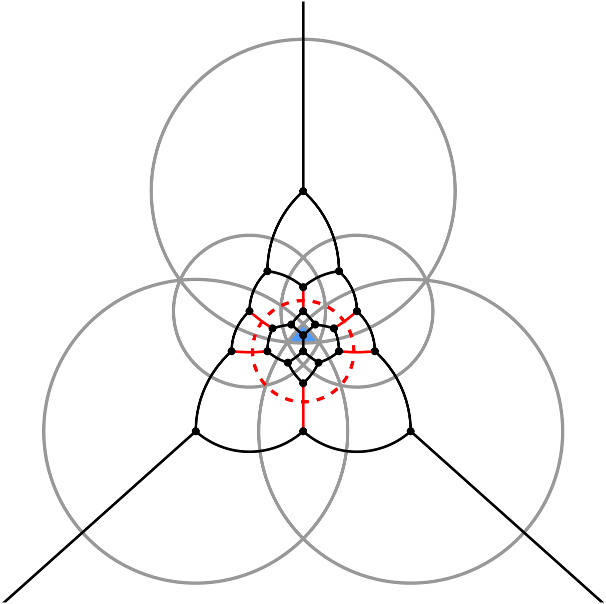

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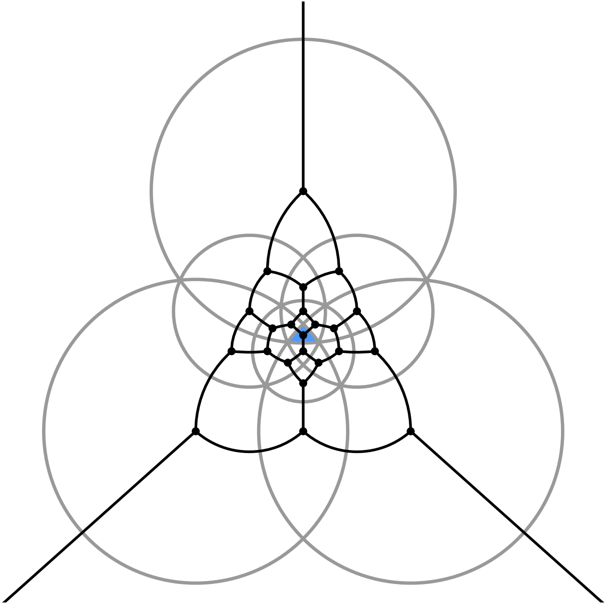

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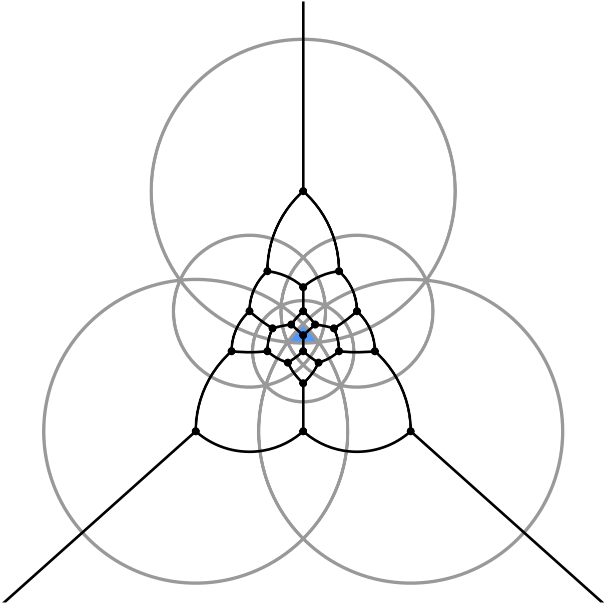

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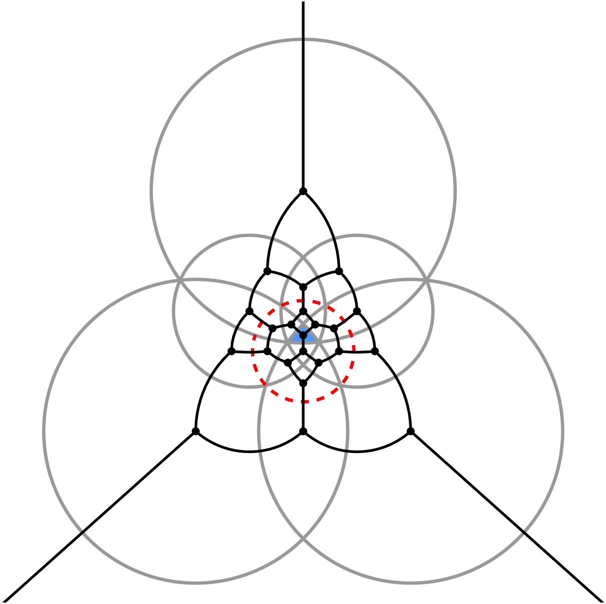

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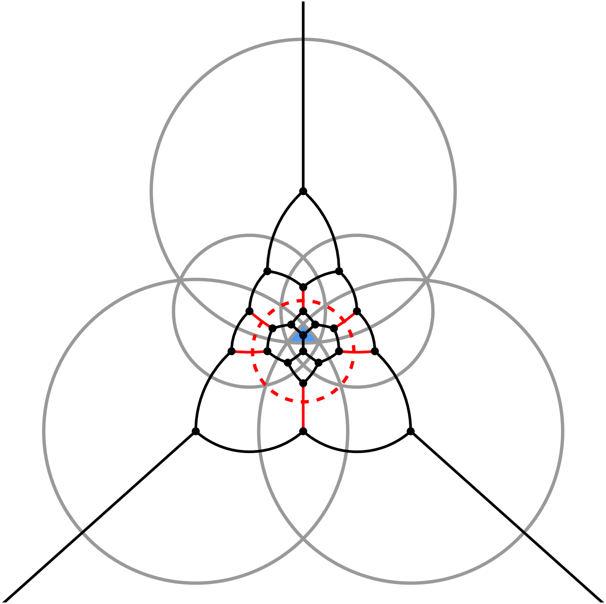

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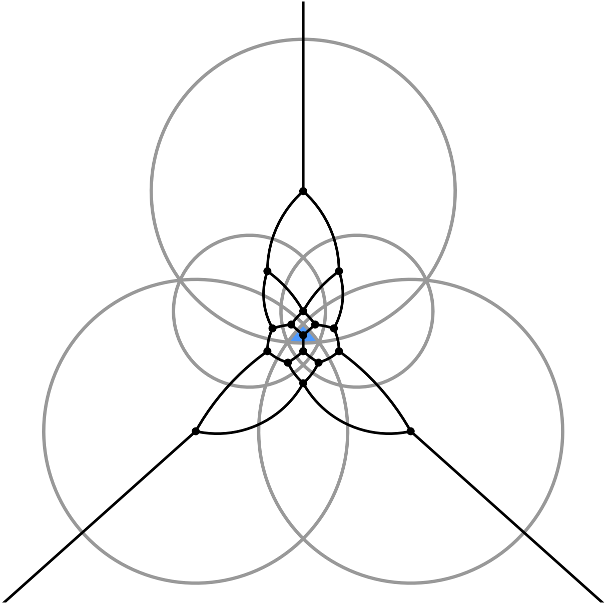

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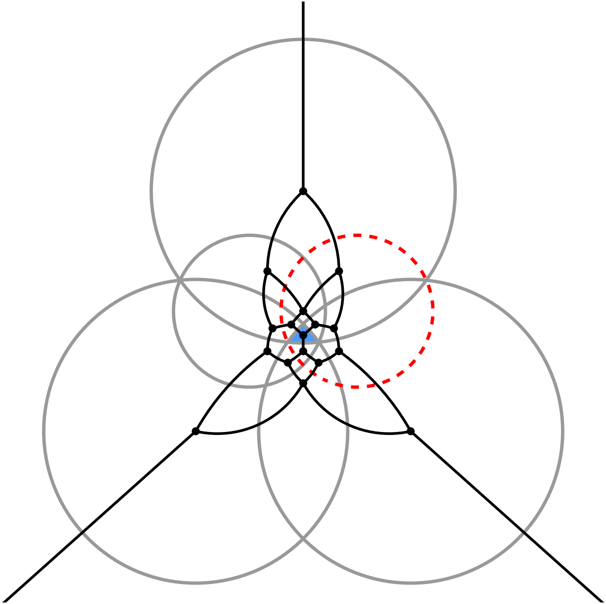

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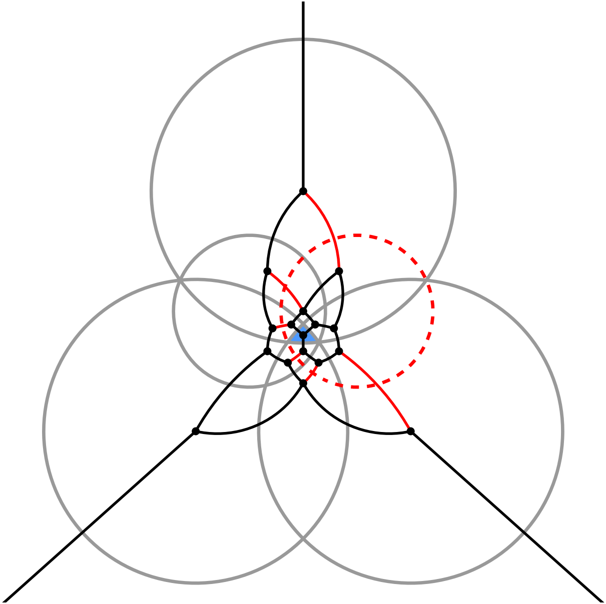

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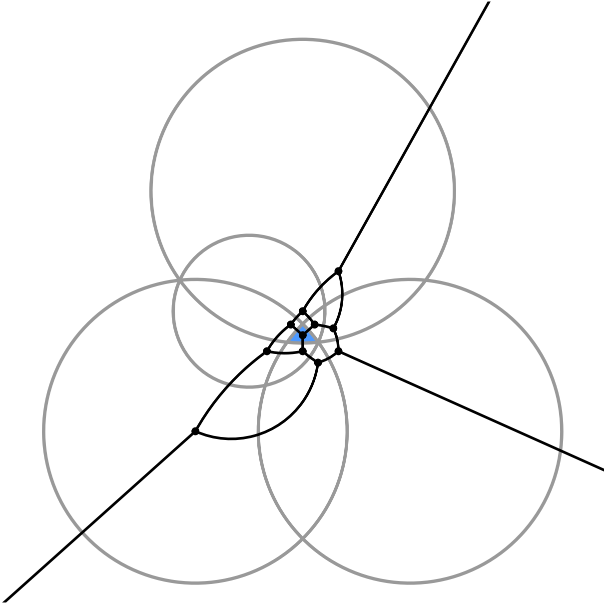

