# Outline for the class

Part I. Lattice congruences for combinatorialists

*The lattice-theoretic "facts of life," emphasizing ideas most relevant to the weak order.*

Part II. Lattice congruences of the weak order

*We apply our knowledge to the weak order, motivated by examples, and develop the combinatorics of congruences/quotients, in general and in specific.*

Part III. The geometry of lattice congruences on posets of regions

*We place the lattice theory in the geometric setting of hyperplane arrangements and "shards."*

# Part I: Lattice congruences for combinatorialists

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NC State University

Algebraic and Geometric Combinatorics of Reflection Groups
CRM/LaCIM Spring School
UQAM, June 1–2, 2017

# Section I.a: Lattice congruences and quotients

# Lattices

A lattice is a set $L$ with two binary operations $\wedge$ ("meet") and $\vee$ ("join") satisfying the axioms:

- $x \vee y = y \vee x$
- $x \wedge y = y \wedge x$
- $x \vee (y \vee z) = (x \vee y) \vee z$
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- $x \vee (x \wedge y) = x$
- $x \wedge (x \vee y) = x$

for all $x, y, z \in L$.

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An example of a lattice:

| $\vee$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ | $1$ |
| $a$ | $a$ | $a$ | $1$ | $1$ | $1$ |
| $b$ | $b$ | $1$ | $b$ | $c$ | $1$ |
| $c$ | $c$ | $1$ | $c$ | $c$ | $1$ |
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|---|---|---|---|---|---|
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| $b$ | $0$ | $0$ | $b$ | $b$ | $b$ |
| $c$ | $0$ | $0$ | $b$ | $c$ | $c$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $0$ |

# Lattices

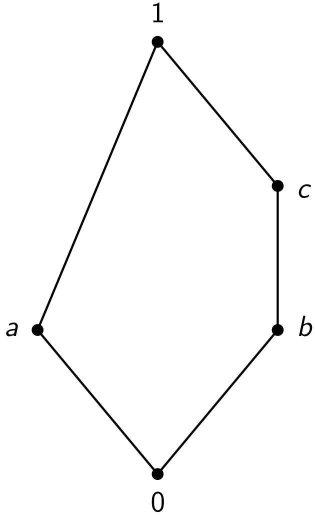
A lattice is a set $L$ with a partial order "$\leq$" such that:

For all finite $S \subseteq L$,

- There exists a unique minimal upper bound for $S$ is $L$, written $\bigvee S$.
- There exists a unique maximal lower bound for $S$ is $L$, written $\bigwedge S$.

# Lattices

An example of a lattice:

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(Universal) algebra

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Combinatorics

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(Universal) algebra

$x \leq y$ iff $x \vee y = y$ iff $x \wedge y = x$

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Combinatorics

$$x \vee y = \bigvee \{x, y\}$$

$$x \wedge y = \bigwedge \{x, y\}$$

# Homomorphisms, congruences, quotients

(Lattice) homomorphism: a map $\eta : L_1 \to L_2$ such that

$$\eta(x \wedge y) = \eta(x) \wedge \eta(y) \text{ and } \eta(x \vee y) = \eta(x) \vee \eta(y).$$

Congruence: an equivalence relation $\equiv$ on $L$ such that

$$(x_1 \equiv x_2 \text{ and } y_1 \equiv y_2) \implies (x_1 \wedge y_1 \equiv x_2 \wedge y_2 \text{ and } x_1 \vee y_1 \equiv x_2 \vee y_2).$$

Quotient: The set $L/\equiv$ of congruence classes with meet and join

$$[x] \vee [y] = [x \vee y] \text{ and } [x] \wedge [y] = [x \wedge y].$$

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$$[x] \vee [y] = [x \vee y] \text{ and } [x] \wedge [y] = [x \wedge y].$$

What do these mean in the order-theoretic definition of lattices?

# Order-theoretic characterization of a lattice congruence

An equivalence relation $\equiv$ on a finite lattice $L$ is a lattice congruence if and only if the following three conditions hold:

(i) Each equivalence class is an interval in $L$.

(ii) The map $\pi_\downarrow$ taking each element to the bottom element of its equivalence class is order-preserving.

(iii) The map $\pi^\uparrow$ taking each element to the top element of its equivalence class is order-preserving.

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$(x \leq y \leq z$ and $x \equiv z) \implies (x \vee y) \equiv (z \vee y)$, i.e. $y \equiv z$.

That's "congruence $\Longrightarrow$ (i)." The rest is similar in spirit.

# Order-theoretic characterization of congruence (continued)

On finite $L$, an equivalence relation $\equiv$ is a lattice congruence iff:

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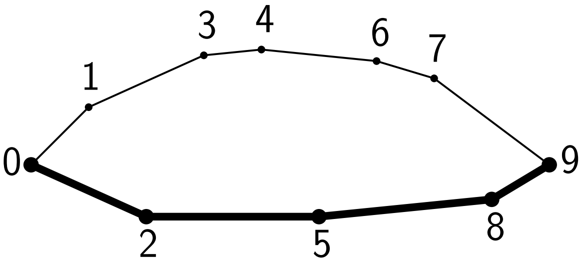
Aside: If you encounter a surjective set map $\eta : L \to S$ (a set):

- Check if the fibers (preimages of el'ts of $S$) are intervals in $L$.
- If so, check (ii) and (iii) on the fibers.
- If these hold, then the fibers of $\eta$ are a congruence $\equiv$, and $\eta$ induces a lattice structure on $S$, isomorphic to $L/\equiv$.

# Example: Permutations-to-triangulations map

Arrange the numbers from 0 to $n+1$ on a polygon such that numbers strictly increase left to right. Begin with a path along the bottom. Modify the path by removing/adding vertices in the order given by $\pi$. The triangulation is the union of the paths.

**Example.** $\pi = 42783165$



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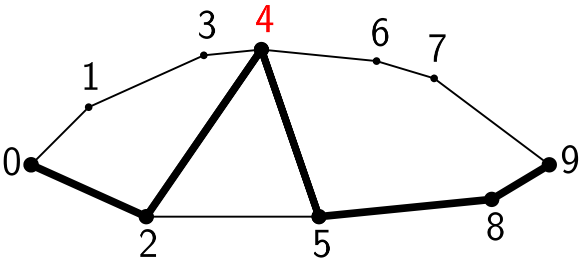
4



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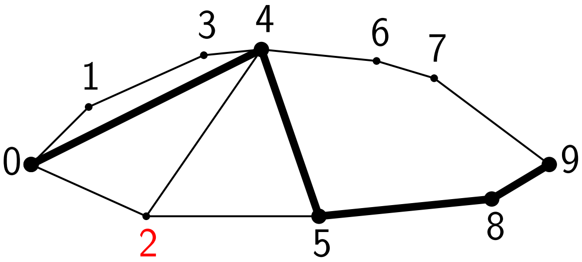
42



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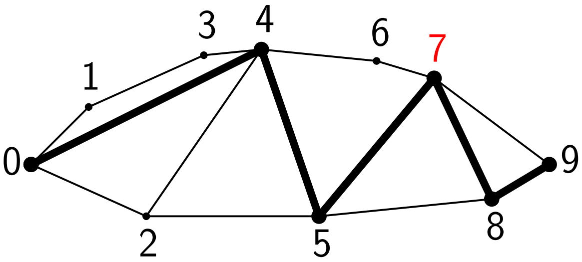
427



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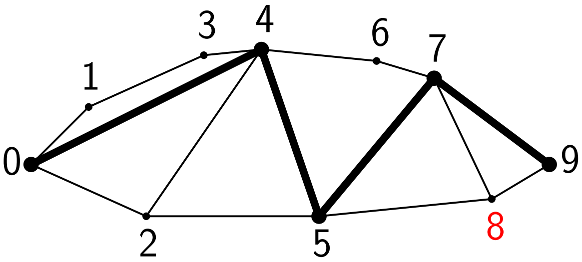
4278



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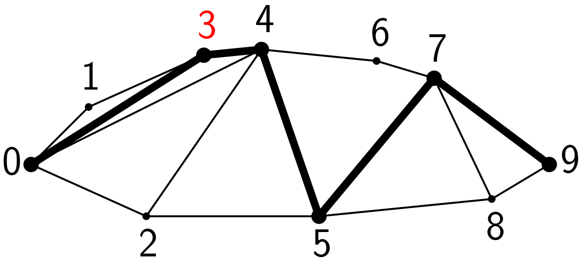

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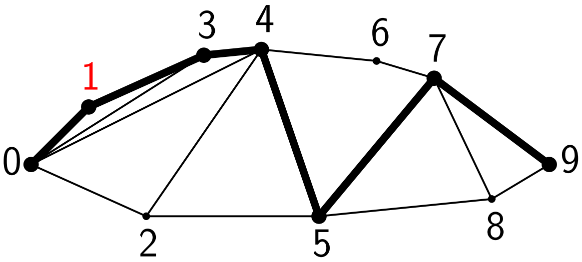

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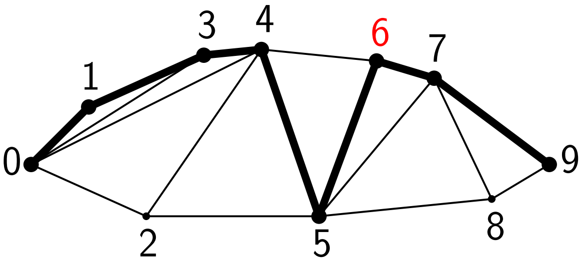

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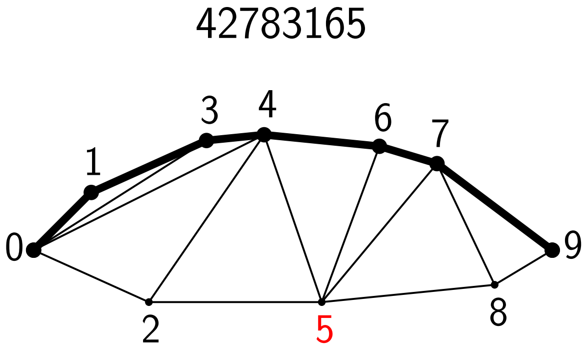

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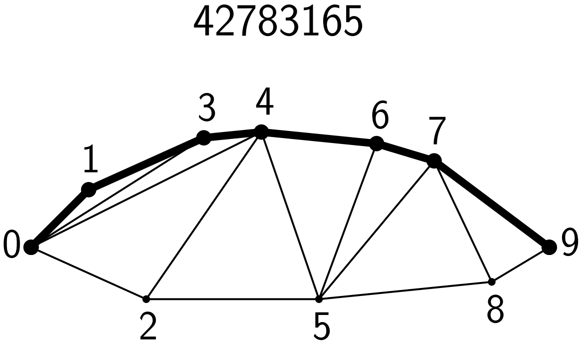
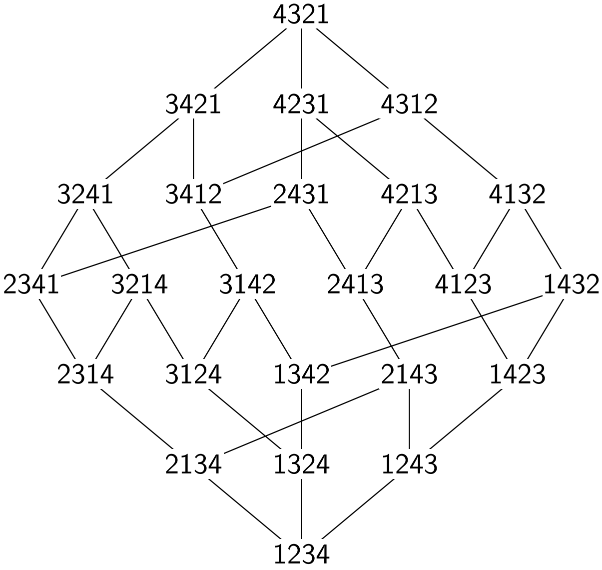


# $S_4$ to triangulations



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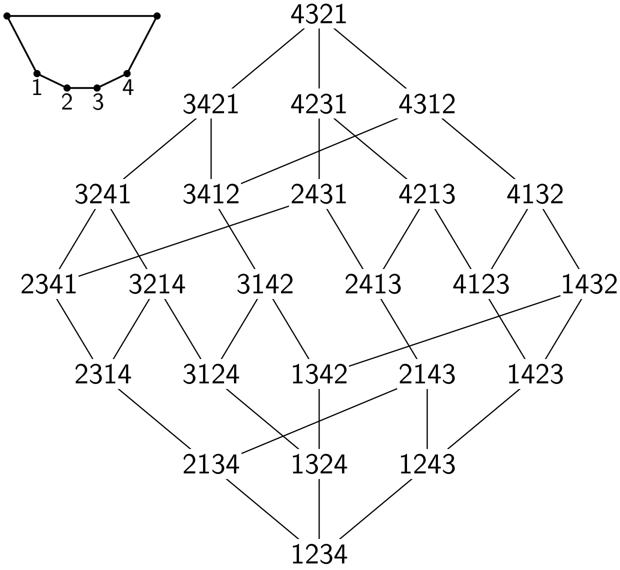

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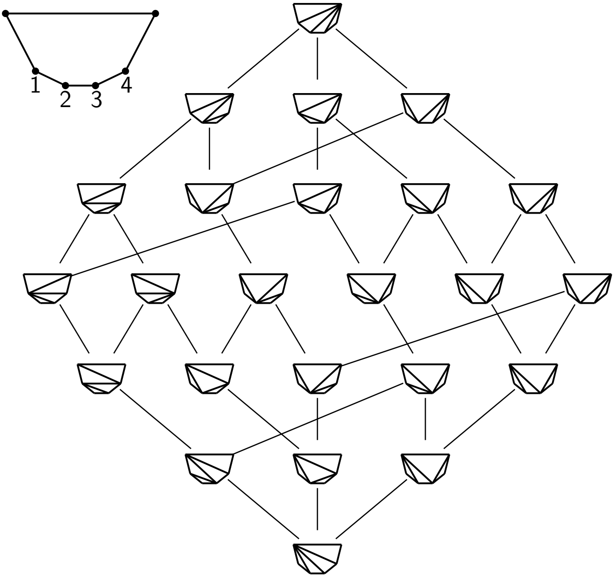

