Second order balanced property on Christoffel words

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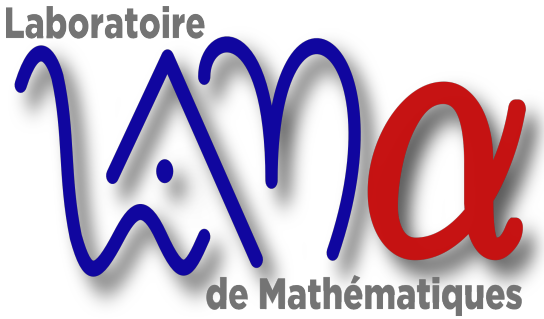
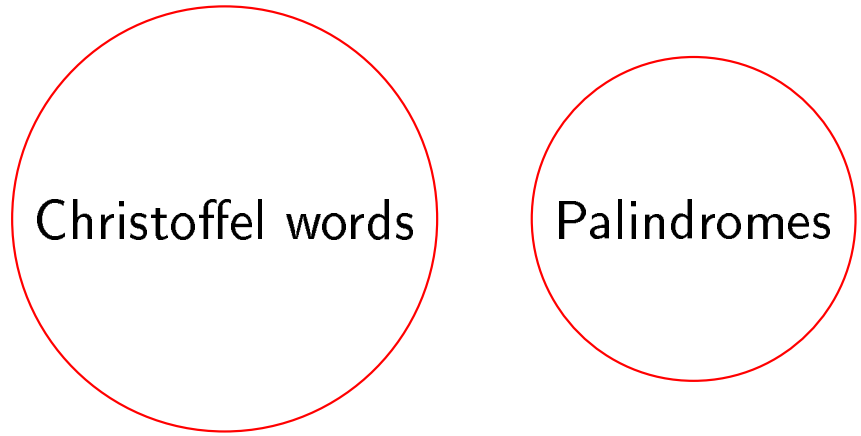
Université Savoie Mont-Blanc