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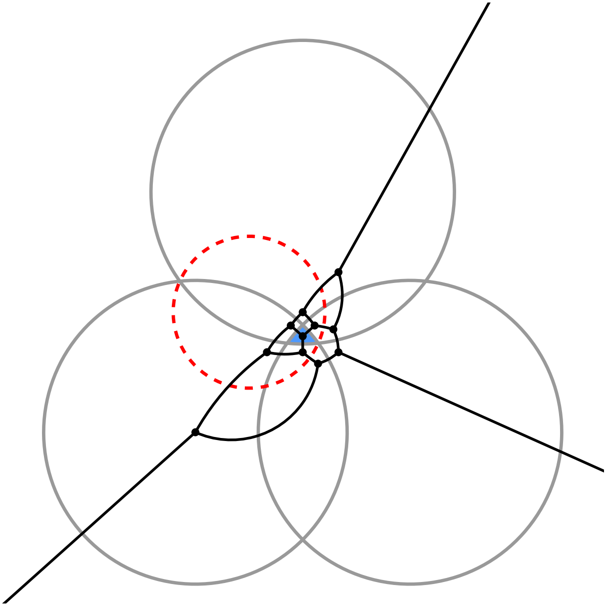

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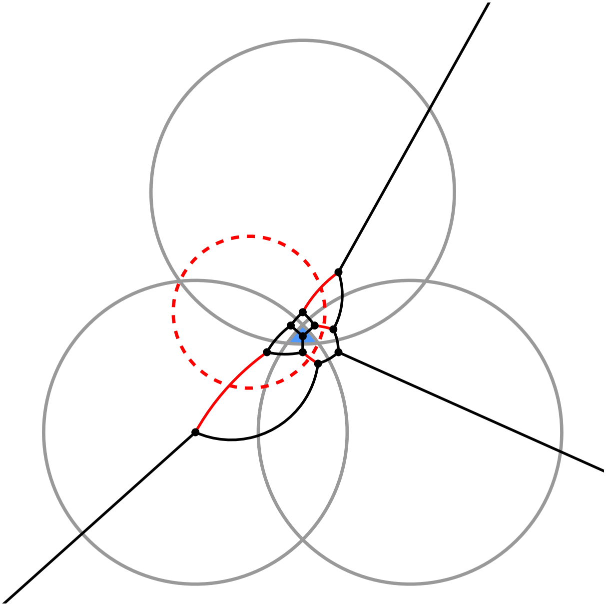

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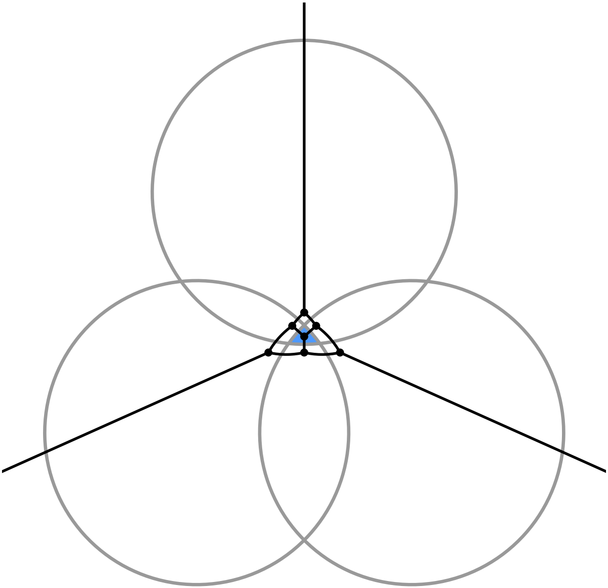

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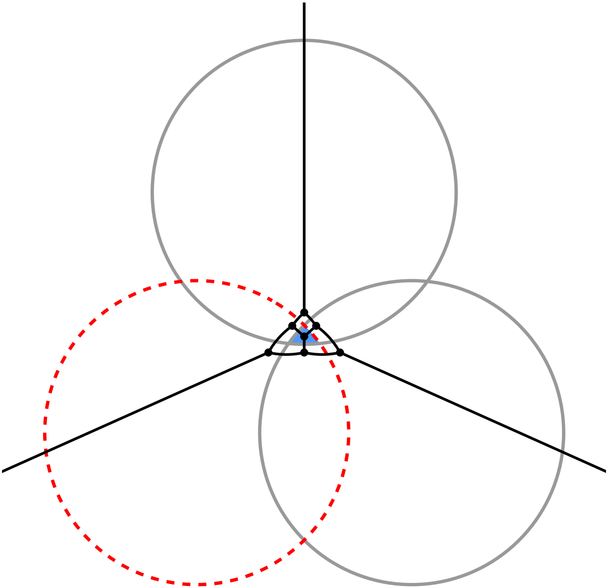

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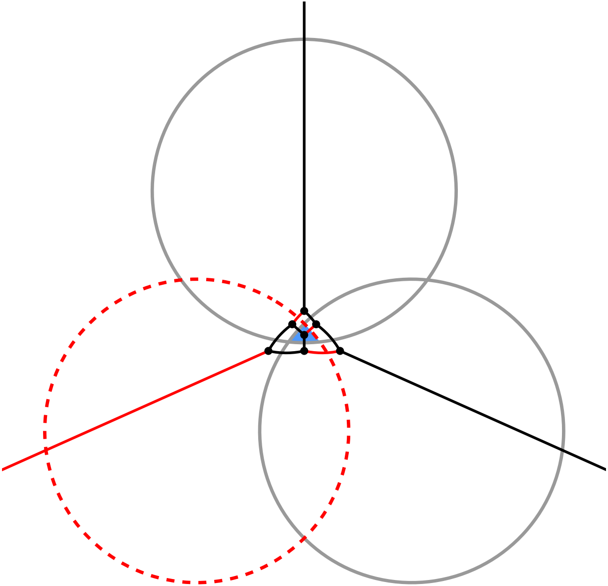

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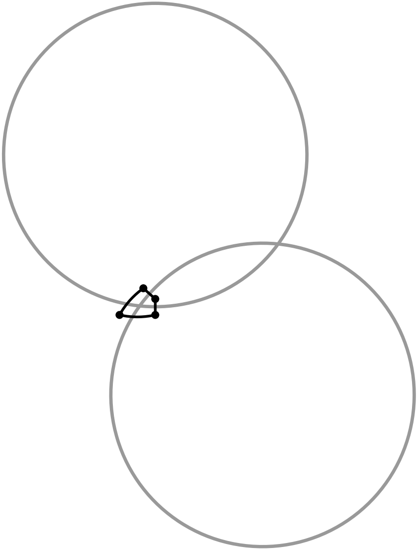

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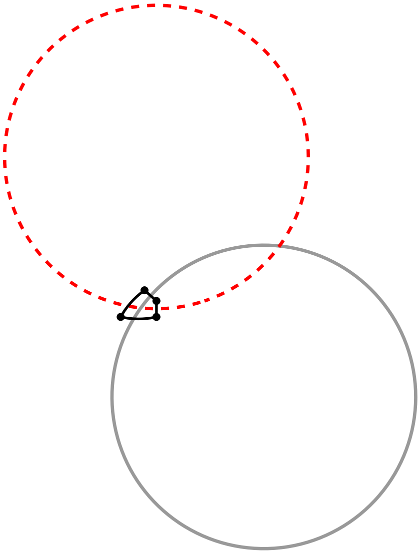

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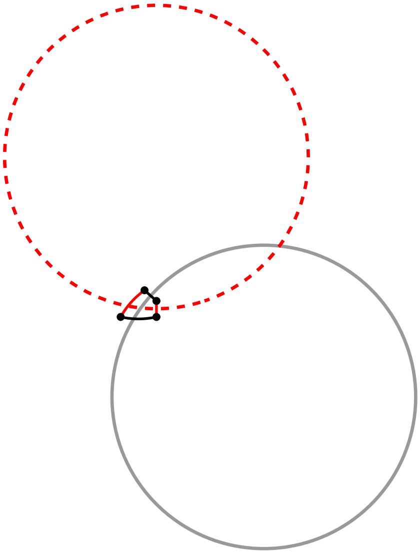

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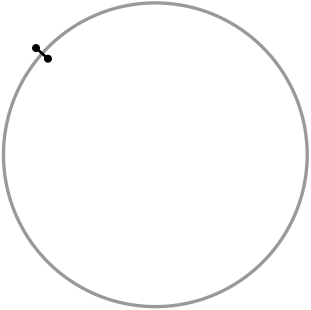

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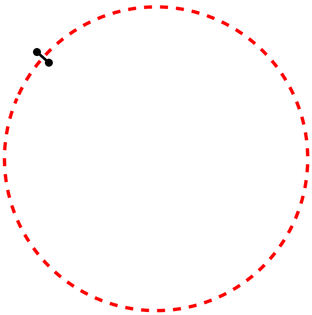

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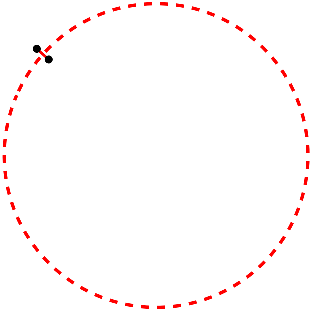

# Example: Weak order on $S_4$ by doubling

# Examples to come...

One practical consequence of polygonality is that we can experiment, especially in rank 3 (i.e. $|S| = 3$).

In fact, most of the examples I'm going to mention came from looking at the rank-3 case and asking "What congruence results from contracting this edge (or these edges)?"

We're run through some of these examples, starting with one general example: parabolic congruences and then moving for a while to the weak order on permutations.

# Homogeneous congruences

A (standard) parabolic subgroup of $W$ is a subgroup $W_K$ generated by some $K \subseteq S$.

By congruence uniformity, each join-irreducible congruence is $\mathrm{con}(j_* \lessdot j)$ for a unique join-irreducible element $j$.

The degree of a join-irreducible element $j$ is $|K|$, where $W_K$ is the smallest standard parabolic subgroup containing $j$.
The degree of a j.i. congruence $\mathrm{con}(j_* \lessdot j)$ is the degree of $j$.