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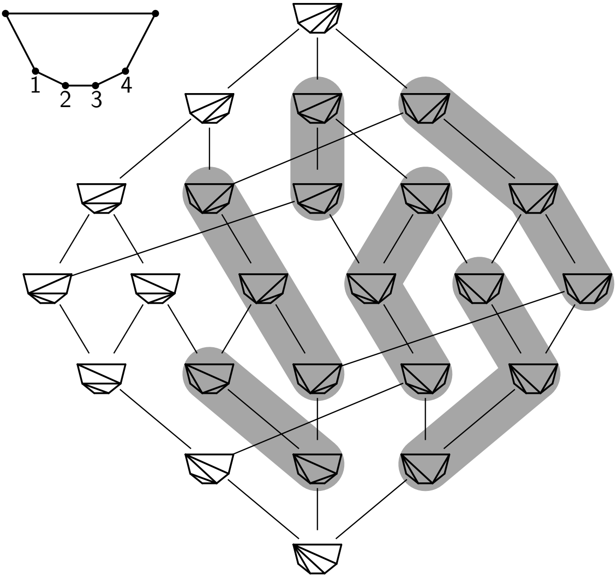

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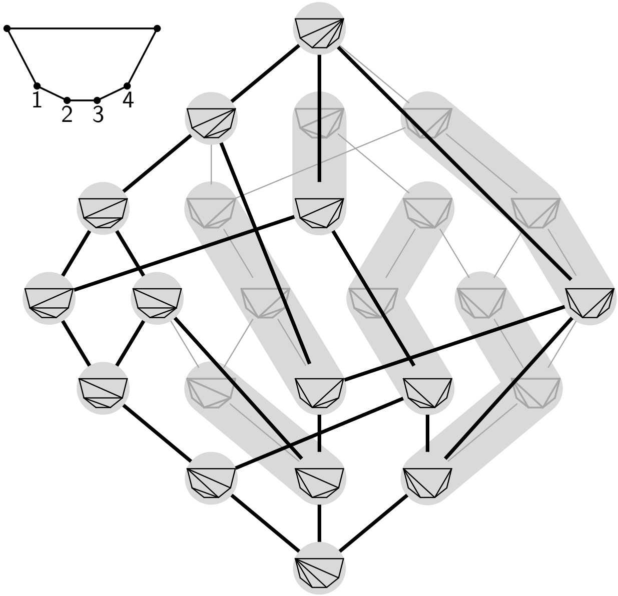

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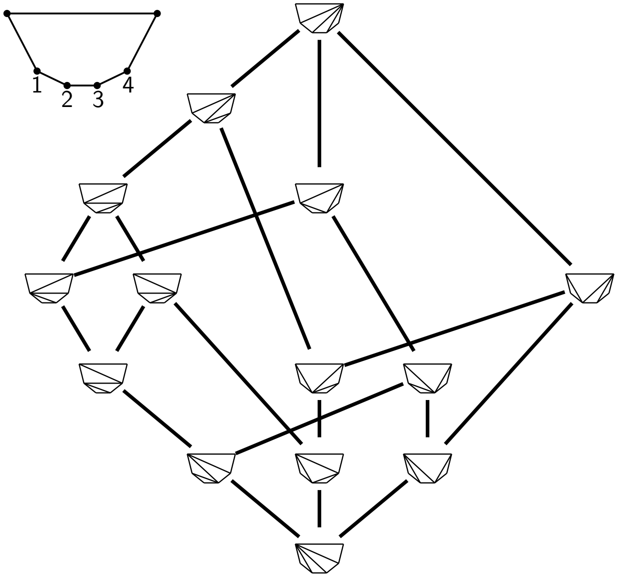
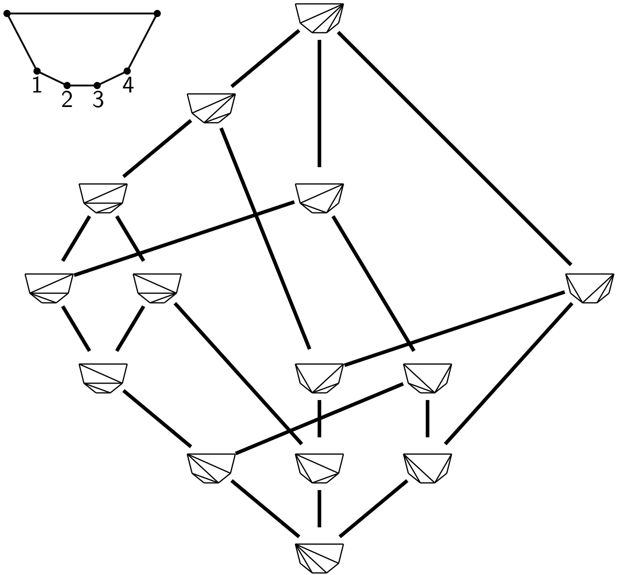
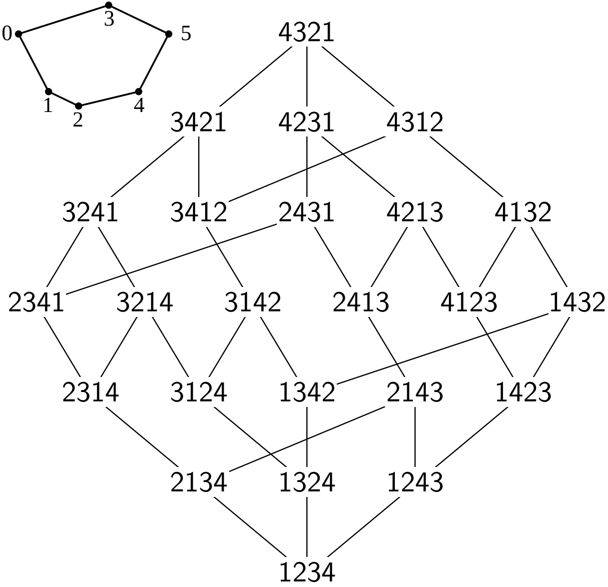


# $S_4$ to triangulations (Quotient is the Tamari lattice)



# $S_4$ to triangulations (for a different polygon)



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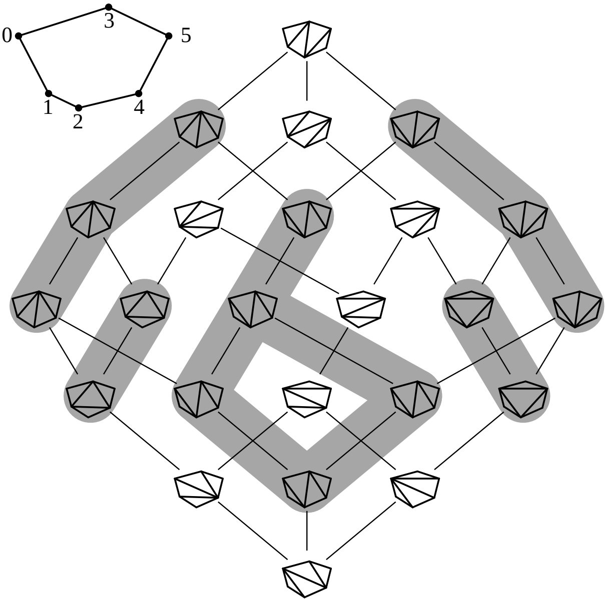

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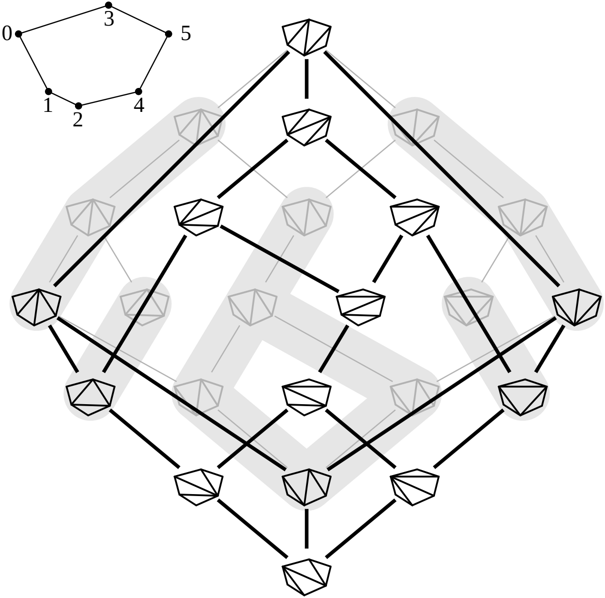

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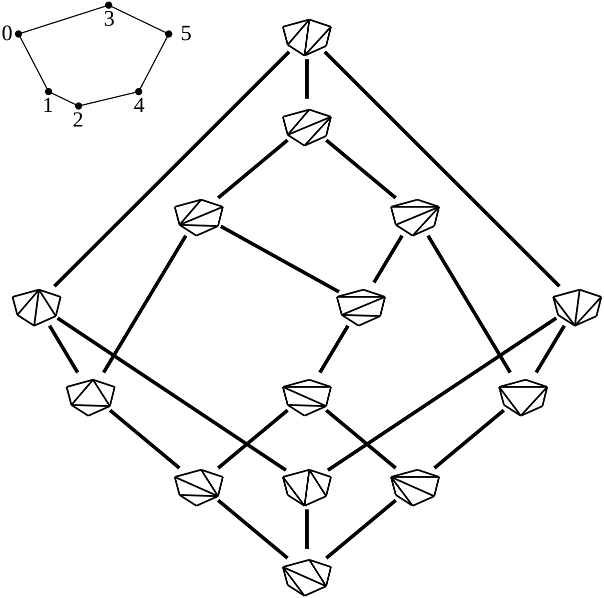
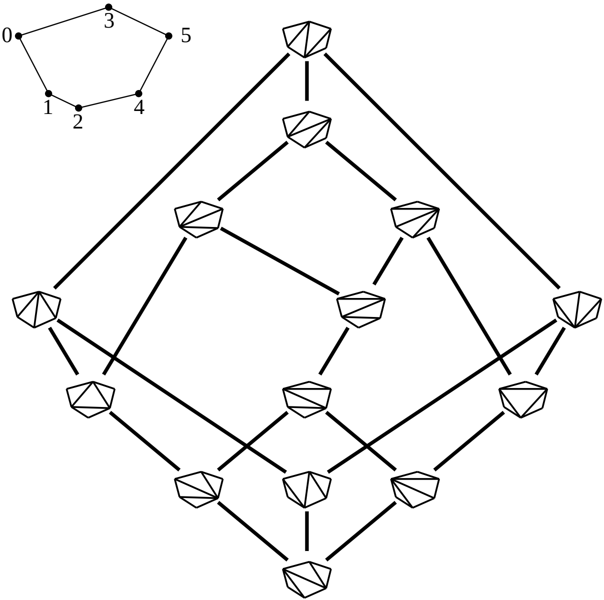


# $S_4$ to triangulations (Quotient is a Cambrian lattice)



# Order-theoretic characterization of congruence (concluded)

Recap of the example: We encountered a surjective map $\eta$ from the weak order on permutations to the set of triangulations. One can check in general (using iterated fiber polytopes):

- Its fibers are intervals in the weak order.
- (ii) and (iii) hold for the fibers.
- Conclude: Fibers of $\eta$ are a congruence $\equiv$, and $\eta$ induces a lattice structure on $S$, isomorphic to $L/\equiv$.

In general, these lattices are "Cambrian lattices of type A." Covers are diagonal flips, and "going up" means increasing the slope of the diagonal. For a special choice of polygon, this is a Tamari lattice.

On finite $L$, an equivalence relation $\equiv$ is a lattice congruence iff:

(i) Each equivalence class is an interval in $L$.

(ii) The map $\pi_\downarrow$ is order-preserving.

(iii) The map $\pi^\uparrow$ is order-preserving.

# Order-theoretic characterization of a lattice quotient

If $L$ is a finite lattice and $\equiv$ is a congruence on $L$ then

- $\pi_\downarrow L$ is a lattice, isomorphic to the quotient lattice $L/\equiv$.
- The map $\pi_\downarrow$ is a lattice homomorphism from $L$ to $\pi_\downarrow L$.

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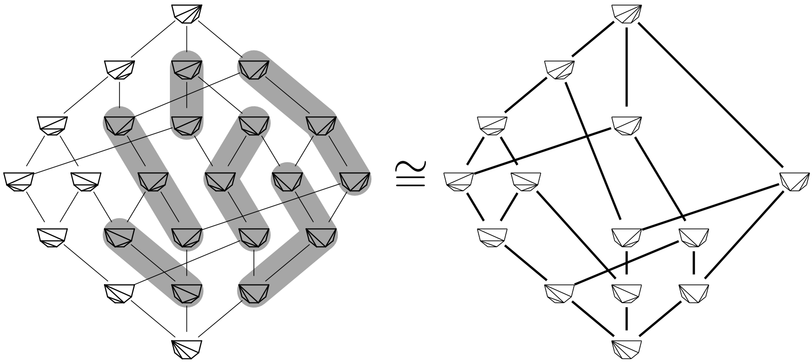
**Example.**



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- The map $\pi_\downarrow$ is a lattice homomorphism from $L$ to $\pi_\downarrow L$.

**Exercise.** $\pi_\downarrow L$ is a join-sublattice of $L$ but can fail to be a sublattice.
(That is, if $x, y \in \pi_\downarrow L$, then $x \vee y \in \pi_\downarrow L$, but possibly $x \wedge y \notin \pi_\downarrow L$.)

The exercise points out an important caveat:

"The map $\pi_\downarrow$ is a lattice homomorphism from $L$ to $\pi_\downarrow L$."
means
$\pi_\downarrow(x \vee_L y) = \pi_\downarrow(x) \vee_{\pi_\downarrow L} \pi_\downarrow(y)$ and $\pi_\downarrow(x \wedge_L y) = \pi_\downarrow(x) \wedge_{\pi_\downarrow L} \pi_\downarrow(y)$

The exercise says we can replace $\vee_{\pi_\downarrow L}$ with $\vee_L$ but usually, we can't replace $\wedge_{\pi_\downarrow L}$ with $\wedge_L$.

# Recap of Section I.a: Lattice congruences and quotients

Lattice: an algebraic object that we can understand combinatorially (order-theoretically).

Homomorphisms, congruences, and quotients are defined as for any (universal) algebraic object. But we can understand them order-theoretically.

We did an example where recognizing a lattice congruence on the weak order allowed us to define a lattice structure on triangulations.