Combinatorics on Words and Tilings- UdeM
07-04-2017



Christoffel words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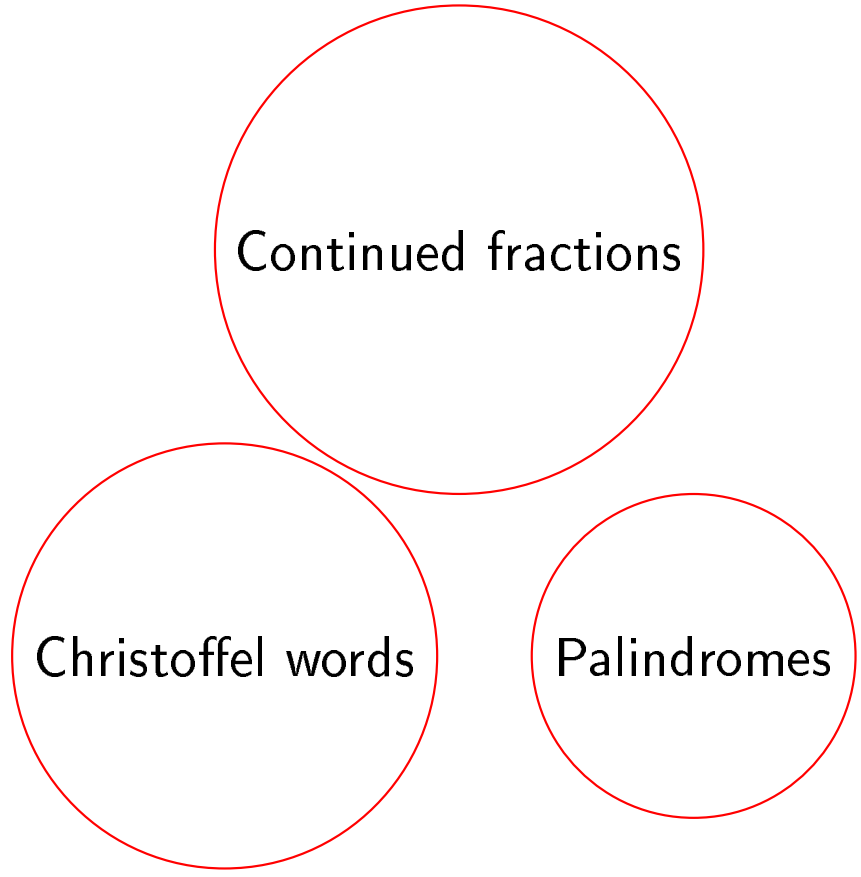
Continued fractions