**Example.** A join-irreducible congruence of degree 1 is $\mathrm{con}(1 \lessdot s_i)$.

**Example.** A join-irreducible congruence of degree 2 looks, for example, like $\mathrm{con}(s_i s_j \lessdot s_i s_j s_i)$.

A congruence is homogeneous of degree $d$ if it is a join of join-irreducible congruences of degree $d$. That is, it is "generated by (contracting) j.i. elements of degree $d$."

# Parabolic congruences

The j.i. elements of degree 1 are the elements of $S$.
The homogeneous congruences of degree 1 are the parabolic congruences $\Theta_K = \bigvee_{s \in S \setminus K} \mathrm{con}(1 \lessdot s)$ for $K \subseteq S$.

**Theorem** (R., 2004). The quotient $W/\Theta_K$ is isomorphic to the weak order on the parabolic subgroup $W_K$.

Consider the map $W \to W_K$ given by factoring $w$ as $w_K \cdot {}^K w$ in the usual way and sending $w$ to $w_K$.

The congruence classes of $\Theta_K$ are the fibers of this map.

In particular, the class of the identity is a set of minimal coset representatives for cosets of $W_K$ in $W$ (a "quotient" in a Coxeter-theoretic sense).

(On way to understand the "usual way" above: $w_K$ is the unique largest element of $W_K$ that is below $w$.)

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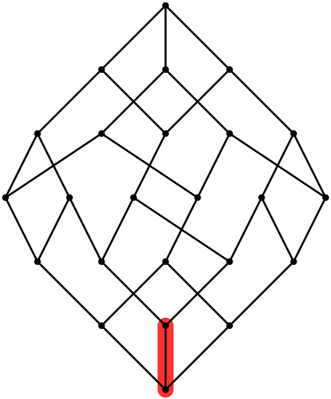
# Parabolic congruence example: $W = S_4$, $S = \{s_1, s_2, s_3\}$

Let $K = \{s_1, s_3\}$, so that $\Theta_K = \mathrm{con}(1 \lessdot s_2)$.



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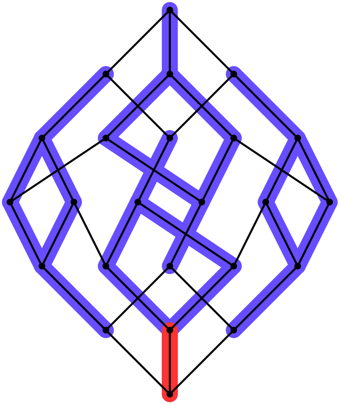

The $\Theta_K$-class of 1 is a set of minimal coset representatives for cosets of $W_{\{s_1,s_3\}}$.

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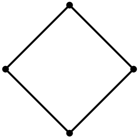


The $\Theta_K$-class of 1 is a set of minimal coset representatives for cosets of $W_{\{s_1,s_3\}}$.

Quotient is weak order on $W_{\{s_1,s_3\}}$.

# Recap of Section II.b: Lattice properties of the weak order

Weak order on a finite Coxeter group is a polygonal, congruence uniform lattice.

Lattice: BEZ lemma and local joins.
Polygonal: $zs_i \vee zs_j = z(s_i s_j s_i \cdots)$
Congruence uniform: Doubling—put in hyperplanes one by one.

Homogenous congruence of degree $d$:
Generated by contracting join-irreducible elements that belong to parabolic subgroups of rank $d$.

Parabolic congruences:
Contracting $s$ has the effect of deleting $s$ from $S$.

Questions?

# Section II.c: Congruences on the weak order on permutations

# Weak order on permutations

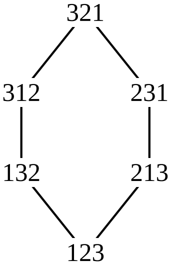
Inversions are

$$\mathrm{inv}(\pi) = \{\text{transpositions } (i\ \ j) : i \text{ comes before } j \text{ in } \pi\},$$

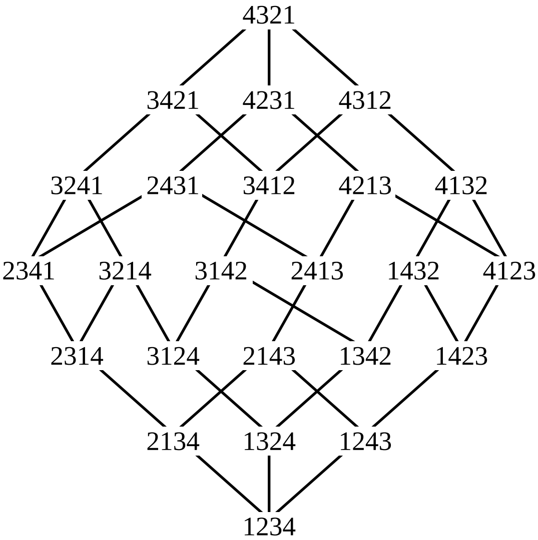
Going "up" by a cover means putting adjacent entries out of numerical order.

The weak order on any finite Coxeter group (including $S_n$) is polygonal.

**Example.** The weak order on $S_3$:



# Weak order on $S_4$

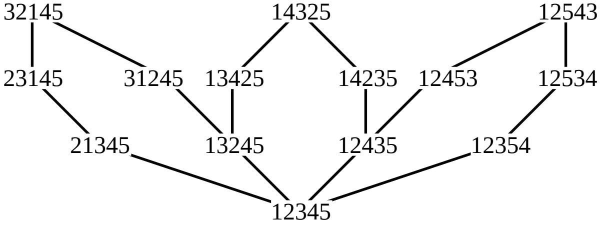


# Homogeneous congruences of degree 2

The standard parabolic subgroups $W_{\{s_i,s_j\}}$ are polygons at the bottom of the weak order on $W$ (specifically $2m(i,j)$-gons).

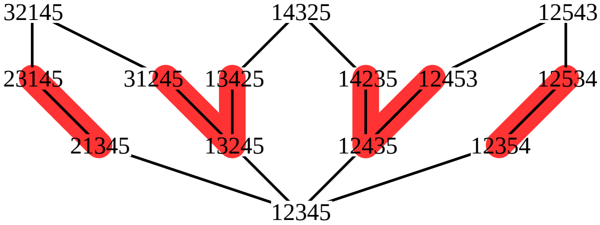
Each bottom edge of one of these polygons is $1 \lessdot s$ for $s \in S$ i.e. $s$ a j.i. element of degree 1. (We have seen that homogeneous congruences of degree 1 are parabolic congruences.)

Each side edge of one of these polygons is $j_* \lessdot j$ for a j.i. element $j$ of degree 2. Homogenous congruences of degree 2 are congruences generated by contracting some of these side edges.



Homogenous congruences of degree 2 are congruences generated by contracting some of the side edges in polygons at the bottom of the weak order. So, what can we try?

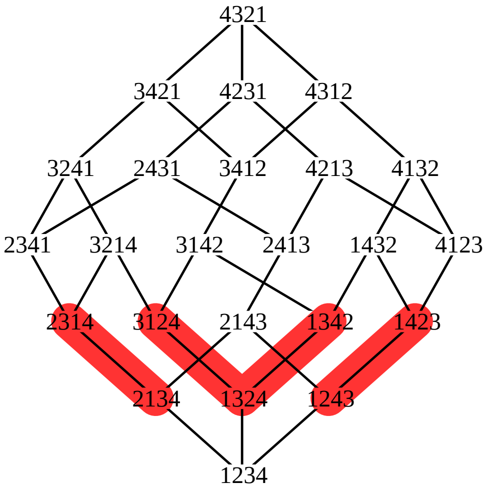
A natural choice it to contact all of them. Let $\Theta_2$ be the congruence generated by contracting all rank-two j.i. elements.




**Proposition.** (In any Coxeter group), the quotient $W/\Theta_2$ is isomorphic to the lattice $2^S$ of subsets of $S$.

# A homogeneous congruence of degree 2 in $S_4$

Let $\Theta_2$ be the congruence generated by contracting all rank-two j.i. elements.



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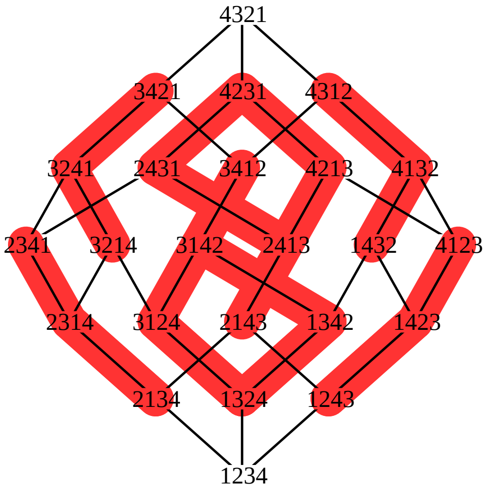
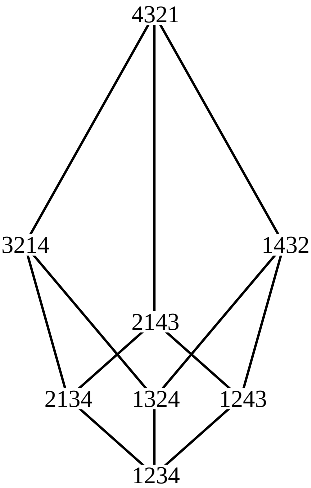


# A homogeneous congruence of degree 2 in $S_4$

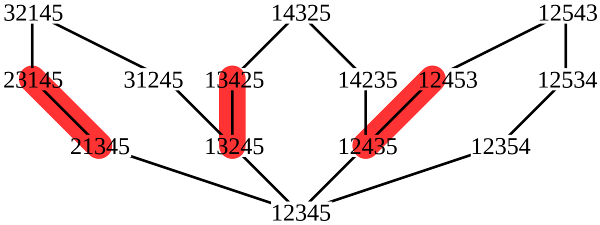
Let $\Theta_2$ be the congruence generated by contracting all rank-two j.i. elements.

Bottom elements are permutations with no descents of size $> 1$.



Homogenous congruences of degree 2 are congruences generated by contracting some of the side edges in polygons at the bottom of the weak order. So, what else can we try?