Questions?

# Section I.b. Join-irreducible congruences

# The lattice of congruences

Con $L$: the set of congruences of $L$, partially ordered as a subposet of the partition lattice. (Refinement order.)

This is in fact a sublattice of the partition lattice.
(Proof: straightforward check.)

Furthermore, it is distributive (and finite if $L$ is).

FTFDL: A finite lattice $L$ is distributive if and only if there exists a poset $P$ such that $L$ is isomorphic to the containment order on order ideals in $P$. If so, then $P \cong \mathrm{Irr}(L)$.

$\mathrm{Irr}(L)$: The subposet of $L$ induced by join-irreducible elements.
Join-irreducible: $x$ is join-irreducible ("j.i.") if and only if it covers exactly one element. Equivalently, if $x = \bigvee S$ then $x \in S$.

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Join-irreducible: $x$ is join-irreducible ("j.i.") if and only if it covers exactly one element. Equivalently, if $x = \bigvee S$ then $x \in S$.

Upshot: To understand Con $L$, we want to understand join-irreducible congruences.

# Join-irreducible congruences

Write $a \lessdot b$ for a cover relation.

A congruence $\Theta$ contracts the edge $a \lessdot b$ if $a \equiv b$ modulo $\Theta$.
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**Proposition.** If $L$ is a finite lattice and $\Theta \in \mathrm{Con}\, L$, TFAE:

(i)
(ii)
(iii)

*Proof.*

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*Proof.* (i) $\Longrightarrow$ (ii): We can write any congruence as a join of congruences $\mathrm{con}(a \lessdot b)$. How? Take every cover relation that is in a congruence class.

(Think about it: Congruence classes are intervals. Join in partition lattice is transitive closure of union.)

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We interrupt this proposition for an example.

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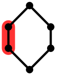
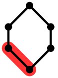
# Example: Con $\Bigg($  $\Bigg)$ We know every join-irreducible congruence is some con$(a \lessdot b)$.

con $\Bigg($  $\Bigg) =$ ?

con $\Bigg($  $\Bigg) =$ ?

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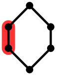
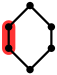
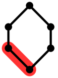
We know every join-irreducible congruence is some $\mathrm{con}(a \lessdot b)$.

con (  ) = 

con (  ) = ?

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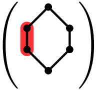
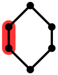
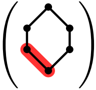
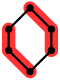
con  $=$ 

con  $=$ 

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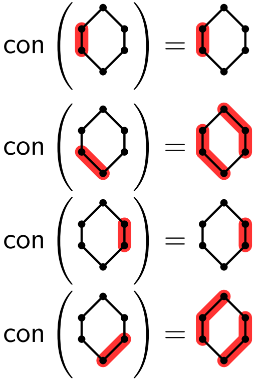

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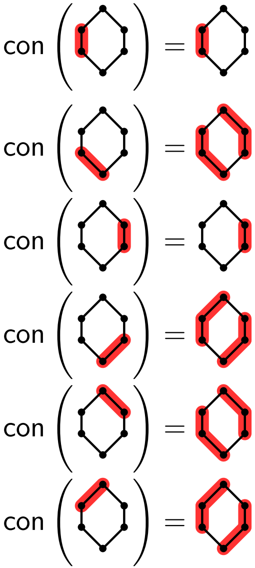

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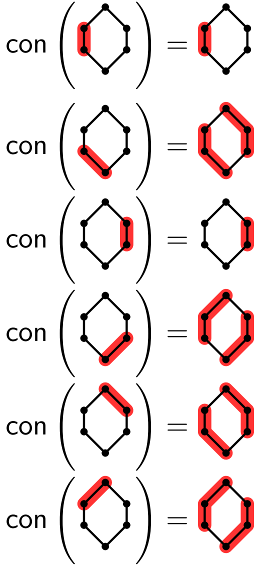


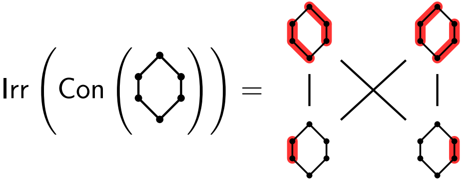

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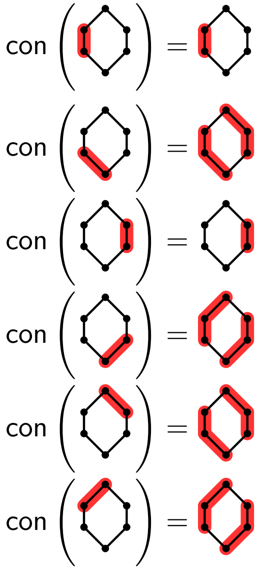


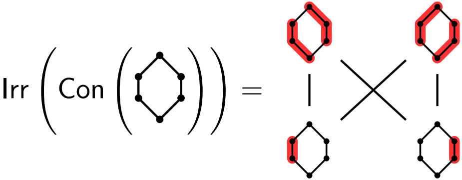


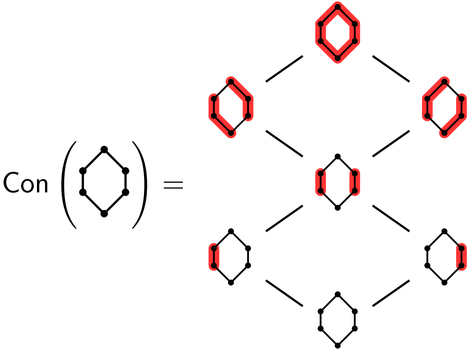

# Join-irreducible congruences

We now return to our regularly scheduled proposition.

# Join-irreducible congruences

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Here $\mathrm{con}(j)$ means $\mathrm{con}(j_* \lessdot j)$ for $j_*$ the element covered by $j$.

The map $j \mapsto \mathrm{con}(j)$ may not be one-to-one. If it is (and if the dual condition holds), then $L$ is called congruence uniform.

# Example: A very not-congruence-uniform lattice

The proposition said $j \mapsto \text{con}(j)$ is a surjective map from join-irreducible elements of $L$ to join-irreducible congruences (join-irreducible elements of $\text{Con}(L)$).

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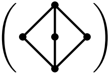
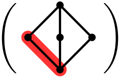
**Example.** Con  con  $=$ ?

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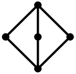
**Example.** Con $\Bigg($  $\Bigg)$ con $\Bigg($  $\Bigg) =$ 

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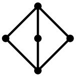
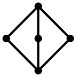
**Example.** Con $\Bigg($  $\Bigg)$ con $\Bigg($  $\Bigg) =$ 

By symmetry, $\mathrm{con}(j)$ is the same congruence for all $j$. This is the unique join-irreducible congruence.

Thus Con $\Bigg($  $\Bigg)$ is the two element lattice.

# Recap of Section I.b: Join-irreducible congruences

Con $L$ is a distributive lattice, sublattice of the partition lattice.

Every join-irreducible congruence is $\mathrm{con}(a \lessdot b)$ for some edge $a \lessdot b$.

Every join-irreducible congruence is $\mathrm{con}(j)$ for some join-irreducible element $j$.

Congruence uniform means $j \mapsto \mathrm{con}(j)$ is one-to-one (and the dual condition holds).

Questions?

# Section I.c. Forcing and polygonal lattices

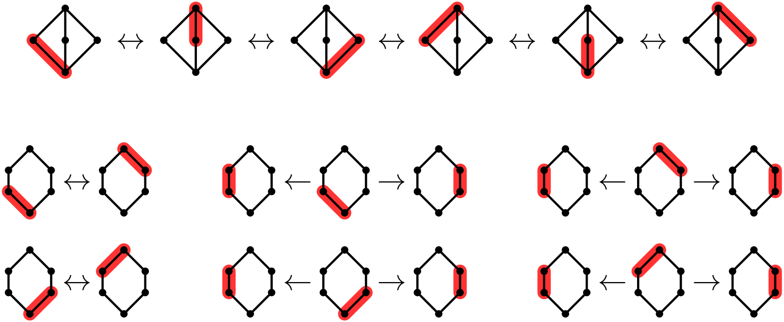
# Forcing among edges

As one might expect, edges cannot be contracted independently.

Say $a \lessdot b$ forces $c \lessdot d$ and write $(a \lessdot b) \to (c \lessdot d)$ if $\mathrm{con}(c \lessdot d) \leq \mathrm{con}(a \lessdot b)$.

That is, every congruence contracting $a \lessdot b$ also contracts $c \lessdot d$.

Examples:



# Forcing among edges (continued)

Forcing $(a \lessdot b) \to (c \lessdot d)$ is not acyclic (unless $L$ is a chain!).

It is a reflexive, transitive relation (a "pre-order" or "quasi-order.")

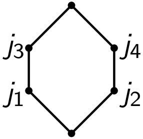
We can make it into a partial order on strongly connected components in the usual way. The result is $\cong \mathrm{Irr}(\mathrm{Con}(L))$, so $\mathrm{Con}(L)$ is isomorphic to the containment order on order ideals this partial order.

When $L$ is congruence uniform, the forcing preorder, restricted to edges $j_* \lessdot j$, is already a partial order, not a pre-order.

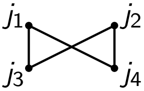
This lets us write $\mathrm{Con}(L)$ as containment order on order ideals in a certain partial order on join-irreducible elements.

**Example.**

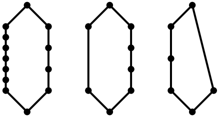
$L =$




$\mathrm{Con}(L) \cong$



# Polygonal lattices

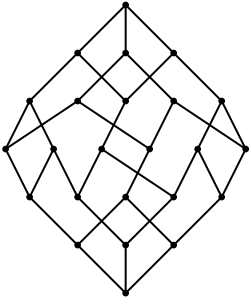
A polygon in a lattice: an interval like  etc.

$L$ may have many polygons or none. It is called polygonal if it has as many polygons as possible. That is:

(i) If distinct elements $y_1$ and $y_2$ both cover an element $x$, then $[x, y_1 \vee y_2]$ is a polygon.

(ii) If an element $y$ covers distinct elements $x_1$ and $x_2$, then $[x_1 \wedge x_2, y]$ is a polygon.

**Example.**



# Forcing in a polygon

Recall: $a \lessdot b$ forces $c \lessdot d$ if every congruence contracting $a \lessdot b$ also contracts $c \lessdot d$.

If $L$ is itself a polygon $[x, y]$, forcing is entirely straightforward.

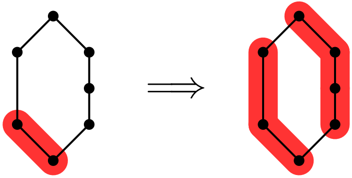
Each edge is a "bottom edge," "top edge," or "side edge."

Each bottom edge forces the opposite top edge and all side edges.

Each top edge forces the opposite bottom edge and all side edges.

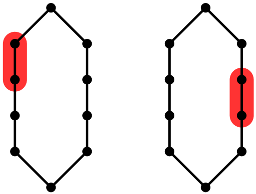
Side edges force nothing.

Up to symmetry, this is the only forcing:

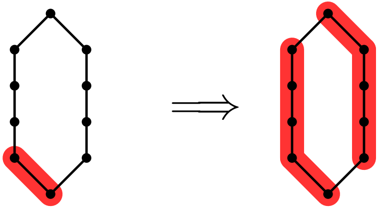


# Forcing in a polygon (rephrased)

A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.



Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

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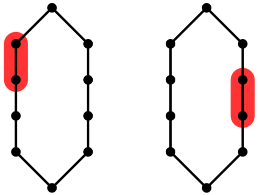

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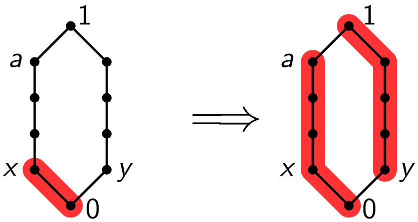


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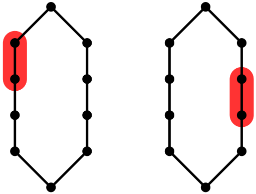

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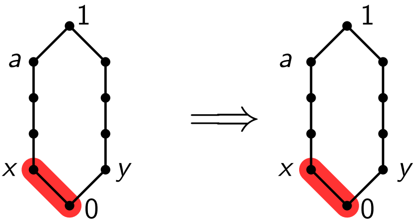


$x \equiv 0$

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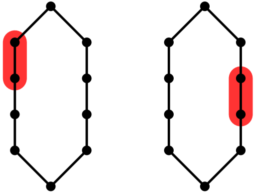

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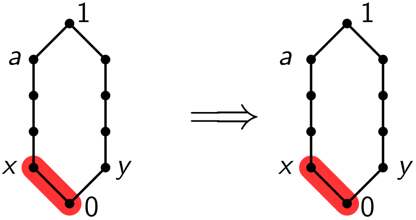


$$\begin{aligned} & x \equiv 0 \\ \Longrightarrow\ & x \vee y \equiv 0 \vee y \end{aligned}$$

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# Forcing in a polygon (rephrased)

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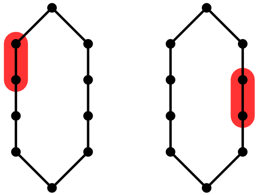

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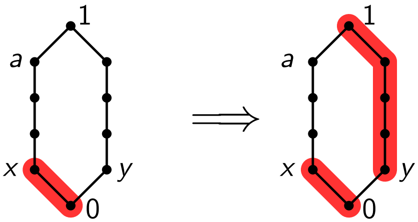


$$\begin{aligned} & x \equiv 0 \\ \Longrightarrow\quad & x \vee y \equiv 0 \vee y \\ \text{i.e.}\quad & 1 \equiv y \end{aligned}$$

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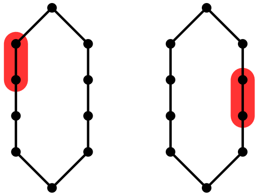

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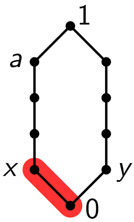
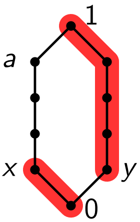



$\Longrightarrow$

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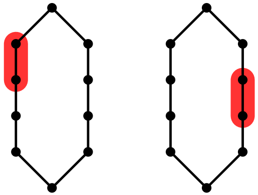
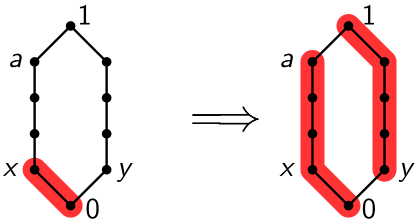


A "bottom" edge forces all side edges and the opposite "top" edge.




$$\begin{array}{ll} & x \equiv 0 \\ \Longrightarrow & x \vee y \equiv 0 \vee y \\ \text{i.e.} & 1 \equiv y \\ \Longrightarrow & a \wedge 1 \equiv a \wedge y \\ \text{i.e.} & a \equiv 0 \end{array}$$

Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

# Forcing in polygonal lattices

The forcing relation in a polygonal lattice is simple and local:

**Proposition.** The forcing relation in a polygonal lattice $L$ is the transitive closure of the forcing relation in each polygon of $L$.