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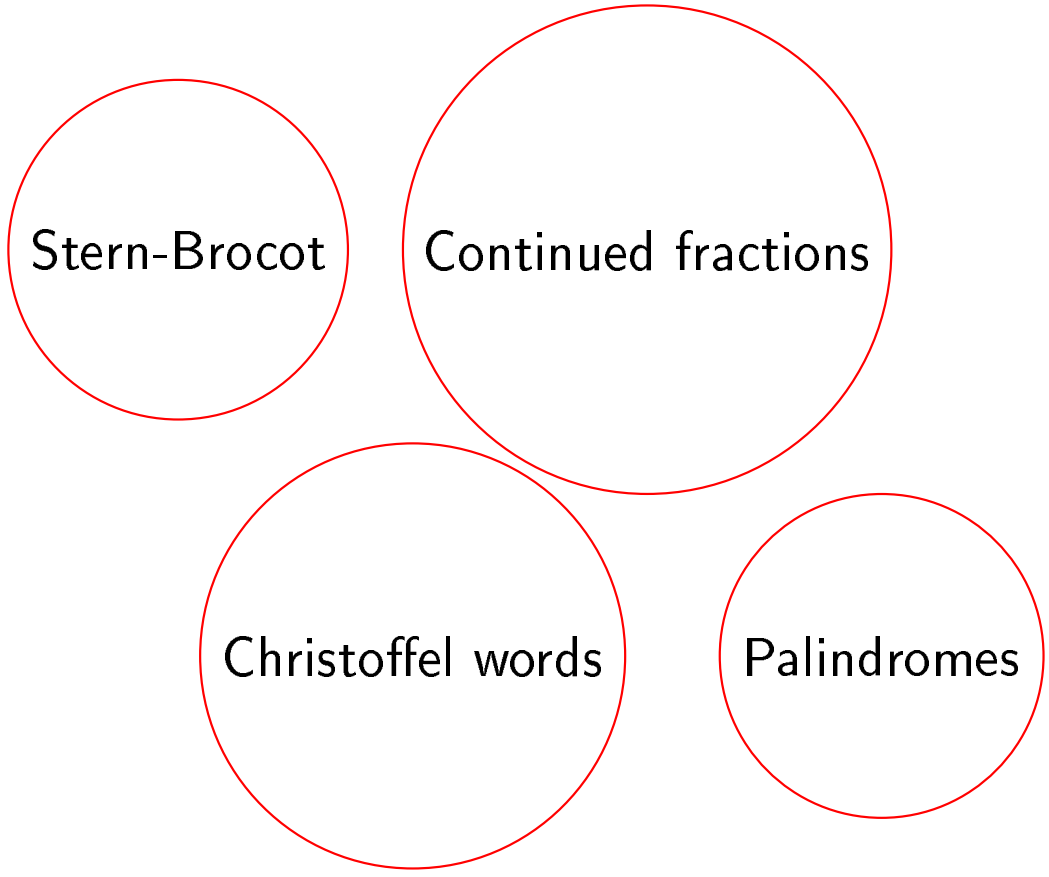
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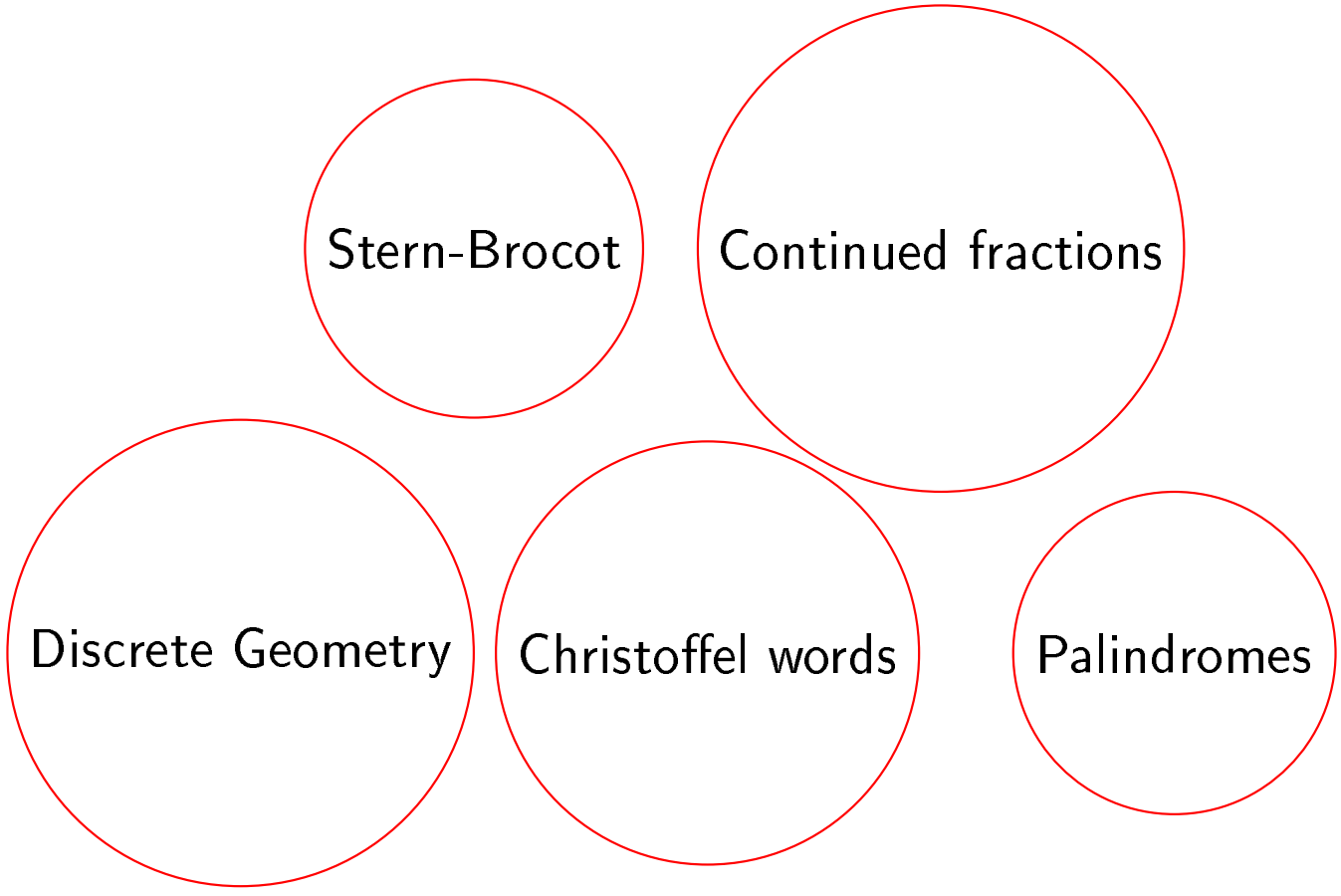
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