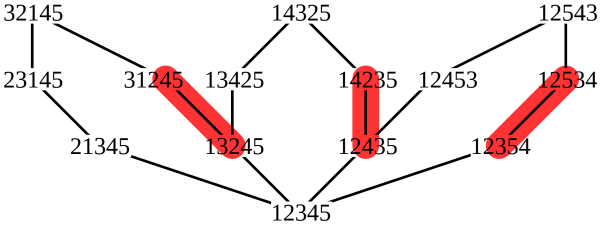
Let $\Theta_{\text{left}}$ be the congruence generated by contracting all rank-two j.i. elements on "left" of hexagons. Similarly $\Theta_{\text{right}}$.




**Proposition.** The quotient $W/\Theta_{\text{left}}$ is isomorphic to the Tamari lattice.

Homogenous congruences of degree 2 are congruences generated by contracting some of the side edges in polygons at the bottom of the weak order. So, what else can we try?

Let $\Theta_{\mathrm{left}}$ be the congruence generated by contracting all rank-two j.i. elements on "left" of hexagons. Similarly $\Theta_{\mathrm{right}}$.

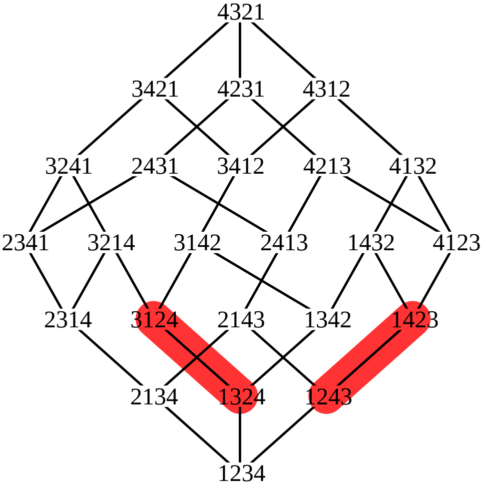



**Proposition.** The quotient $W/\Theta_{\mathrm{right}}$ is isomorphic to the Tamari lattice.

# Tamari lattice example

Let $\Theta_{\text{right}}$ be the congruence generated by contracting all rank-two j.i. elements on "right" of hexagons.

We've seen this before...



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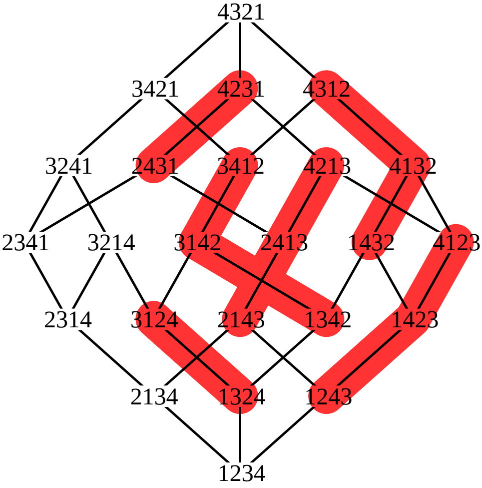
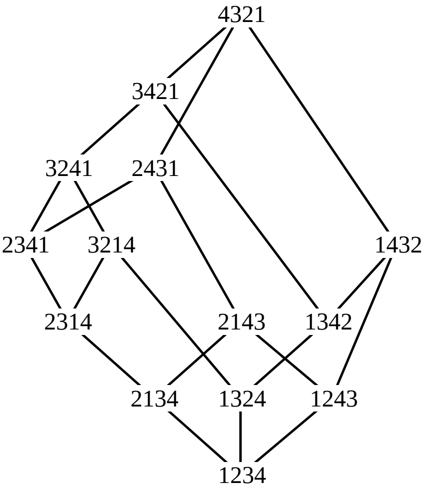


# Tamari lattice example

Let $\Theta_{\text{right}}$ be the congruence generated by contracting all rank-two j.i. elements on "right" of hexagons.

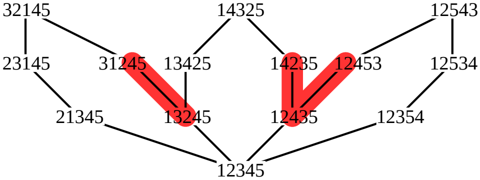
We've seen this before...

Bottom elements are 312-avoiding permutations.



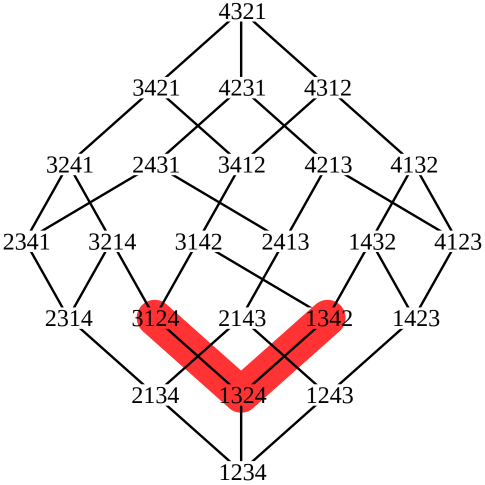
Homogenous congruences of degree 2: contract some side edges in polygons at the bottom of weak order. So, what else can we try?

Let $c$ stand (vaguely for now) for a choice, for each hexagon, to contract either the left j.i. or the right j.i. Let $\Theta_c$ be the congruence generated by contracting those j.i.'s.



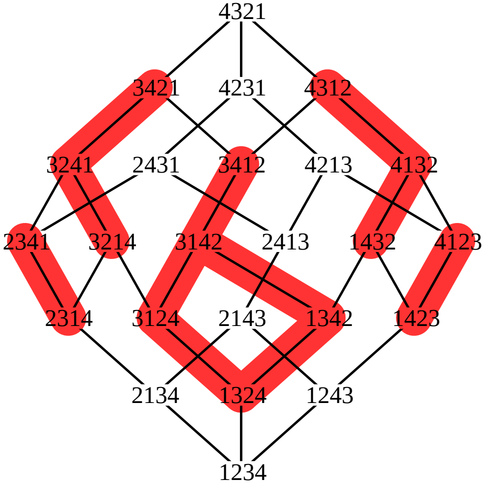

**Theorem** (R., 2006). The quotient $W/\Theta_c$ is a lattice whose Hasse diagram is isomorphic to the graph of the associahedron. In particular, it has Catalan many elements. These are the Cambrian lattices. (More later.)

A non-Tamari example.



# Cambrian lattice example

A non-Tamari example.

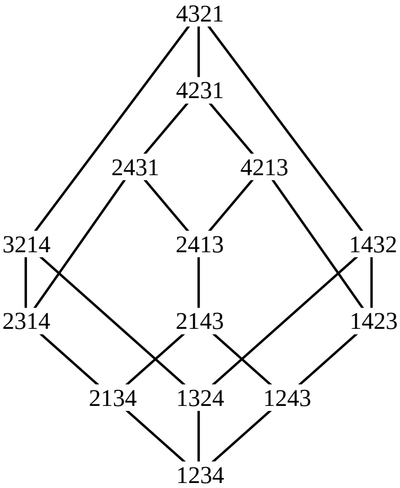


# Cambrian lattice example

A non-Tamari example.

14 bottom elements (Catalan).

Hasse diagram is the 1-skeleton of an associahedron.



# Aside: Multitriangulations (Pilaud, 2015)

For each $k$, there is a homogeneous degree-$(k+1)$ congruence on the weak order on permutations whose quotient is a lattice of $k$-triangulations, generalizing the Tamari lattice.

A $k$-triangulation of a convex $(n+2k)$-gon is a maximal set of diagonals such that no $k+1$ of them are pairwise crossing.

# Aside: Permutrees (Pilaud-Pons, 2016)

Pilaud and Pons' construction of permutrees can be seen as a uniform, simultaneous construction of all homogeneous degree-2 congruences on the weak order on permutations (together with the trivial congruence).

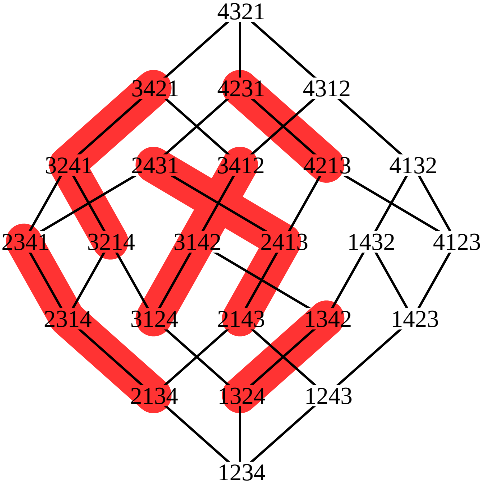
The decoration on a permutree indicates which side(s) of each hexagon are contracted in the associated congruence.

The set of linear extensions of the permutree is the corresponding congruence class.

# BiCambrian congruences

A biCambrian congruence is the meet (in the congruence lattice!) of two opposite Cambrian congruences.

For example, $\Theta_{\text{left}} \wedge \Theta_{\text{right}}$



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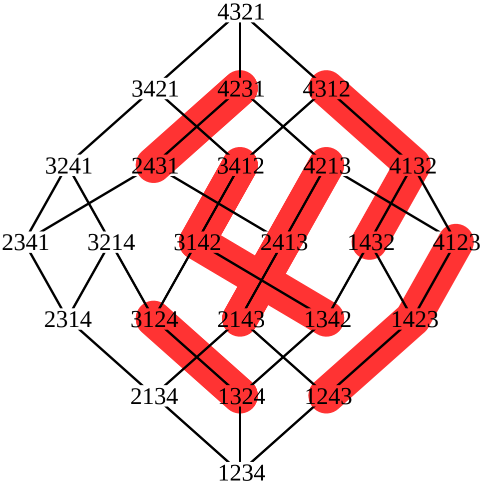

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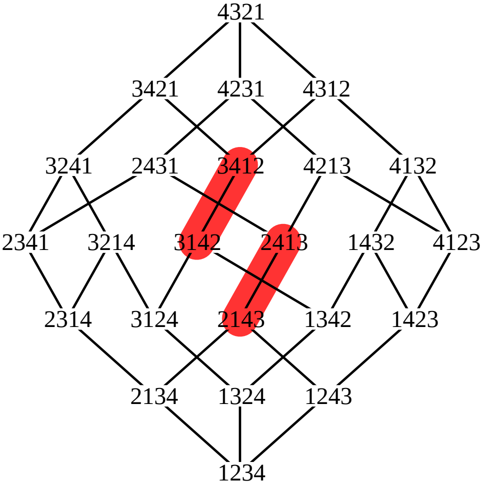
Homogeneous of degree 3



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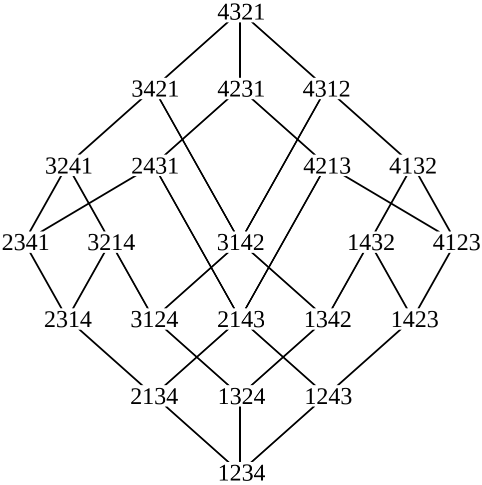
Homogeneous of degree 3

The bottom elements are twisted Baxter permutations, counted by the Baxter number.