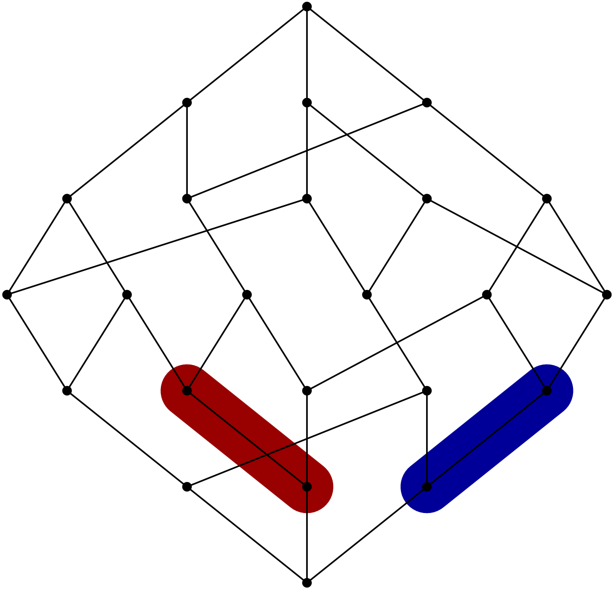
Proof idea: Every relation in the transitive closure is a forcing relation in $L$: easy (forcing is transitive).
Every forcing relation in $L$ is in the transitive closure: Show that every set of edges that is closed under forcing in polygons defines a congruence (using order-theoretic characterization of congruence).

As a result, we can compute examples easily by hand.

Terminology: We'll compute the congruence generated by contracting a set of edges.

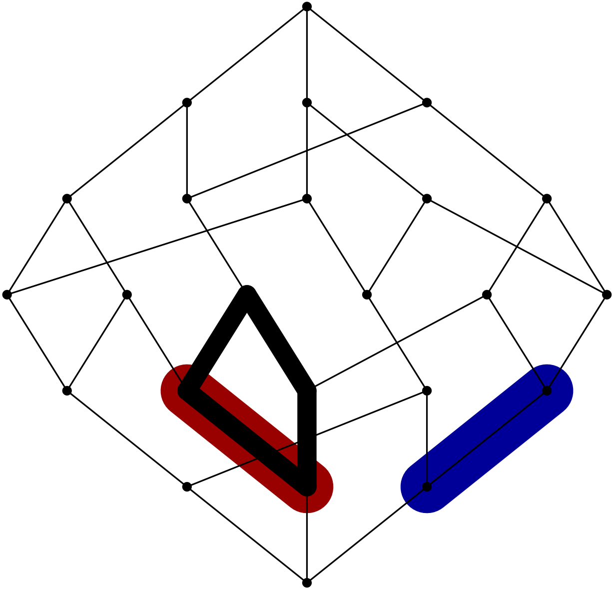
# Example of forcing in a polygonal lattice

The congruence generated by contracting the **red** and **blue** edges.



# Example of forcing in a polygonal lattice

The congruence generated by contracting the **red** and **blue** edges.



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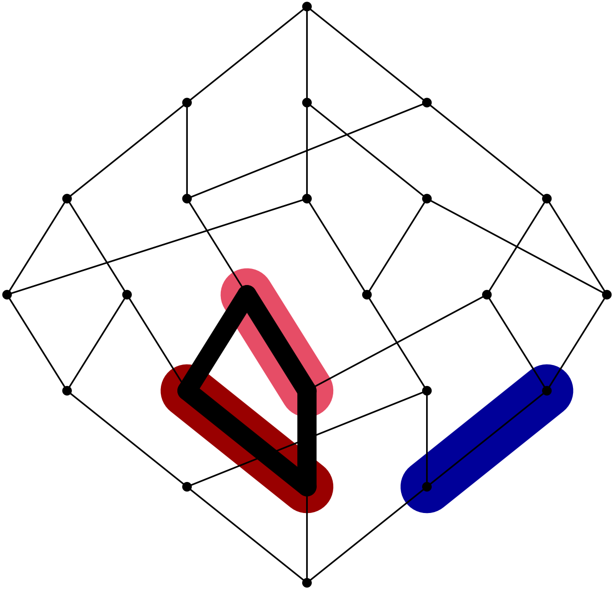

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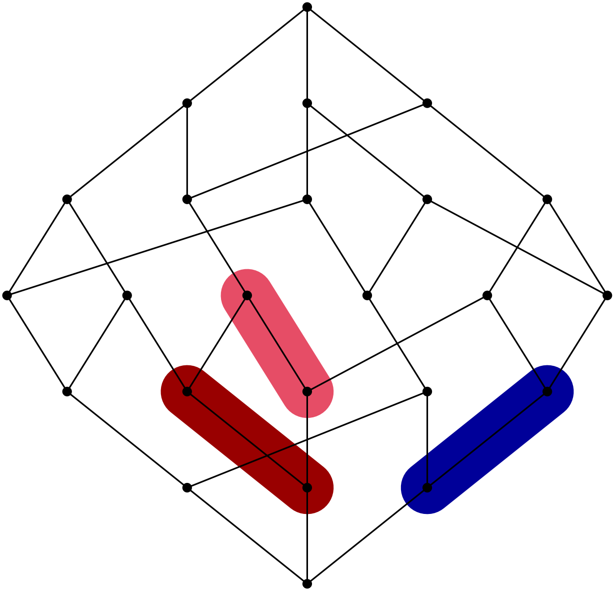

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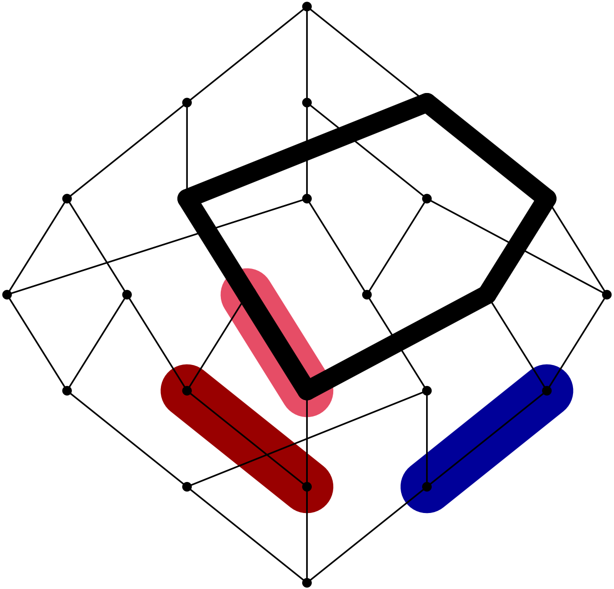

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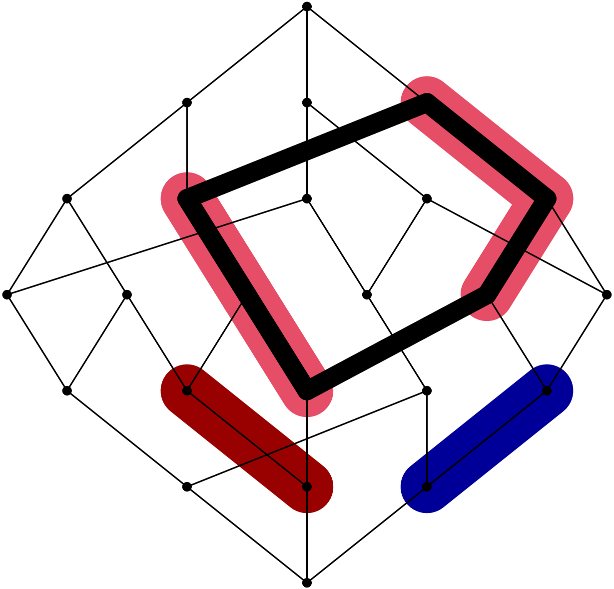

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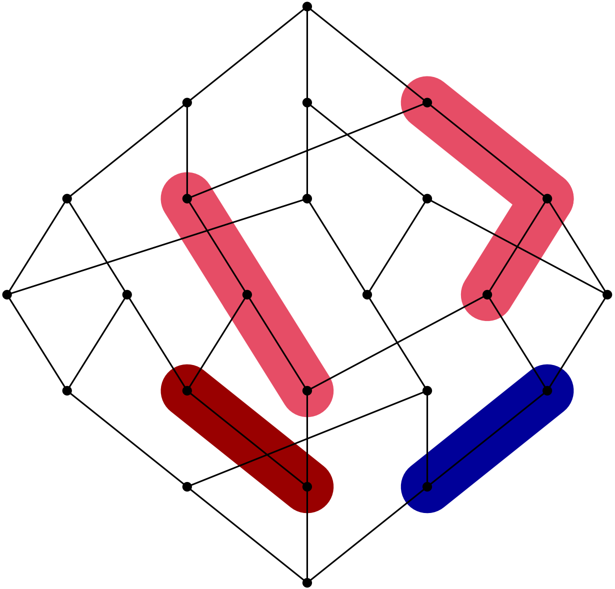

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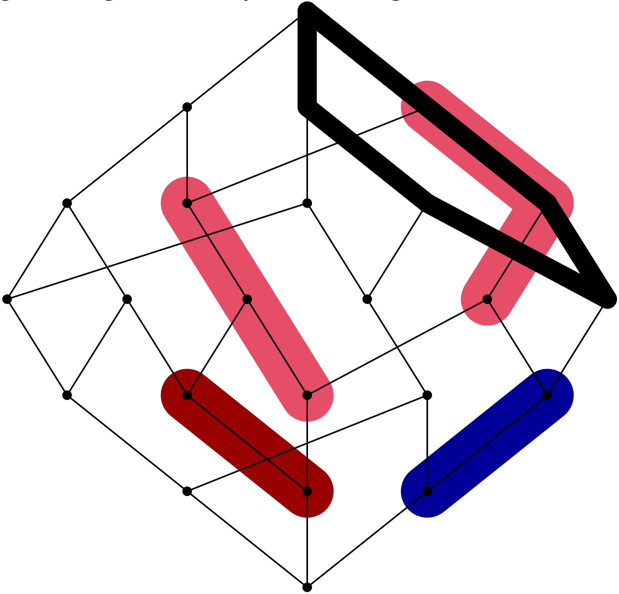

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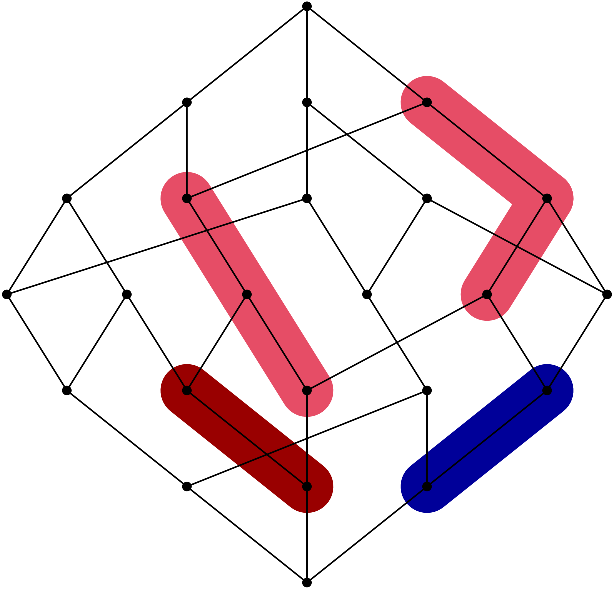

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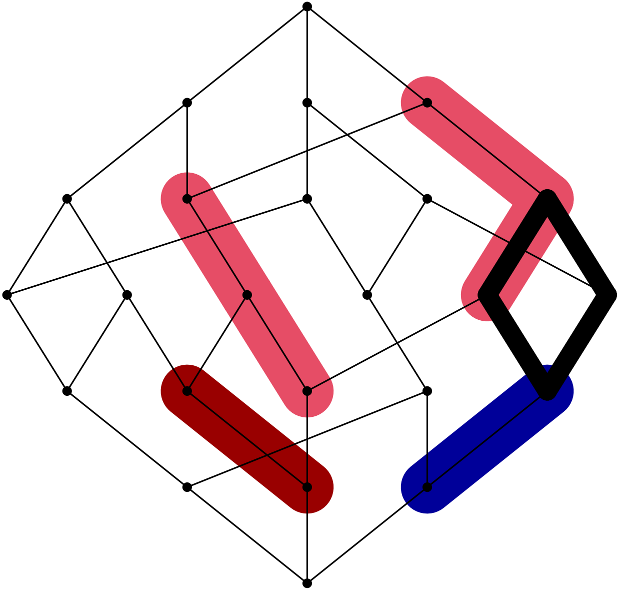

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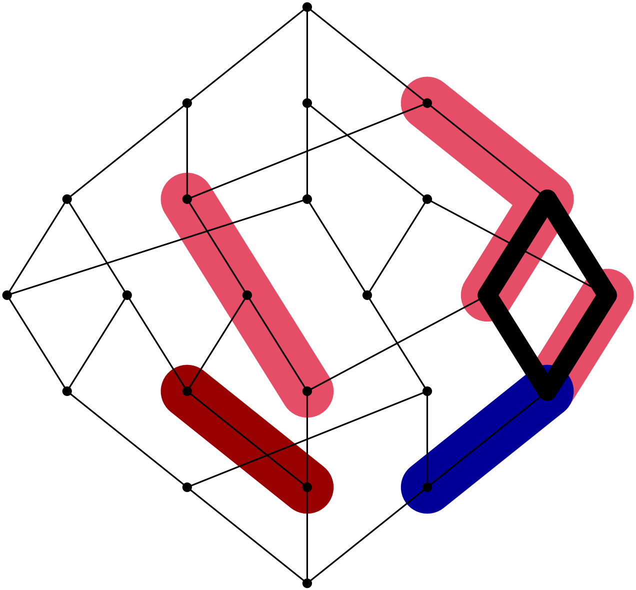