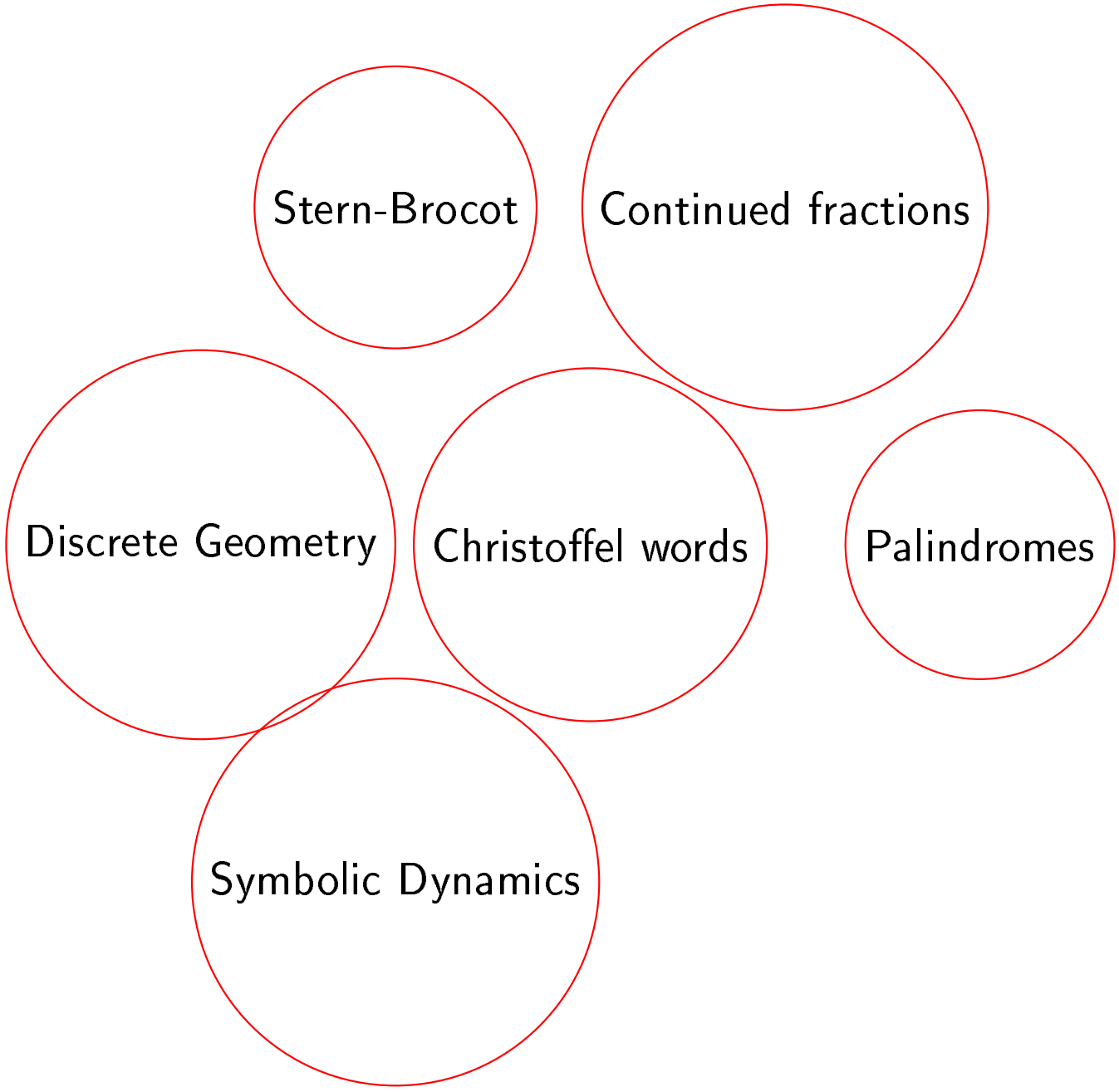
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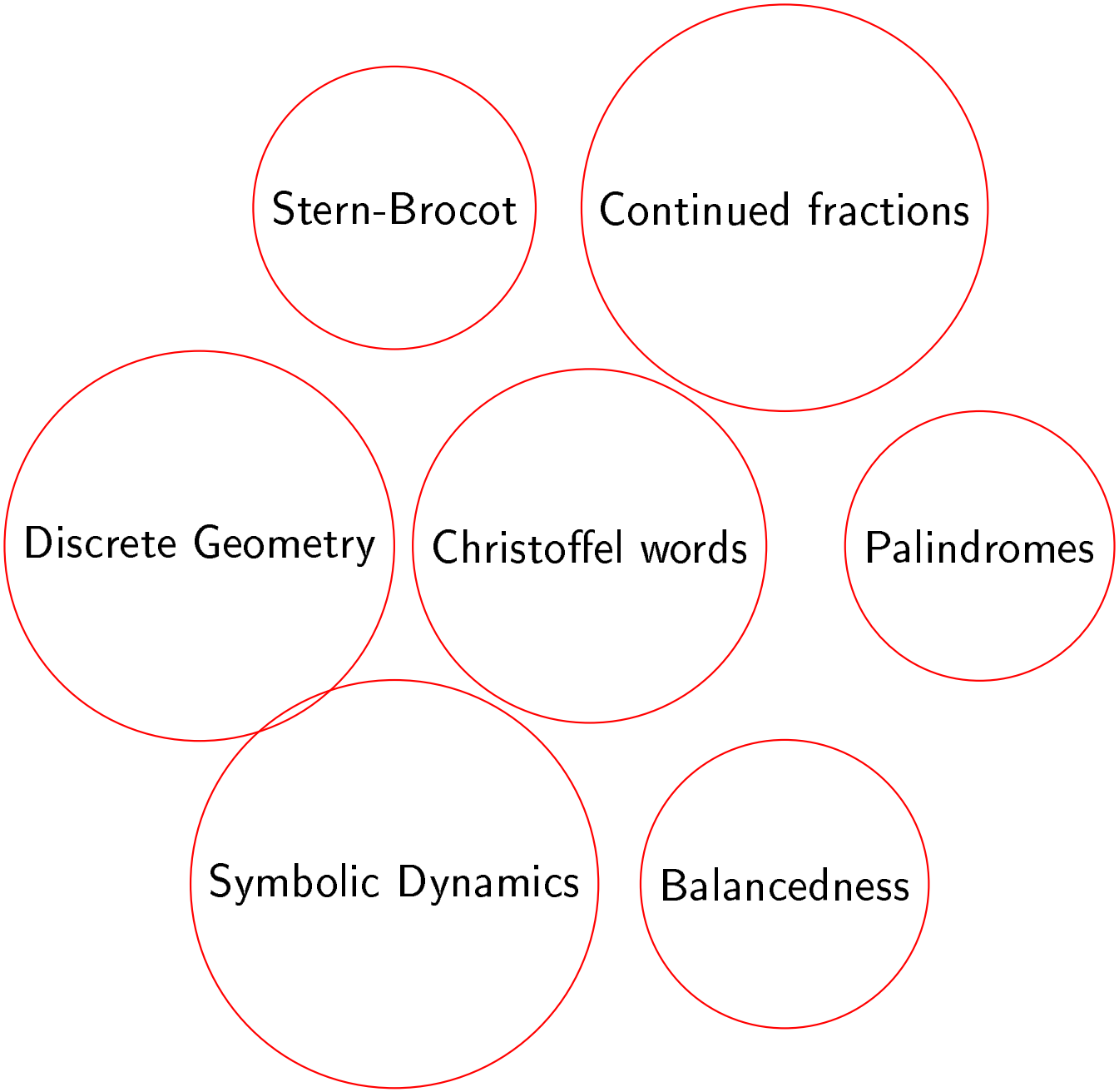
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Balancedness