# BiCambrian congruences

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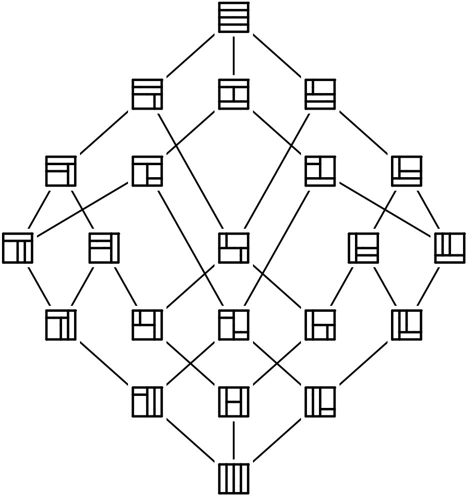
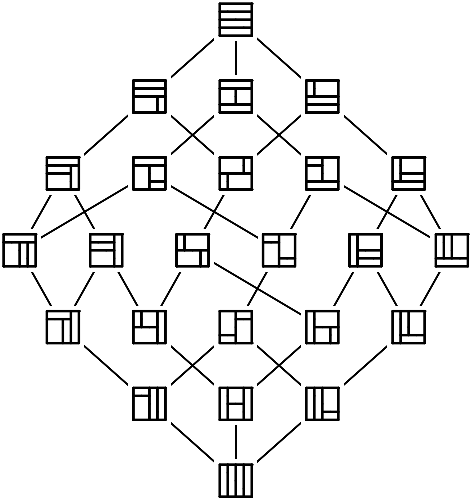
Homogeneous of degree 3

The bottom elements are twisted Baxter permutations, counted by the Baxter number.

A lattice of diagonal rectangulations, induced by a "permutations-to-rectangulations map."



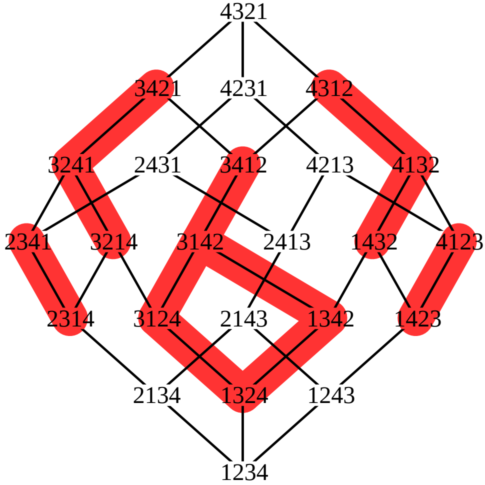
There is a coarser congruence than $\Theta_{\text{left}} \wedge \Theta_{\text{right}}$ that gives rise to a lattice of generic rectangulations. It's homogeneous of degree 4.



# Bipartite biCambrian congruences

A bipartite Cambrian congruence comes from choosing alternating left-right- $\cdots$ sides of hexagons.

The bipartite biCambrian congruence is the meet of two opposite bipartite Cambrian congruences.



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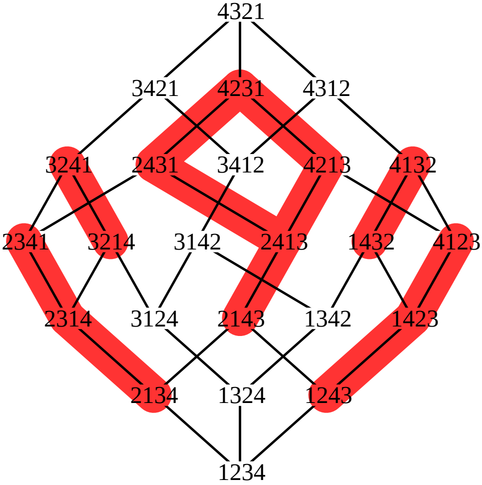

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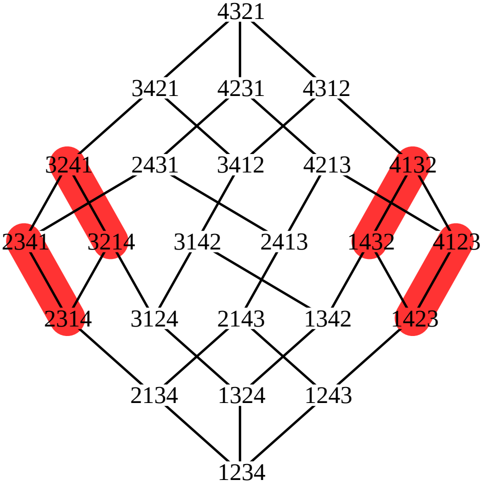
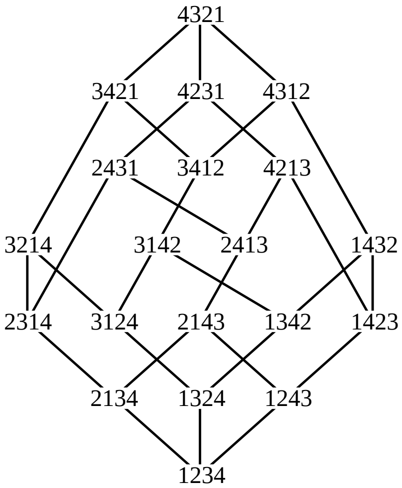


# Bipartite biCambrian congruences

A bipartite Cambrian congruence comes from choosing alternating left-right- $\cdots$ sides of hexagons.

The bipartite biCambrian congruence is the meet of two opposite bipartite Cambrian congruences.

The bottom elements are counted by central binomial coefficients. (Châtel-Pilaud 2014, Barnard 2015).



# Recap of Section II.c: Congruences on the weak order on permutations

Homogeneous congruences (for general Coxeter groups):

Degree 1: parabolic congruences—contract elements of $S$.

Degree 2: contract side edges of polygons at bottom of $W$.

Homogenous congruence examples for $W = S_n$:

Lattice of subsets

Tamari lattice

Cambrian lattices

biCambrian congruence: Meet of opposite Cambrian congruence

Baxter case (diagonal rectangulations), degree 3

bipartite case—central binomial coefficients.

A certain degree-4 congruence leads to generic rectangulations.

Pilaud (2015): lattice of $k$-triangulations (quotient of a homogeneous degree-$(k+1)$ congruence).

Questions?

# Section II.d: Noncrossing arc diagrams

# Combinatorial models

Recall that weak order on a Coxeter group is congruence uniform. That is, join-irreducible congruences biject with j.i. elements.

That also means the CJR of $w \in W$ is $\bigvee \{j_{u \lessdot w} : u \lessdot w\}$.

Recall also that the canonical join complex of a congruence uniform (or just semidistributive) lattice is flag.

This is crying out for a combinatorial model. Specifically, we would like a set of objects

- in bijection with join-irreducible elements of $W$.
- with a compatibility relation modeling edges of the CJC (so pairwise compatible sets of objects are in bijection with $W$).
- with forcing among j.i. elements read off combinatorially.

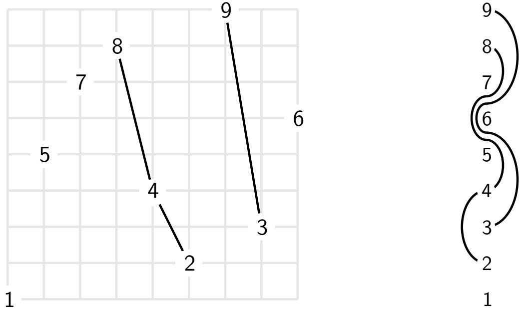
In the case of the symmetric group, we have a very nice model of this kind: Noncrossing arc diagrams.

# Noncrossing arc diagrams

It's easiest to start with the bijection $\delta$ from permutations to pairwise compatible sets of objects:

1. Graph $\pi_1 \cdots \pi_n$ by writing $\pi_i$ at the point $(i, \pi_i)$ in the plane.
2. Connect the descents with lines.
3. Move the numbers to a single vertical line. Lines connecting descents become curves ("arcs").

**Example.** $\delta(157842936)$.



# Noncrossing arc diagrams (continued)

Put $1, \ldots, n$ on a vertical line

Arcs connect the point (monotone up/down). Consider arcs up to combinatorics (endpoints and what points it passes left/right of).

Compatibility of arcs: non-intersecting except possibly at their endpoints, and they don't share the same upper endpoint or the same lower endpoint.

Noncrossing arc diagram: a collection of pairwise-compatible arcs.

**Theorem** (R., 2015). The map $\delta$ is a bijection from permutations in $S_n$ to noncrossing arc diagrams on $n$ points. The map $\delta$ restricts to a bijection between join-irreducible permutations and arcs. The arcs in $\delta(\pi)$ give the canonical join representation of $\pi$.

# Forcing and subarcs

Recall that we wanted a set of objects

- in bijection with join-irreducible elements of $W$.
- with a compatibility relation modeling edges of the CJC (so pairwise compatible sets of objects are in bijection with $W$).
- with forcing among j.i. elements read off combinatorially.

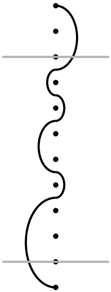
What's left is the third point: Describing forcing in terms of arcs.

The description is in terms of subarcs.

We define subarcs by a picture:

# Forcing and subarcs (continued)

We define subarcs by a picture:



# Forcing and subarcs (continued)

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We define subarcs by a picture:

**Theorem** (R., 2015). A join-irreducible permutation $j_1$ forces another j.i. permutation $j_2$ if and only if $\delta(j_1)$ is a subarc of $\delta(j_2)$.

The proof uses the geometry of shards.