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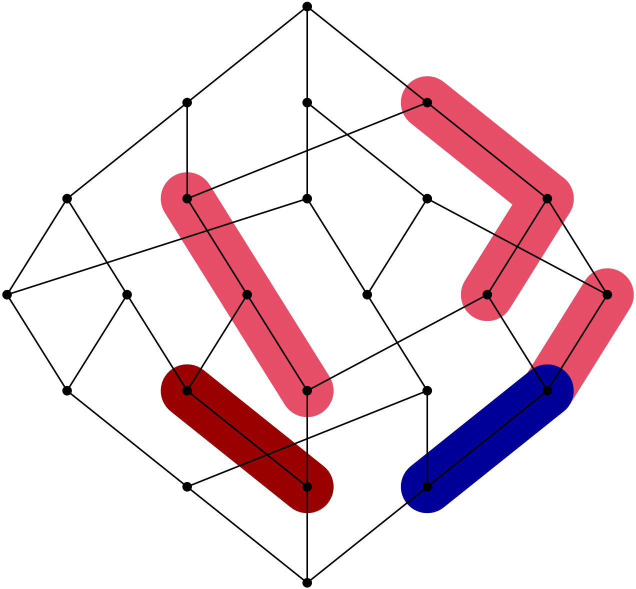

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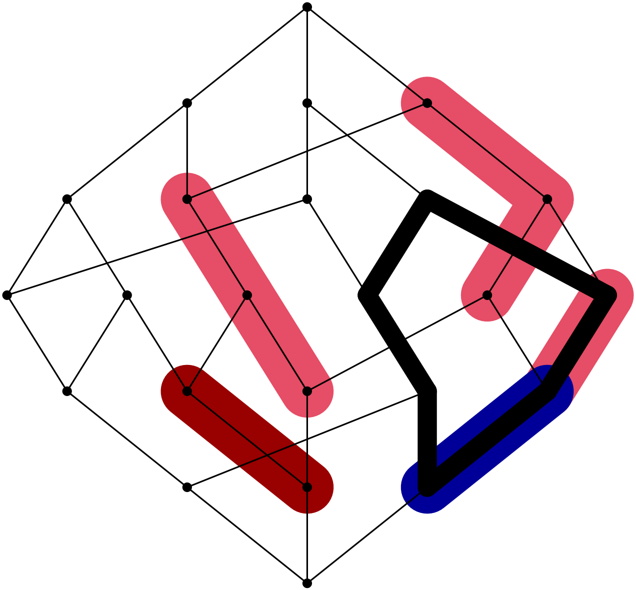

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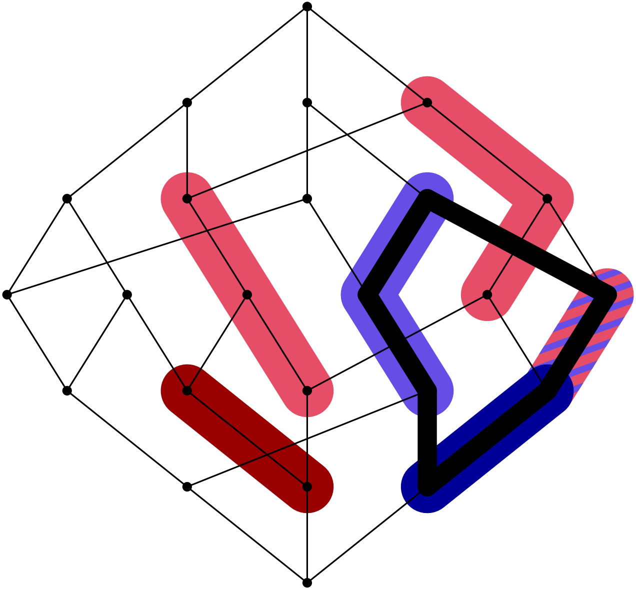

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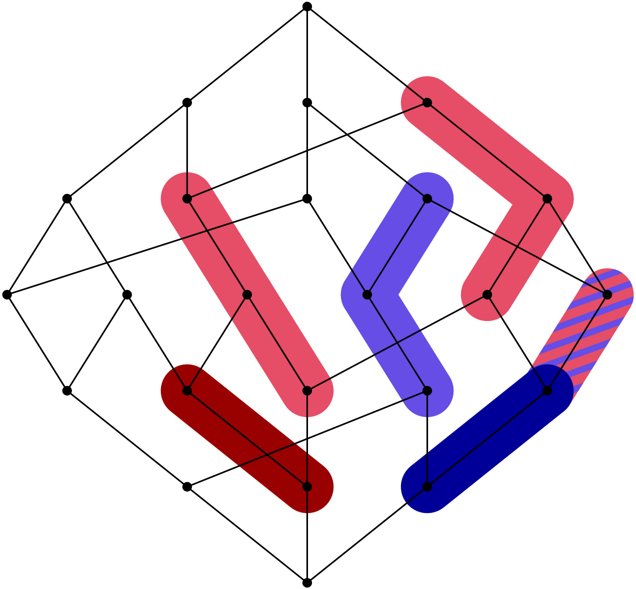

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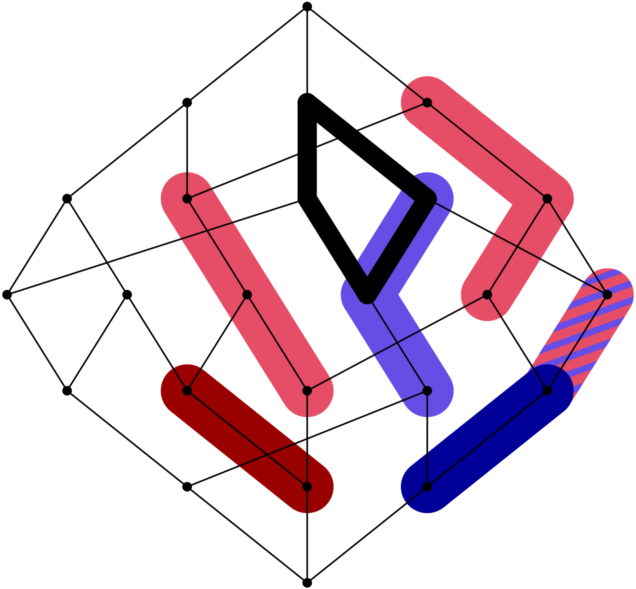

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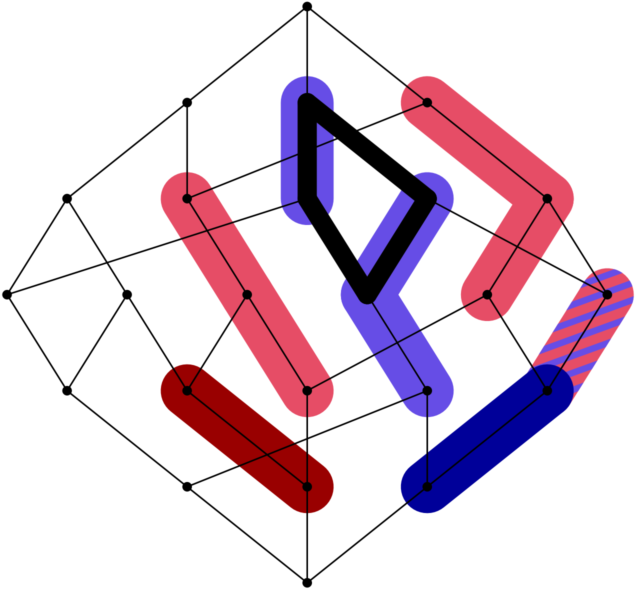

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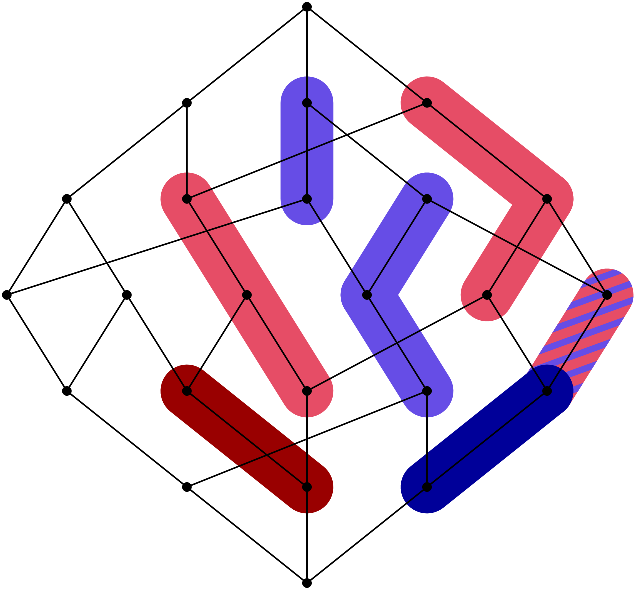

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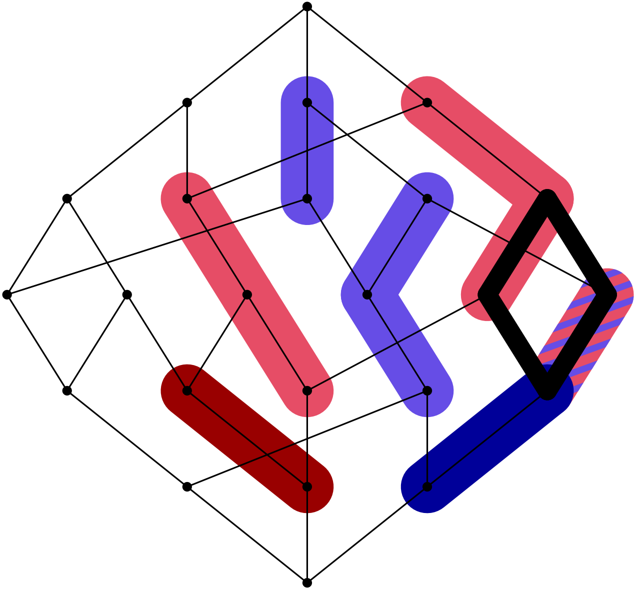

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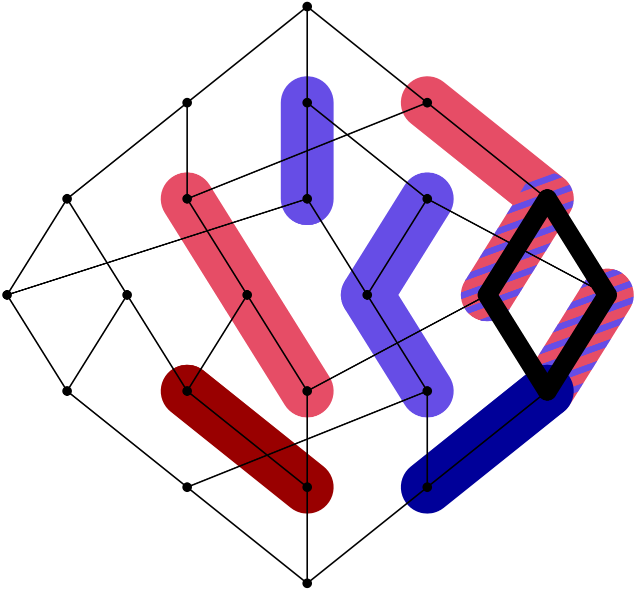

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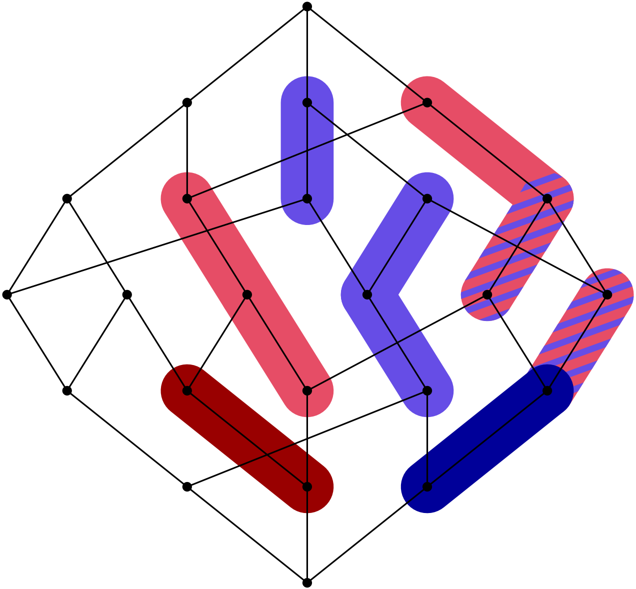
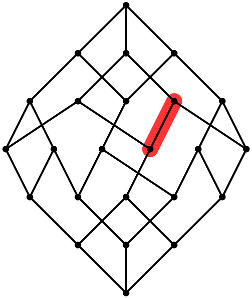
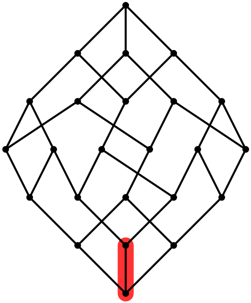
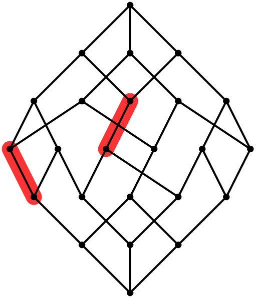


# Examples for you of forcing in a polygonal lattice

Find the congruence generated by the red edges. Find the quotient.