# Forcing and subarcs (continued)

**Theorem.** A join-irreducible permutation $j_1$ forces another j.i. permutation $j_2$ if and only if $\delta(j_1)$ is a subarc of $\delta(j_2)$.

**General lattice fact.** The canonical join complex $\Gamma(\pi_\downarrow L)$ of a quotient is the subcomplex of $\Gamma(L)$ induced by j.i. elements not contracted by the congruence.

Recall that "$j_1$ forces $j_2$" **means** that a congruence that contracts $j_1$ must also contract $j_2$

Upshot: Given a set $U$ of arcs, TFAE:

(i) The set of noncrossing arc diagrams that can be made from arcs in $U$ is the CJC of some lattice quotient of $S_n$.

(ii) $U$ is closed under passing to subarcs.

# Example: Left-right noncrossing arc diagrams

The set of all right arcs (arcs that don't pass left of any points) is closed under passing to subarcs. The corresponding congruence is associated to the Tamari lattice. (Rotate the diagrams $90^\circ$ to get noncrossing partitions.)

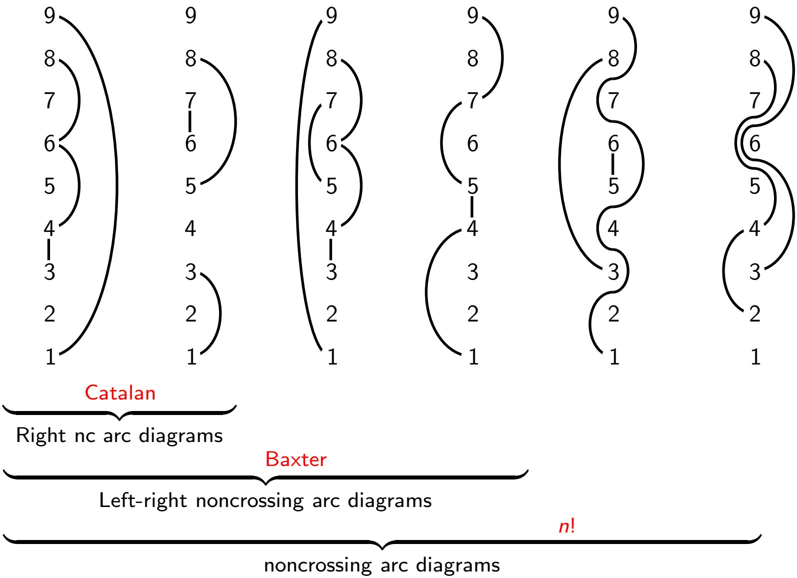
Similarly left arcs don't pass right of any points.

The set $\{\text{left arcs}\} \cup \{\text{right arcs}\}$ is closed under passing to subarcs. A left-right noncrossing arc diagram is a diagram made from left and right arcs.

It's not hard to translate excluded arcs into a pattern-avoidance condition. In this case, elements of $\pi_\downarrow S_n$ are the twisted Baxter permutations (permutations avoiding 2-41-3 and 3-41-2).

Conclusion: left-right noncrossing arc diagrams are counted by the Baxter number.

# Noncrossing arc diagrams



# Example: Alternating arc diagrams

The left-right diagrams example was a biCambrian congruence (the meet of two opposite "linear" Cambrian congruences).

Recall: a bipartite Cambrian congruence comes from choosing alternating left-right- $\cdots$ sides of hexagons.

The bipartite biCambrian congruence is the meet of the two opposite bipartite Cambrian congruences. The corresponding quotient is the bipartite biCambrian lattice.

The canonical join complex of the bipartite biCambrian lattice is the set of alternating arc diagrams, made from arcs that never pass left of two adjacent points and never pass right of two adjacent points.

**Theorem** (E. Barnard, 2015). There are $\binom{2n}{n}$ alternating arc diagrams on $n$ points.

# Recap of Section II.d: Noncrossing arc diagrams

Noncrossing arc diagrams are a model for permutations that is particularly suited to understanding lattice congruences.

- Arcs are in bijection with join-irreducible permutations.
- pairwise compatible sets of arcs are in bijection with permutations (and are faces of the CJC).
- forcing relations is subarc relation.

Quotients of the weak order are modeled by sets $U$ of arcs closed under subarcs.

Examples:

Right arcs: Tamari lattice
Left-right arcs: Twisted Baxter permutations/rectangulations
Alternating arcs: central binomial coefficients

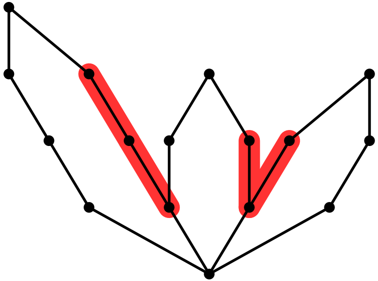
Questions?

# Section II.e: Cambrian lattices

# Cambrian lattices

In more general finite Coxeter groups, there may be larger polygons than hexagons at the bottom of the weak order on $W$.

The definition of a Cambrian congruence still calls for choosing one side of each polygon and contracting all side edges on that side.



As before, a Cambrian lattice is the quotient of the weak order on $W$ modulo a Cambrian congruence.

# Types A and B

We have seen that type-A Cambrian lattices are lattices of triangulations and that their Hasse diagrams are diagonal-flip graphs (1-skeleta of associahedra).

In type B, we get centrally symmetric triangulations and thus cyclohedra.

# Types A and B and beyond

We have seen that type-A Cambrian lattices are lattices of triangulations and that their Hasse diagrams are diagonal-flip graphs (1-skeleta of associahedra).

In type B, we get centrally symmetric triangulations and thus cyclohedra.

In general, Hasse diagrams of Cambrian lattices are graphs of generalized associahedra.

These are exchange graphs of cluster algebras of finite type and are also one of the central objects in Coxeter-Catalan combinatorics.

# (Coxeter-)Catalan combinatorics (various researchers, 1980–present)

There are hundreds of families of combinatorial objects known to be counted by the Catalan numbers $\frac{1}{n+2}\binom{2n+2}{n+1}$, including:

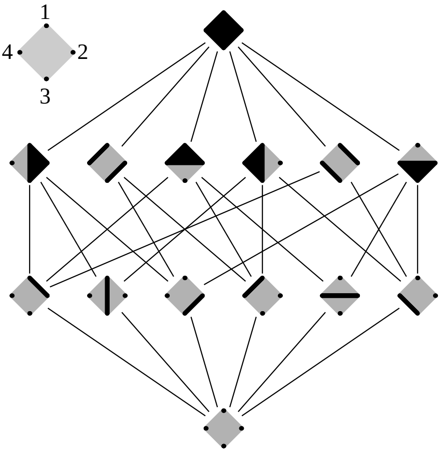
- Noncrossing partitions of an $(n+1)$-cycle
- Triangulations of an $(n+3)$-gon
- Dyck paths from $(0,0)$ to $(0,2(n+1))$
- 312-avoiding permutations.

Each is the type-$A_n$ case of definition that depends on a finite Coxeter group $W$. The generalized objects are each counted by the $W$-Catalan number

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}$$

The $h$ (Coxeter number) and the $e_i$'s (exponents) are fundamental numerical invariants of $W$.

# Noncrossing (NC) partitions (Kreweras, 1972)




Partitions of $(n+1)$-cycle with noncrossing parts.

Bijection with certain elements of $S_{n+1}$: parts become cycles (clockwise).

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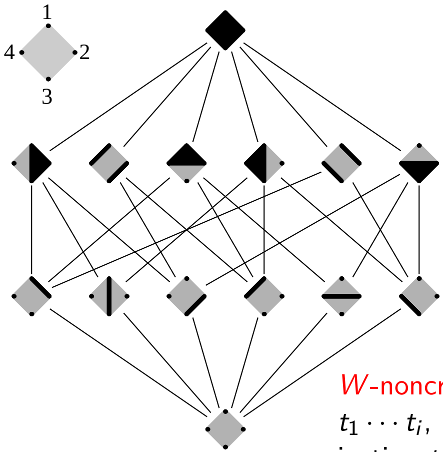


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Bijection with certain elements of $S_{n+1}$: parts become cycles (clockwise).

Generalization:
Factor the Coxeter element $c = s_1 \cdots s_n$ as a product of $n$ reflections $t_1 \cdots t_n$.

*W*-noncrossing partitions: elements $t_1 \cdots t_i$, as both $i \leq n$ and the factorization $t_1 \cdots t_n$ vary.

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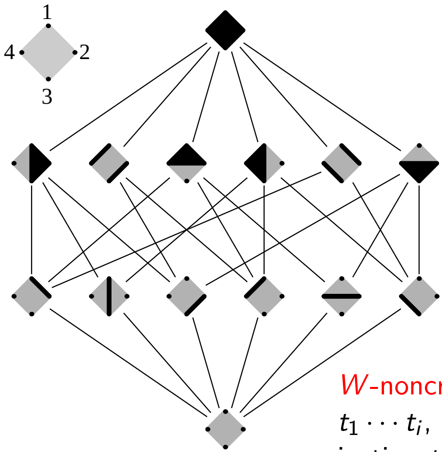


Partitions of $(n+1)$-cycle with noncrossing parts.

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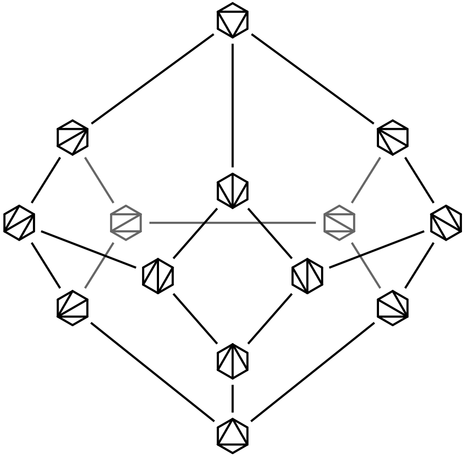
$W$-noncrossing partitions: elements $t_1 \cdots t_i$, as both $i \leq n$ and the factorization $t_1 \cdots t_n$ vary.

Why do this?

1. Eilenberg-MacLane spaces (and more) for Artin groups (e.g. the braid group).
2. Interesting algebraic combinatorics.

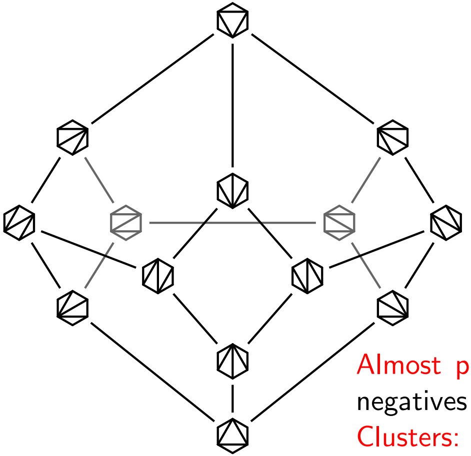


Vertices: Triangulations of $(n+3)$-gon. (max'l collections of noncrossing diagonals.)

Edges: "diagonal flips."

Associahedron: simple convex polytope with this graph.

# Associahedron (Haiman, Lee, Milnor, Stasheff, 1963–1989)



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Generalized associahedron (Fomin, Zelevinsky, 2003):

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**Example.** $W = S_n$

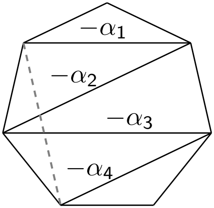
Simples: $\alpha_1, \ldots \alpha_{n-1}$

Positives: $\alpha_i + \cdots + \alpha_i$, $i \leq j$

$\binom{n+1}{2} - 1$ diagonals of $(n+3)$-gon $\leftrightarrow$ "almost positive roots."

Compatible = noncrossing

Exchanges = diagonal flips.



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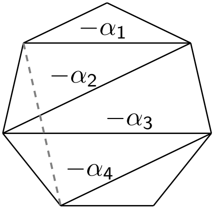
Why do this?

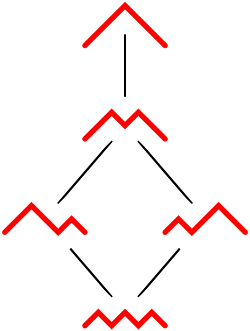
1. Cluster algebras of finite type.
2. Interesting polytopes.
3. Interesting algebraic combinatorics.



# Dyck paths

These are paths from $(0,0)$ to $(0,2(n+1))$ with Northeast and Southeast steps of length $\sqrt{2}$, never passing below the horizontal axis.

**Example** $(n = 2)$.



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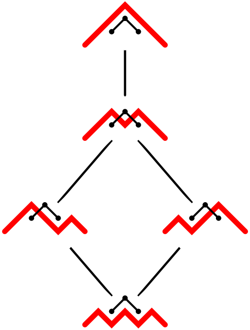
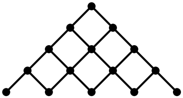


Equivalently, order ideals (or antichains) in a poset of this form with $n+1$ minimal elements:



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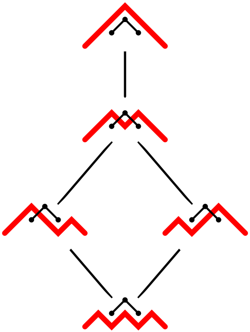

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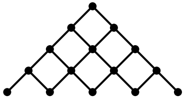

More generally, antichains in the root poset are counted by $\mathrm{Cat}(W)$.

# 312-avoiding permutations

So far, $\mathrm{Cat}(W)$ counts:

- $W$-noncrossing partitions
- clusters of almost positive roots
- antichains in the root poset

Our other Catalan objects were the 312-avoiding permutations.

These are permutations having no subsequence *cab* with $a < b < c$.

**Example.** Not 312-avoiding:

$$3762451$$

These are generalized as sortable elements.

# Sorting words

Fix some reduced word $s_1 \cdots s_n$ for a Coxeter element $c$. Form an infinite word

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$$

The *c*-sorting word for $w$ is the lexicographically first (i.e. leftmost) subword of $c^\infty$ which is a reduced word for $w$.

**Example.** $W = B_4$

$$s_1 \overset{4}{\text{———}} s_2 \text{———} s_3 \text{———} s_4$$

For $c = s_1 s_2 s_4 s_3$,

$$c^\infty = s_1 s_2 s_4 s_3 | s_1 s_2 s_4 s_3 | s_1 s_2 s_4 s_3 | \cdots$$

The element $w = s_4 s_2 s_1 s_2 s_3 s_2 s_1 s_2 s_1$ has $c$-sorting word

$$s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 s_2.$$

# Sorting words in $S_{n+1}$

Multiplying a permutation $\pi$ on the left by an adjacent transposition $s_i := (i \;\; i{+}1)$ swaps the entries $i$ and $i + 1$ in $\pi$.

Do this repeatedly, always putting entries into numerical order, and record the sequence of $s_i$'s. Result: a reduced word for $\pi$.

Fix an order on the adjacent transpositions, and write a reduced word for $\pi$ by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for $\pi$. (C.f. "bubble sort.")

**Example:** $W = S_4, \quad c = s_1 s_2 s_3, \quad \pi = 4231$

| Step | $s_i$ tried | Sorting word | "Remainder" |
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| 5 | $s_2$ | $s_1 s_2 s_3 \vert s_2$ | 2134 |

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| 7 | $s_1$ | $s_1 s_2 s_3 \vert s_2 \vert s_1$ | 1234 |

# Sortable elements of a Coxeter group $W$

In general, to find the $c$-sorting word for $w \in W$:
Try the generators cyclically according to $c$.
Place a divider "|" every time a pass through $S$ is completed.
A $c$-sorting word can be interpreted as a sequence of sets
(sets of letters between dividers "$|$").
If the sequence is nested then $w$ is $c$-sortable.

**Example:** $\pi = 4231$ with $c$-sorting word $s_1 s_2 s_3 | s_2 | s_1$
$\pi$ is not $c$-sortable because $\{s_1\} \not\subseteq \{s_2\}$.

**Example:** $W = B_2, \quad c = s_1 s_2$

$c$-sortable: $1,\ s_1,\ s_1 s_2,\ s_1 s_2 | s_1,\ s_1 s_2 | s_1 s_2,\ s_2$

not $c$-sortable: $s_2 | s_1,\ s_2 | s_1 s_2$

# Example: Sortable elements in $S_{n+1}$

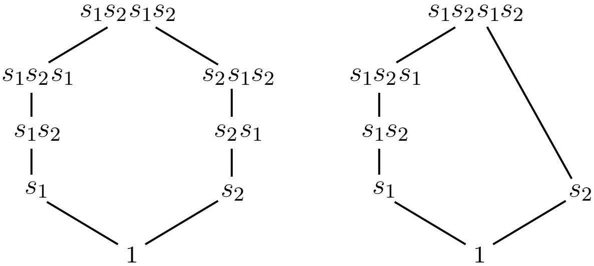
$W = S_{n+1}$,
For $c = s_1 s_2 \cdots s_n$, the $c$-sortable elements are the 312-avoiding permutations.

| | | | |
|---|---|---|---|
| $1$ | 1234 | $s_1 s_2 s_3 \vert s_2$ | 2431 |
| $s_1$ | 2134 | $s_1 s_2 \vert s_1$ | 3214 |
| $s_1 s_2$ | 2314 | $s_1 s_3$ | 2143 |
| $s_1 s_2 s_3$ | 2341 | $s_2$ | 1324 |
| $s_1 s_2 s_3 \vert s_1$ | 3241 | $s_2 s_3$ | 1342 |
| $s_1 s_2 s_3 \vert s_1 s_2$ | 3421 | $s_2 s_3 \vert s_2$ | 1432 |
| $s_1 s_2 s_3 \vert s_1 s_2 \vert s_1$ | 4321 | $s_3$ | 1243 |

For $c = s_n s_{n-1} \cdots s_1$, the $c$-sortable elements are the 231-avoiding or stack-sortable permutations.

For other Coxeter elements, the condition is more complicated, blending the two avoidance conditions.

A Coxeter element $c$ defines a $c$-Cambrian congruence $\Theta_c$: For each polygon $[1, s_i \vee s_j]$ at the bottom of the weak order such that $s_i$ precedes $s_j$ in $c$, contract the side edges on the $s_j$ side.




**Theorem.** The bottom elements of $\Theta_c$ are exactly the $c$-sortable elements. In particular, the $c$-Cambrian lattice $W/\Theta_c$ has $\mathrm{Cat}(W)$ elements.

**Theorem.** The $c$-Cambrian lattice is a sublattice of $W$.

# The bijection to clusters of almost-positive roots

Let $v$ be $c$-sortable with $c$-sorting word $a_1 a_2 \cdots a_k$.

Given $s_i \in S$, the *last reflection* for $s$ in $v$ is $a_1 \cdots a_{j-1} a_j a_{j-1} \cdots a_1$, where $a_i$ is the rightmost occurrence of $s$ in $a_1 a_2 \cdots a_k$.

Define $\mathrm{cl}_c^{s_i}(v)$ to be the positive root associated to this last reflection. That is, $\mathrm{cl}_c^{s_i}(v)$ is $a_1 \cdots a_{j-1} \alpha_j$.

If $s$ doesn't occur in $a_1 a_2 \cdots a_k$, define $\mathrm{cl}_c^{s_i}(v) = -\alpha_i$.

Define $\mathrm{cl}_c(v) = \{\mathrm{cl}_c^{s_i}(v) : i = 1, \ldots, n\}$.

**Example:** $W = B_4, \quad c = s_1 s_2 s_4 s_3$, quad $v = s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 s_2$

$\mathrm{cl}_c(v) =$

$\{ \qquad \qquad \qquad \qquad \qquad \qquad \}.$

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**Example:** $W = B_4, \quad c = s_1 s_2 s_4 s_3$, quad $v = s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 {\color{red} s_2}$
$\mathrm{cl}_c(v) =$
$\{s_1 s_2 s_4 s_3 s_1 s_2 s_3 \alpha_1, s_1 s_2 s_4 s_3 s_1 s_2 s_3 s_1 {\color{red} \alpha_2} \qquad\qquad \}.$

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**Example:** $W = B_4, \quad c = s_1 s_2 s_4 s_3$, quad $v = s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 s_2$
$\mathrm{cl}_c(v) =$
$\{s_1 s_2 s_4 s_3 s_1 s_2 s_3 \alpha_1, s_1 s_2 s_4 s_3 s_1 s_2 s_3 s_1 \alpha_2, s_1 s_2 s_4 s_3 s_1 s_2 \alpha_3, s_1 s_2 \alpha_4\}$.

# The bijection to clusters of almost-positive roots

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Define $\mathrm{cl}_c^{s_i}(v)$ to be the positive root associated to this last reflection. That is, $\mathrm{cl}_c^{s_i}(v)$ is $a_1 \cdots a_{j-1} \alpha_j$.