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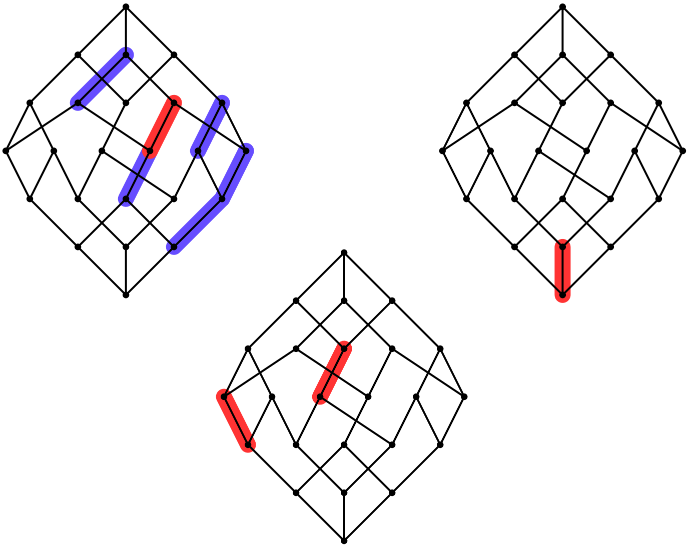

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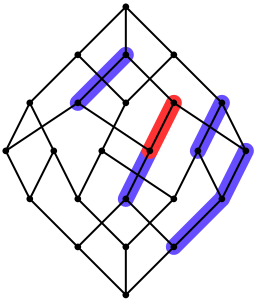

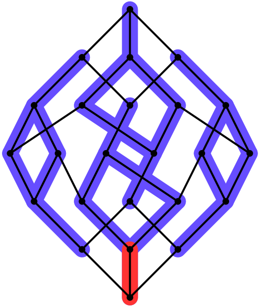

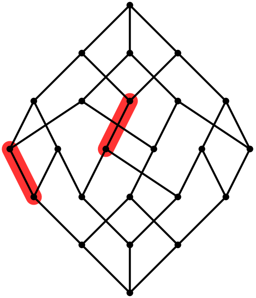

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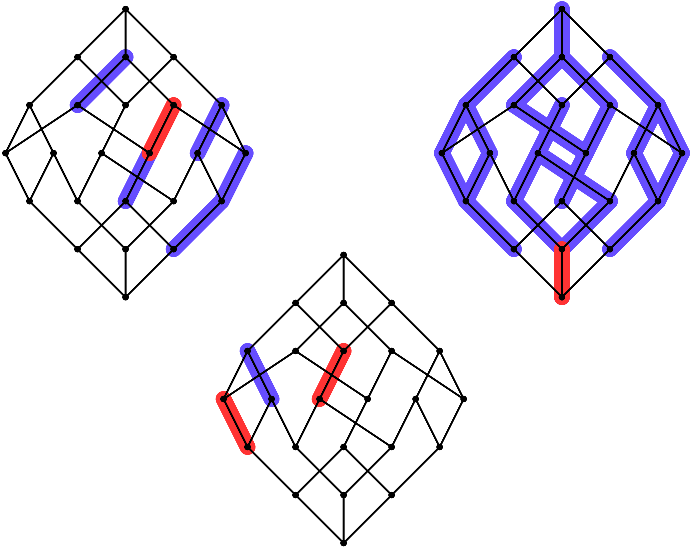

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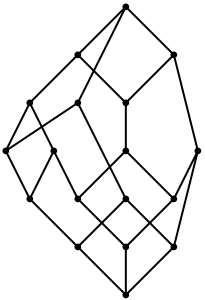

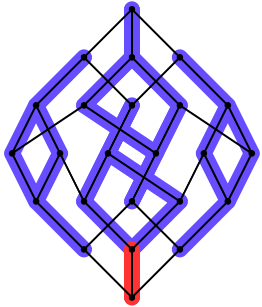

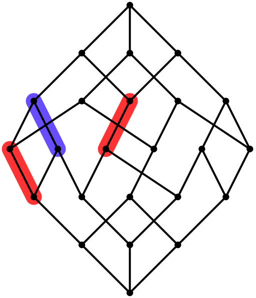

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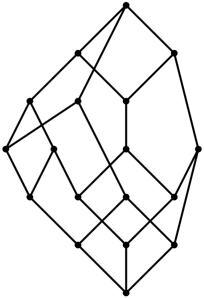

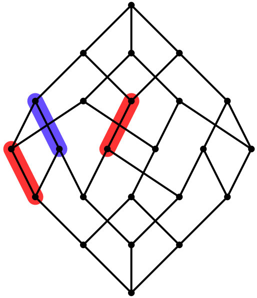

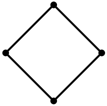

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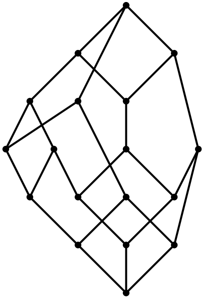
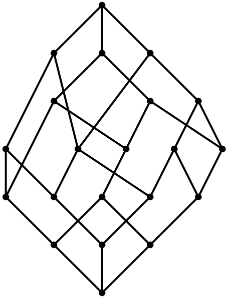
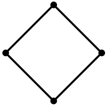






# Recap of Section I.c: Forcing and polygonal lattices

If a congruence contracts a given edge, it may be "forced" to contract others.

Forcing is a pre-order on edges. It restricts to a pre-order on join-irreducible elements (or to an order if $L$ is congruence uniform).

Forcing in a polygon is easy.

A polygonal lattice contains as many polygons as possible. In a polygonal lattice, all forcing can be understood locally, by forcing in polygons.

Questions?

# Section I.d. Canonical join representations

# Canonical join representations

The canonical join representation of $x \in L$ is the lowest way of writing $x$ as a join. More precisely:

A join representation for $x \in L$: an expression $x = \bigvee U$.
It is irredundant if $\nexists U' \subsetneq U$ with $x = \bigvee U'$. ($\therefore U$ is an antichain.)

For antichains $U$ and $V$ of $L$, write $U \ll V$ if the order ideal generated by $U$ is contained in the order ideal generated by $V$. This is a partial order on antichains.

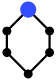
The canonical join representation (CJR) of $x$, if it exists, is the unique minimal antichain $U$ in this order, among antichains joining to $x$. Elements of $U$ are canonical joinands of $x$.

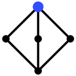
**Exercise.** Canonical joinands are join-irreducible.

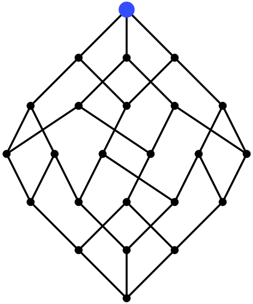
**Exercise.** $x$ is join-irreducible if and only if its CJR is $\{x\}$.

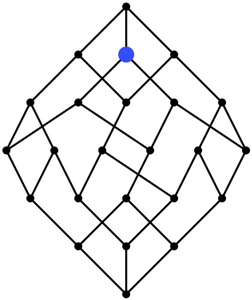
# Examples of canonical join representations

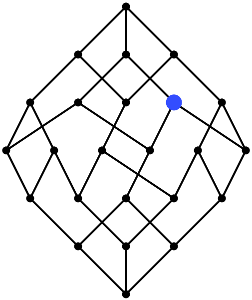
Find the canonical join representation of the blue element.











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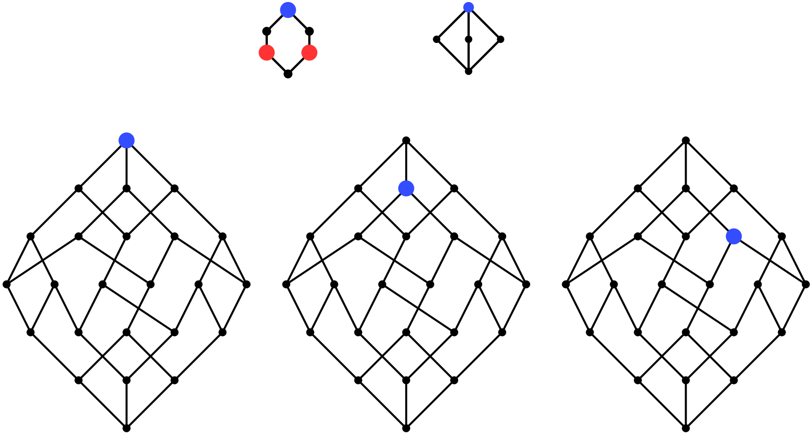

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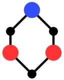
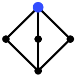
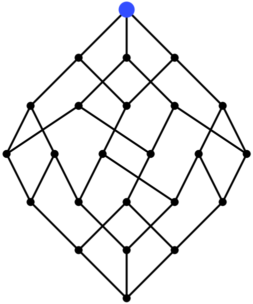
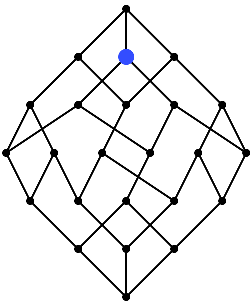
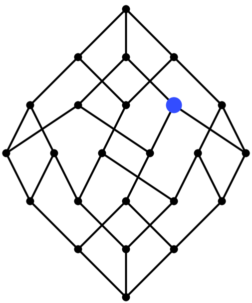




$\nexists$







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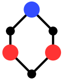

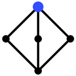

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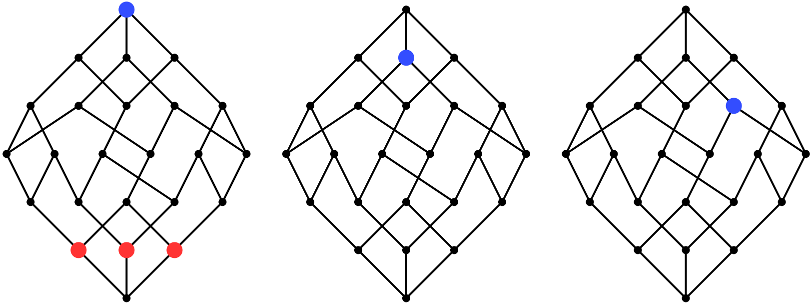

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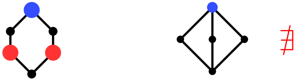


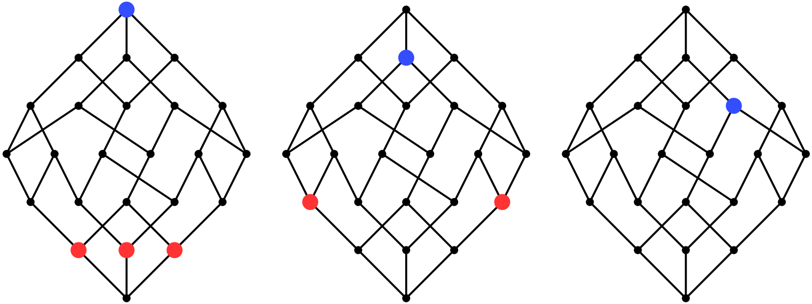

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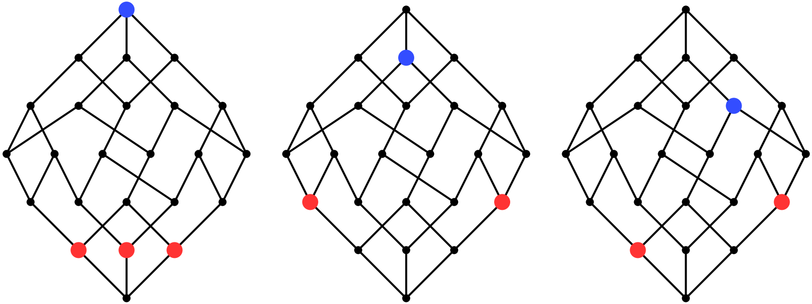
$\nexists$



# Semi-distributive lattices

$L$ is join-semidistributive if

$$x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y.$$

It is meet-semidistributive if the dual condition holds and semidistributive if both conditions hold.

**Theorem.** A finite lattice $L$ is join-semidistributive if and only if every element of $L$ has a canonical join representation.

**Example.** Distributivity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ implies semidistributivity. FTFDL says a finite distributive lattice $L$ is containment on order ideals in $\mathrm{Irr}(L)$. CJR of an element is the set of maximal elements of the corresponding ideal.

# Canonical join reps in congruence uniform lattices

**Exercise.** Suppose $L$ is a finite lattice and $a \lessdot b$ is a cover relation in $L$. Each minimal element of $\{x \in L : x \le b,\ x \not\le a\}$ is a join-irreducible element $j$ and has $\mathrm{con}(a \lessdot b) = \mathrm{con}(j_* \lessdot j)$.

**Exercise.** Suppose $L$ is a finite congruence uniform lattice and $a \lessdot b$ is a cover relation. The unique join-irreducible element of $L$ with $\mathrm{con}(a \lessdot b) = \mathrm{con}(j_* \lessdot j)$ is $j = \bigwedge \{x \in L : x \le b,\ x \not\le a\}$. Furthermore, $j \le b$ but $j \not\le a$.

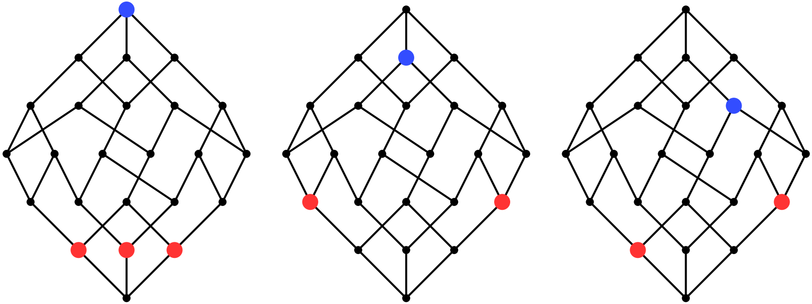
Write $j_{a \lessdot b}$ for $\bigwedge \{x \in L : x \le b,\ x \not\le a\}$.

**Exercise.** Suppose $L$ is a finite congruence uniform lattice. The canonical join representation of an element $x$ is $\bigvee \{j_{a \lessdot x} : a \lessdot x\}$.

These exercises (and their duals) say that a finite congruence uniform lattice is semidistributive.