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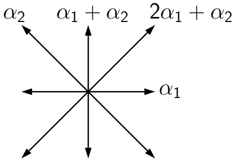
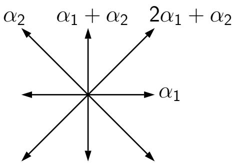
**Theorem.** For any finite Coxeter group and any Coxeter element $c$, the map $\mathrm{cl}_c$ is a bijection from $c$-sortable elements to $c$-clusters of almost positive roots. (Bipartite $c$ gives usual clusters.)

# Clusters example: $W = B_2, \quad c = s_1 s_2$



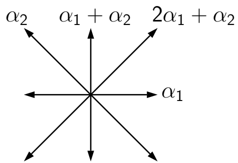

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | |
| | $s_2$ | |
| $s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



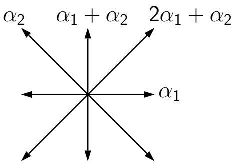

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | |
| $s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



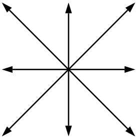

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



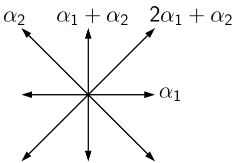

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | |
| $s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



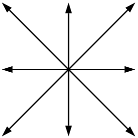

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



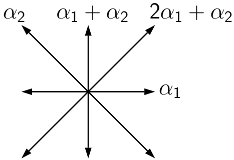

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| 1 | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



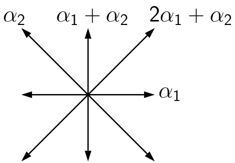

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



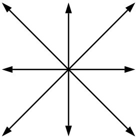

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1 \alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



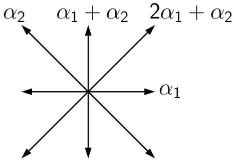

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1 \alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 \alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$



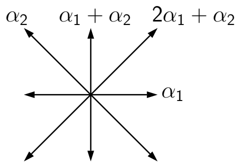

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2$, $c = s_1 s_2$



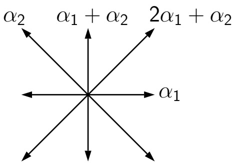

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 s_2 s_1 \alpha_2 = \alpha_2$ |
| $s_2$ | $s_1$ | |
| | $s_2$ | |

# Clusters example: $W = B_2$, $c = s_1 s_2$



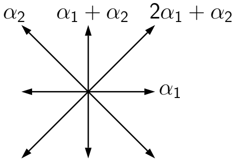

| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 s_2 s_1 \alpha_2 = \alpha_2$ |
| $s_2$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | |

# Clusters example: $W = B_2, \quad c = s_1 s_2$




| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 s_2 s_1 \alpha_2 = \alpha_2$ |
| $s_2$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $\alpha_2$ |

# Clusters example: $W = B_2, \quad c = s_1 s_2$




| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1 \alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
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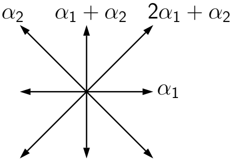
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 s_2 s_1 \alpha_2 = \alpha_2$ |
| $s_2$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $\alpha_2$ |

# Clusters example: $W = B_2, \quad c = s_1 s_2$




| $v$ | $s_i$ | $\mathrm{cl}_c^{s_i}(v)$ |
|---|---|---|
| $1$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $-\alpha_2$ |
| $s_1 s_2$ | $s_1$ | $\alpha_1$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1\alpha_2 = 2\alpha_1 + \alpha_2$ |
| $s_1 s_2 s_1 s_2$ | $s_1$ | $s_1 s_2 \alpha_1 = \alpha_1 + \alpha_2$ |
| | $s_2$ | $s_1 s_2 s_1 \alpha_2 = \alpha_2$ |
| $s_2$ | $s_1$ | $-\alpha_1$ |
| | $s_2$ | $\alpha_2$ |

Aside: Clusters of almost roots give denominator vectors of clusters of cluster variables in cluster algebras of finite type.

# The bijection to noncrossing partitions

A cover reflection of $w \in W$ is an inversion $t$ of $w$ such that $tw = ws$ for some $s \in S$.

("Cover reflection" because $w \gtrdot tw$ in the weak order. Also, every cover $ws \lessdot w$ is associated to a cover reflection $wsw^{-1}$ of $w$.)

$\text{cov}(w) = \{\text{cover reflections of } w\}$.

**Example.** In $S_n$: $\text{cov}(\pi) = \{\text{transpositions } (\pi_{i+1}\ \pi_i) : \pi_i > \pi_{i+1}\}$.

Let $v$ be $c$-sortable with $c$-sorting word $a_1 a_2 \cdots a_k$. Each $t \in \text{cov}(v)$ has a unique $i \in [k]$ with $tv = s_1 s_2 \cdots \hat{s}_i \cdots s_k$. Let $\mathrm{nc}_c(v)$ be the product of the cover reflections of $v$, ordered by increasing $i$.

**Example.** $W = B_4, \;\; v = s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 s_2$.
Cover reflections: $ws_1 w^{-1} = s_1$ and $ws_2 w^{-1} = s_2 s_3 s_4 s_3 s_2$, corresponding to $\hat{s}_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 \hat{s}_2$.

Thus $\mathrm{nc}_c(w) = s_1 \cdot s_2 s_3 s_4 s_3 s_2$.

# The bijection to noncrossing partitions (cont'd)

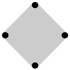
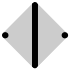
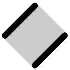
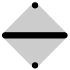
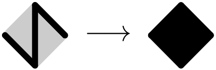
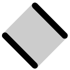
**Theorem:** The map $\mathrm{nc}_c$ is a bijection between $c$-sortable elements and non-crossing partitions.

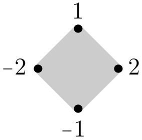
**Example.** In $S_n$, we already discussed this bijection from 312-avoiding permutations to noncrossing partitions.

**Example.** $W = B_2$, next slide.

# Example: $W = B_2$, $\quad c = s_1 s_2 = (1\ 2\ \ -1\ \ -2)$

$s_1 = (-1\ 1)$, $\quad s_2 = (1\ 2)(-1\ \ -2)$

| $v$ | $\mathrm{nc}_c(v)$ | |
|---|---|---|
| $1$ | $1$ |  |
| $\hat{s}_1$ | $s_1$ |  |
| $s_1 \hat{s}_2$ | $s_1 s_2 s_1$ |  |
| $s_1 s_2 \mid \hat{s}_1$ | $s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2$ |  |
| $\hat{s}_1 s_2 \mid s_1 \hat{s}_2$ | $s_1 \cdot s_2$ |  |
| $\hat{s}_2$ | $s_2$ |  |



# Proof technique: induction on length and rank

A given Coxeter element $c$ may have several reduced words. They are all equivalent by transpositions of commuting elements of $S$.

A generator $s \in S$ is initial in $c$ if there is a reduced word for $c$ having $s$ as its first letter. Similiarly, $s$ is final in $c$ if it is the last letter of some reduced word for $c$. In either case, the element $scs$ is another Coxeter element.

Example: $W = S_4$
If $c = s_1s_3s_2 = s_3s_1s_2$ then $s_1$ and $s_3$ are initial and $s_2$ is final.
If $c = s_1s_2s_3$ then $s_1$ is initial and $s_3$ is final.

Passing from $c \leftrightarrow scs$, for $s$ initial or final, is a source-sink move or BGP reflection functor.

**Lemma.** Let $s$ be initial in $c$ and suppose $w \not\geq s$. Then $w$ is $c$-sortable if and only if it is an $sc$-sortable element of $W_{\langle s \rangle}$.

**Lemma.** Let $s$ be initial in $c$ and suppose $w \geq s$. Then $w$ is $c$-sortable if and only if $sw$ is $scs$-sortable.

Both become obvious on inspection of the definition, and staring at:

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$$

Since the identity element is $c$-sortable for any $c$, the lemmas are a recursive characterization of $c$-sortability, by induction on the length $\ell(w)$ and on the rank of $W$ (the cardinality of $S$).

# Proof technique: induction on length and rank (continued)

**Lemma.** Let $s$ be initial in $c$ and suppose $w \not\geq s$. Then $w$ is $c$-sortable if and only if it is an $sc$-sortable element of $W_{\langle s \rangle}$.

**Lemma.** Let $s$ be initial in $c$ and suppose $w \geq s$. Then $w$ is $c$-sortable if and only if $sw$ is $scs$-sortable.

Both become obvious on inspection of the definition, and staring at:

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$$

Since the identity element is $c$-sortable for any $c$, the lemmas are a recursive characterization of $c$-sortability, by induction on the length $\ell(w)$ and on the rank of $W$ (the cardinality of $S$).

This form of induction is the most important proof technique for sortable elements. For example, it is used to connect sortable elements to Cambrian lattices.

# Aside: Coxeter-biCatalan combinatorics

Lattice theory leads to interesting new combinatorics...

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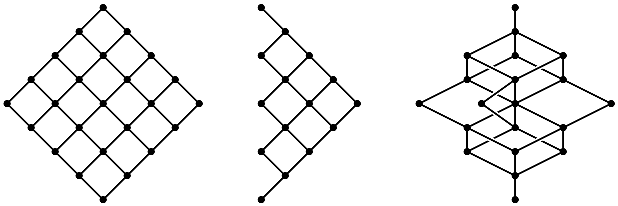
A part of Coxeter-Catalan combinatorics: elements of the Cambrian lattice are in bijection with antichains in the root poset.

The doubled root poset is the root poset and an upside-down copy of the root poset, identified at the simple roots.



**Theorem** (E. Barnard, R., 2016). Antichains in the doubled root poset are in bijection with elements of the bipartite biCambrian lattice. (Recall that this is $W/(\Theta_c \wedge \Theta_{c^{-1}})$ for bipartite $c$.)

# Recap of Section II.e: Cambrian lattices

Cambrian lattice is $W/\Theta_c$ (contract one side of each polygon at the bottom of weak order).

Coxeter Catalan combinatorics

| Type A | General $W$ |
|---|---|
| Noncrossing partitions of a cycle | "prefixes" of $c$ |
| triangulations | clusters of almost positive roots |
| Dyck paths | antichains in the root poset |
| 312-avoiding permutations | sortable elements |

Nice bijections using combinatorics of sorting words.

Sortable elements are bottom el'ts of the Cambrian congruence and are a sublattice of the weak order (the Cambrian lattice).

Coxeter-biCatalan combinatorics

Questions?

# References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