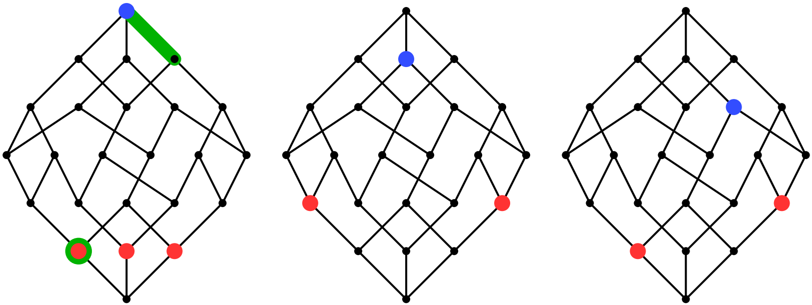
# Examples of CJRs in congruence uniform lattices

For $L$ a finite congruence uniform lattice, the CJR of $x \in L$ is $\bigvee \{j_{a \lessdot x} : a \lessdot x\}$, where $j_{a \lessdot b} = \bigwedge \{x \in L : x \le b,\ x \not\le a\}$.



# Examples of CJRs in congruence uniform lattices

For $L$ a finite congruence uniform lattice, the CJR of $x \in L$ is $\bigvee \{j_{a \lessdot x} : a \lessdot x\}$, where $j_{a \lessdot b} = \bigwedge \{x \in L : x \leq b,\, x \not\leq a\}$.



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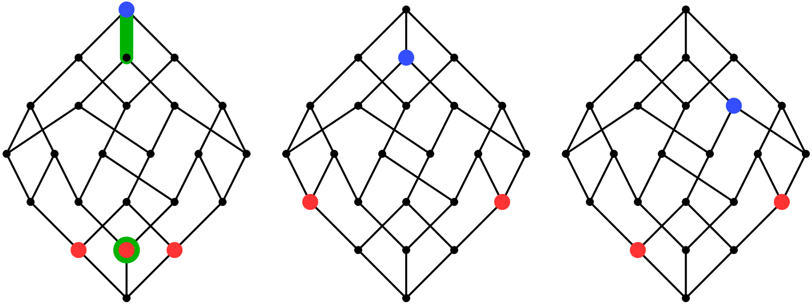

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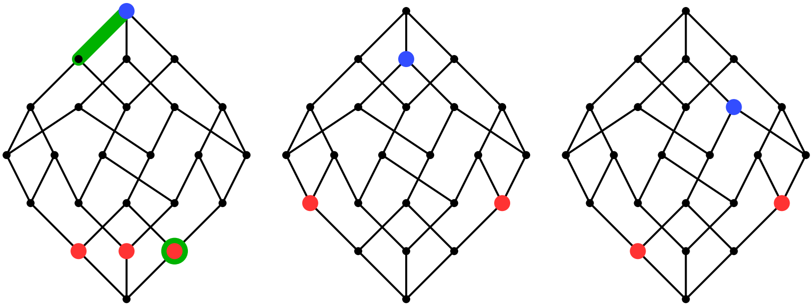

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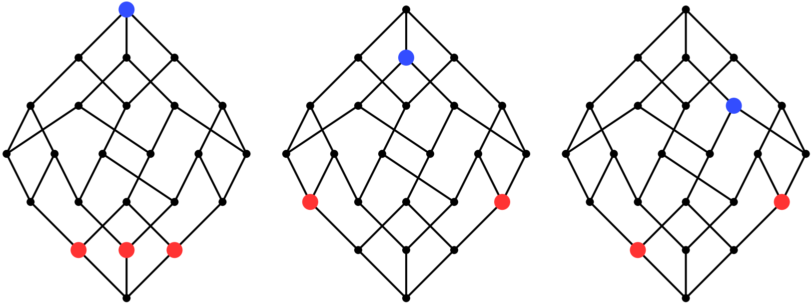

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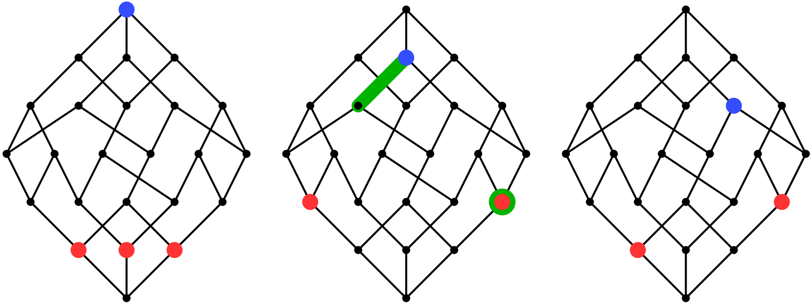

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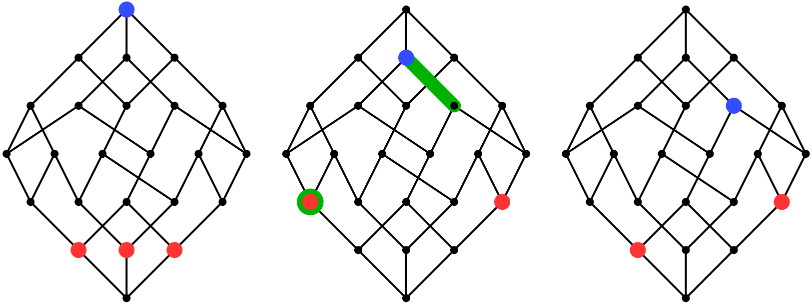

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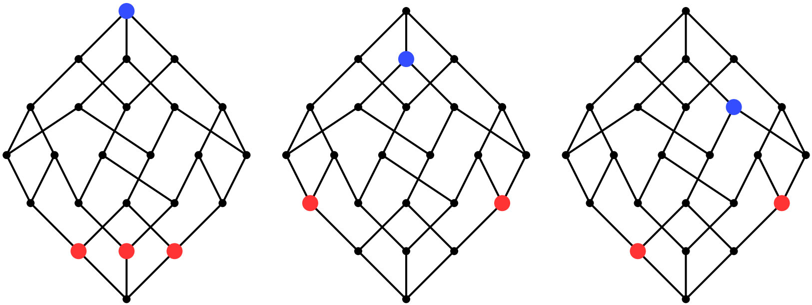

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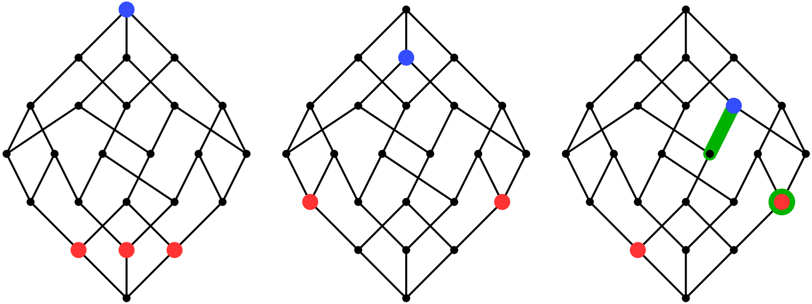

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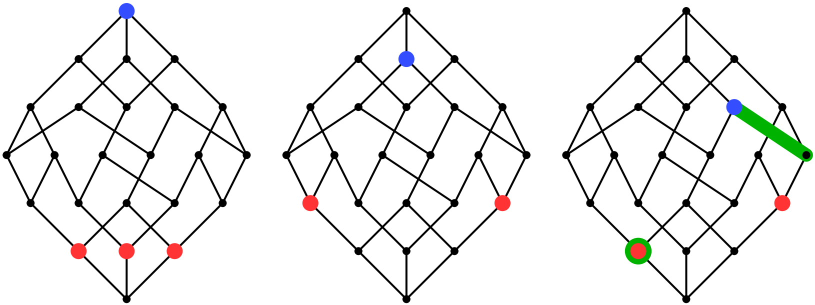

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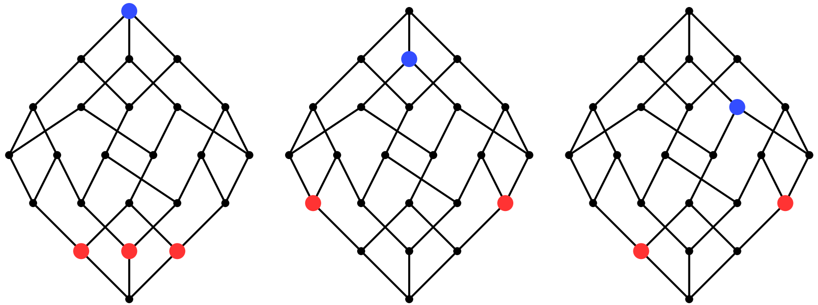


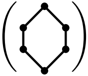
# The canonical join complex

**Exercise.** If $x \in L$ has CJR $x = \bigvee S$ and $S' \subseteq S$, then there exists $x' \in L$ with CJR $x' = \bigvee S'$.

Suppose $L$ is join-semidistributive (i.e. every element has a CJR). The canonical join complex (CJC) of $L$ is

$$\Gamma(L) = \left\{ S \subseteq L : \exists x \in L \text{ with CJR } x = \bigvee S \right\}.$$

**Exercise.** $\Gamma(L)$ is an abstract simplicial complex with vertex set $\{$join-irreducible elements of $L\}$. Its faces are in bijection with the elements of $L$.

**Example.** $\Gamma$  $=$ 

# The canonical join complex (continued)

A simplicial complex is flag if each of its minimal non-faces has exactly two elements. Equivalently, it is the set of cliques in its 1-skeleton.

**Theorem** (E. Barnard, 2016). Suppose $L$ is join-semidistributive. Then the canonical join complex $\Gamma(L)$ is flag if and only if $L$ is semidistributive.

Upshot for us: If $L$ is semidistributive (e.g. if it is congruence uniform), then to understand its CJC, we only need to understand which pairs of join-irreducible elements are "compatible" in the sense of "can participate in a CJR together."

Examples very soon...

# Recap of Section I.d: Canonical join representations

The canonical join representation (CJR) of an element $x \in L$ is the lowest way of writing $x$ as a join.

The canonical join complex (CJR) is the collection of all canonical join representations.

Join-semidistributive means (for us) that every element has a CJR. In this case, the CJC is an abstact simplicial complex on the join-irreducible elements of $L$.

Semidistributive means (for us) that the CJC is flag.

In the congruence uniform case, we gave an explicit formula for the CJR of $x$ with one canonical joinand for each element covered by $x$.

Questions?

# Section I.e. Polygonal, congruence uniform lattices in nature

# Weak order on a finite Coxeter group

**Theorem.** The weak order on a finite Coxeter group is a congruence uniform (therefore semidistributive), polygonal lattice.

Semidistributivity: C. Le Conte de Poly-Barbut, 1994.

Congruence uniformity: N. Caspard, C. Le Conte de Poly-Barbut, and M. Morvan, 2002. (Special case: Caspard, 2000.)

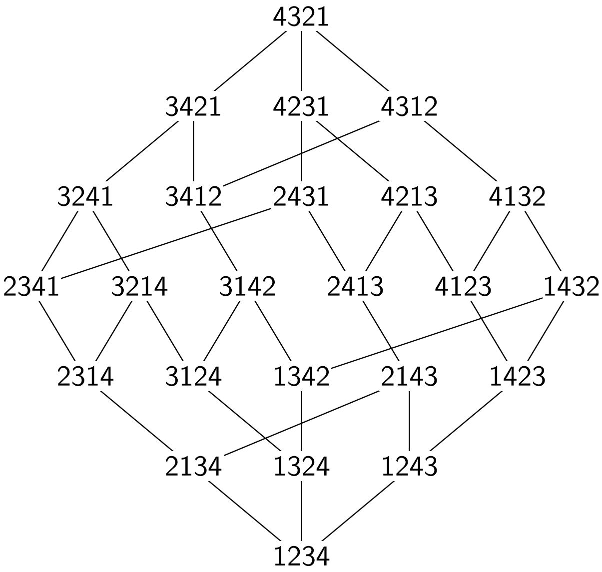
Polygonality (in different terminology): N. Caspard, C. Le Conte de Poly-Barbut, and M. Morvan, 2002.

Examples soon (comprising much of Part II).

# The Tamari lattice

Recall the permutations-to-triangulations map from earlier.

# $S_4$ to triangulations



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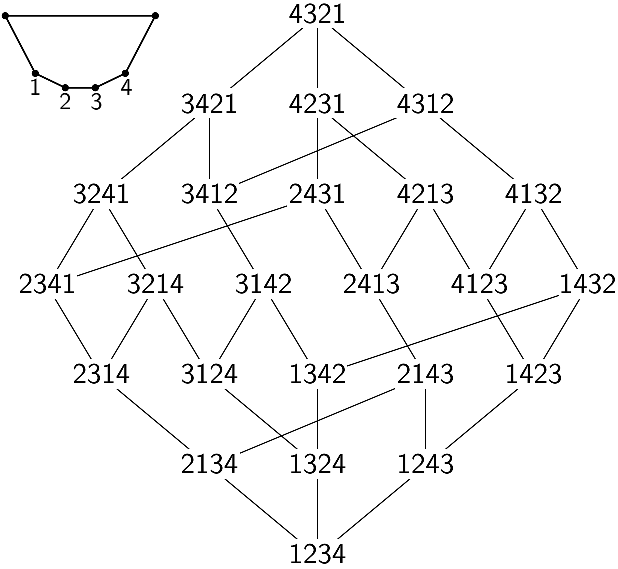

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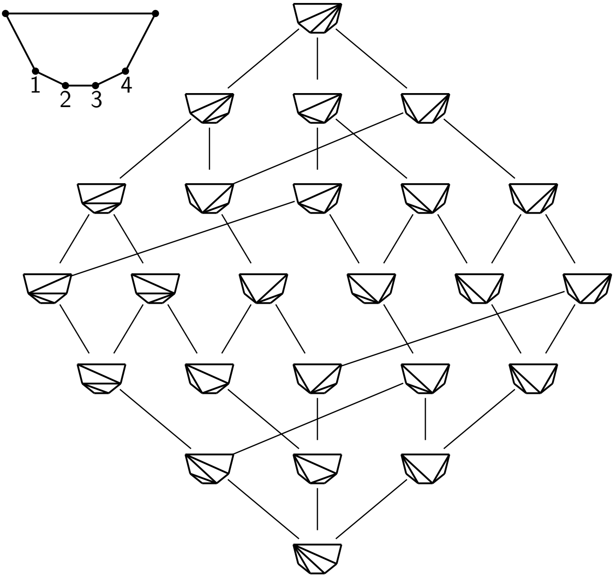

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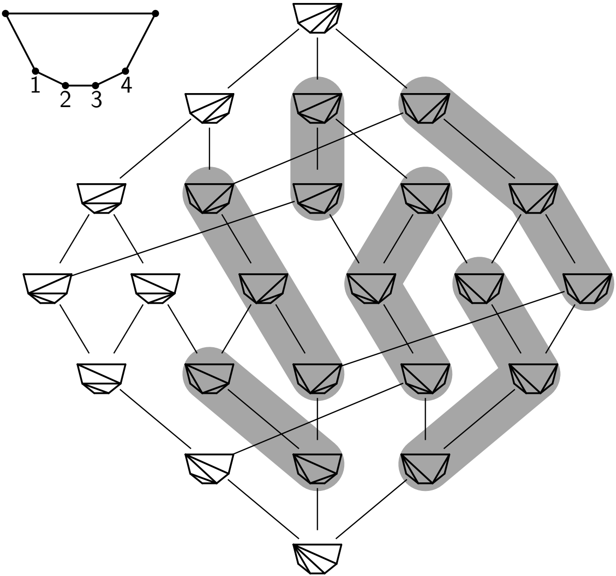

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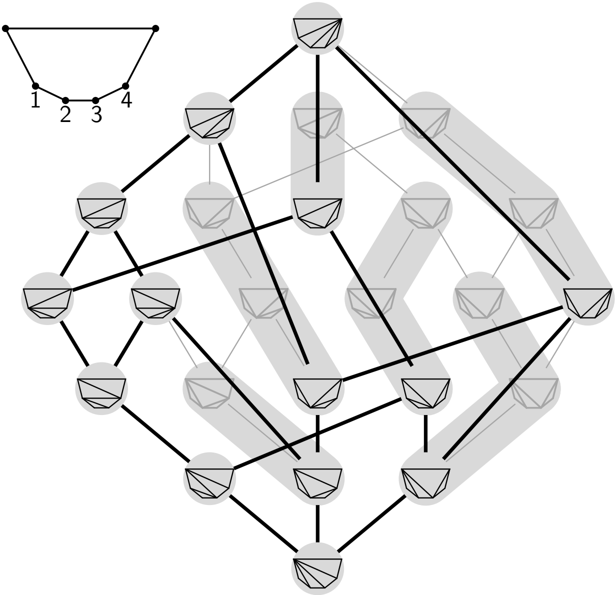

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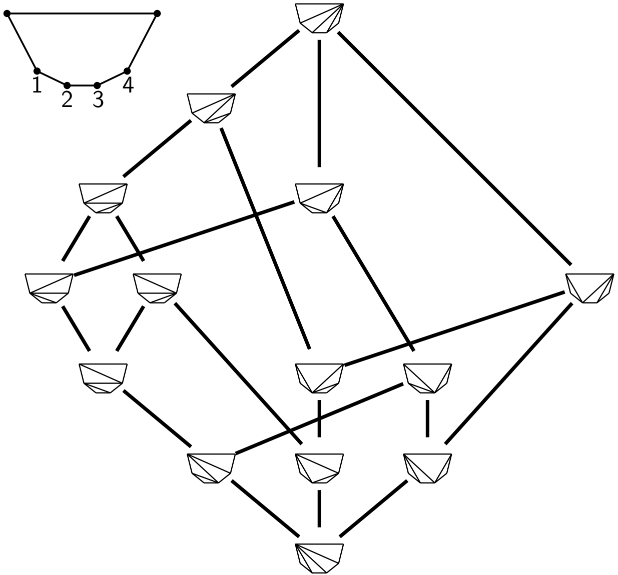

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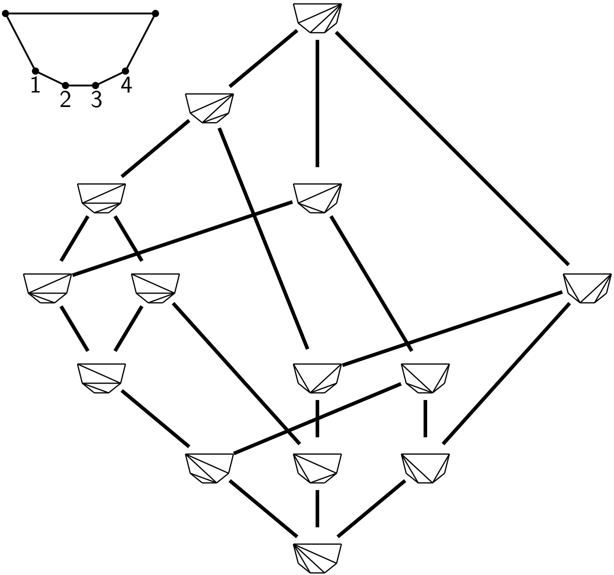

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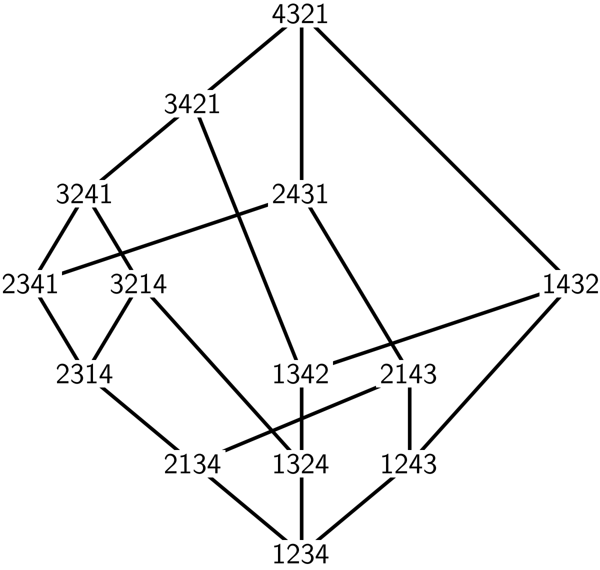

# The Tamari lattice

In the permutations-to-triangulations map, if the polygon has all vertices "on the bottom," the quotient lattice is the Tamari lattice.

Bottom elements of congruence classes are exactly 312-avoiding permutations, so we recover the fact that the Tamari lattice is the weak order restricted to 312-avoiding permutations.
(A. Björner and M. Wachs, 1994. They had all the "combinatorial lattice theory" ingredients without the lattice theory.)

Congruence uniformity and polygonality are inherited by quotients of finite lattices. Thus:

**Theorem.** The Tamari lattice is a congruence uniform (therefore semidistributive), polygonal lattice.

Congruence uniformity: W. Geyer, 1994.

# Join-irreducible elements in the Tamari lattice

We'll continue realizing the Tamari lattice as the weak order restricted to 312-avoiding permutations.

How to go down by a cover from a 312-avoider: Undo a descent, then do 312 $\to$ 132-moves until you hit another 312-avoider.

**Example.** 235879641 $\xrightarrow{\text{undo descent}}$ 235879461

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Questions before the example goes away?

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Conclusion: Join-irreducible elements of the Tamari lattice are 312-avoiding permutations with exactly one descent.

For each pair $1 \le a < b \le n$, there is exactly one 312-avoiding permutation whose only descent is $ba$. Specifically:

$$1\,2 \cdots (a-1)(a+1)(a+2) \cdots (b-1)\, b\, a\, (b+1)(b+2) \cdots n$$

# Canonical join representations in the Tamari lattice

Since the Tamari lattice is congruence uniform, the CJR of $x$ is $\bigvee \{j_{w \lessdot x} : w \lessdot x\}$, where $j_{w \lessdot x} = \bigwedge \{u \in L : u \leq x,\, u \not\leq w\}$.

$x$ is a 312-avoiding permutation. We already say that covers $w \lessdot x$ come from descents of $x$. Suppose $w \lessdot x$ is coming from a descent $ba$ in $x$. One can show that $j_{w \lessdot x}$ is the (unique!) join-irreducible element with descent $ba$.

Conclusion: The canonical join representation of an element of the Tamari lattice is essentially its set of descent-pairs.

**Example.** CJR(236759841) is $\{75, 98, 84, 41\}$, where, for example, 84 represents 123567849.

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But we haven't yet seen the point...

# CJRs in the Tamari lattice (continued)

The CJR of an element of the Tamari lattice is its set of descent-pairs. Since the Tamari lattice is congruence uniform (and therefore semi-distributive), its canonical join-complex is flag.

Easy: Two descent-pairs *ba* and *dc* can participate in the same 312-avoider if and only if

(i) Not $a < c < b < d$ and not $c < a < d < b$, and

(ii) $a \neq c$ and $b \neq d$.

Put $1, \ldots, n$ on a horizontal line and represent a pair *ba* by an arc above the line connecting $a$ to $b$. A CJR is a collection of such arcs that (pairwise) don't cross, don't share left endpoints and don't share right endpoints.

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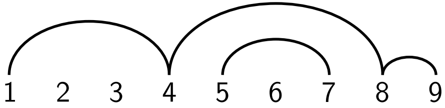

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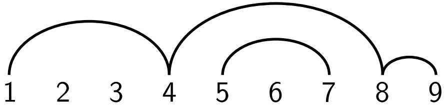


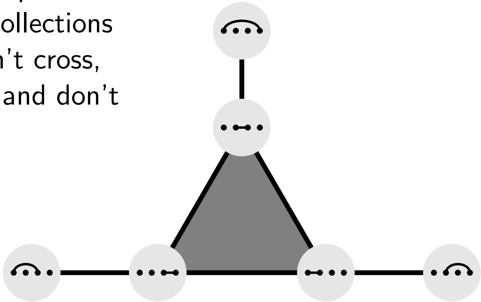

CJRs of elements of the Tamari lattice are noncrossing partitions!

# The canonical join complex of the Tamari lattice

Put $1, \ldots, n$ on a horizontal line (again).

Faces of canonical join-complex of the Tamari lattice are collections of arcs that (pairwise) don't cross, don't share left endpoints and don't share right endpoints.

**Example.** $n = 4$



**Theorem** (E. Barnard, 2017). The CJC of the Tamari lattice is shellable. It is contractible when $n$ is even and homotopy equivalent to a wedge of Catalan($r$) many spheres, all of dimension $r - 1$, when $n = 2r + 1$.

# Lattices of torsion classes

Context: Representation theory of finite-dimensional algebras.
I will just *mention* these as an indication that congruence uniform, polygonal lattices show up in various contexts.

$A$: An associative, finite-dimensional algebra with identity.

mod$A$: The category of finitely-generated left $A$-modules.

A torsion class of $A$ is a full subcategory of mod$A$ that is closed under factor modules, isomorphisms, and extensions.

**Theorem.** The set of all torsion classes of $A$, ordered by inclusion, is a lattice. When finite, it is congruence uniform and polygonal.

Lattice: O. Iyama, I. Reiten, H. Thomas, G. Todorov, 2015.
Semidistributive: A. Garver and T, McConville, 2015.
Congruence uniform and polygonal: L. Demonet, 2017

# Examples from the work of McConville and Garver

Grid-Tamari orders. Santos, Stump, and Welker generalized Tamari lattices to "Grassmann-Tamari orders" and conjectured that they are lattices. McConville generalized and proved the conjecture to show that "grid-Tamari orders" are congruence uniform lattices. Technique: Constructed a larger congruence uniform lattice (analogous to the weak order on permutations) and constructed grid-Tamari order as a quotient (analogous to the permutations-to-triangulations map).

McConville and Garver: Biclosed sets of acyclic paths in a graph form a congruence uniform, polygonal lattice.

McConville and Garver: Oriented Flip Graphs and Noncrossing Tree partitions ...

# Recap of Section I.e: Polygonal, congruence uniform lattices in nature

Weak order on a finite Coxeter group is polygonal and congruence uniform. (More coming in Part II.)

The Tamari lattice is polygonal and congruence uniform. Canonical join representations are noncrossing partitions.

Finite lattices of torsion classes are polygonal and congruence uniform.

Examples from McConville and Garver.

Questions?

# References


E. Barnard, *The canonical join complex.* arXiv:1610.05137

A. Garver and T. McConville, *Lattice Properties of Oriented Exchange Graphs and Torsion Classes.* arXiv:1507.04268

A. Garver and T. McConville, *Oriented Flip Graphs and Noncrossing Tree Partitions.* arXiv:1604.06009

O. Iyama, I. Reiten, H. Thomas, G. Todorov, *Lattice structure of torsion classes for path algebras.* Bull. LMS (2015).

T. McConville, *Lattice structure of Grid-Tamari orders.* JCTA 2017.

N. Reading, *Noncrossing arc diagrams and canonical join representations.* SIAM J. Discrete Math. (2015).

N. Reading, *Lattice Theory of the Poset of Regions*, Chapter 9 in Lattice Theory: Special Topics and Applications, Volume 2, ed. G. Grätzer and F. Wehrung. Especially Section 9-5.

# Exercises (gathered into one place)

**Exercise.** $\pi_\downarrow L$ is a join-sublattice of $L$ but can fail to be a sublattice. (That is, if $x, y \in \pi_\downarrow L$, then $x \vee y \in \pi_\downarrow L$, but possibly $x \wedge y \notin \pi_\downarrow L$.)

**Exercise.** Canonical joinands are join-irreducible.

**Exercise.** $x$ is join-irreducible if and only if its CJR is $\{x\}$.

**Exercise.** Suppose $L$ is a finite lattice and $a \lessdot b$ is a cover relation in $L$. Each minimal element of $\{x \in L : x \leq b,\ x \not\leq a\}$ is a join-irreducible element $j$ and has $\mathrm{con}(a \lessdot b) = \mathrm{con}(j_* \lessdot j)$.

**Exercise.** Suppose $L$ is a finite congruence uniform lattice and $a \lessdot b$ is a cover relation. The unique join-irreducible element of $L$ with $\mathrm{con}(a \lessdot b) = \mathrm{con}(j_* \lessdot j)$ is $j = \bigwedge \{x \in L : x \leq b,\ x \not\leq a\}$. Furthermore, $j \leq b$ but $j \not\leq a$.

**Exercise.** Suppose $L$ is a finite congruence uniform lattice. The canonical join representation of an element $x$ is $\bigvee \{j_{a \lessdot x} : a \lessdot x\}$.

**Exercise.** If $x \in L$ has CJR $x = \bigvee S$ and $S' \subseteq S$, then there exists $x' \in L$ with CJR $x' = \bigvee S'$.

**Exercise.** $\Gamma(L)$ is an abstract simplicial complex with vertex set $\{$join-irreducible elements of $L\}$. Its faces are in bijection with the elements of $L$.