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- Two words w and w' are *conjugate* and denoted $w \equiv_k w'$ iff $\exists\ u, v;\ |u| = k$ *with* $w = uv$ *and* $w' = vu$.

Example

Let

$$w = balancedness$$

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- We restrict the work to the binary alphabet $A = \{0, 1\}$.

Christoffel path

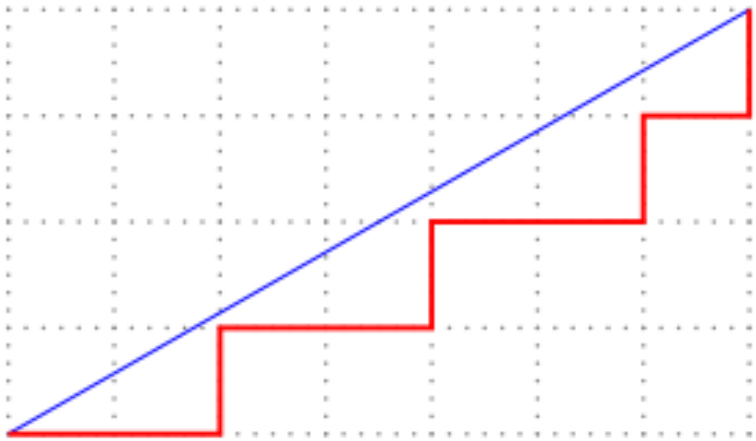

Figure: The Christoffel path of the line segment joining $O(0,0)$ to (b,a).

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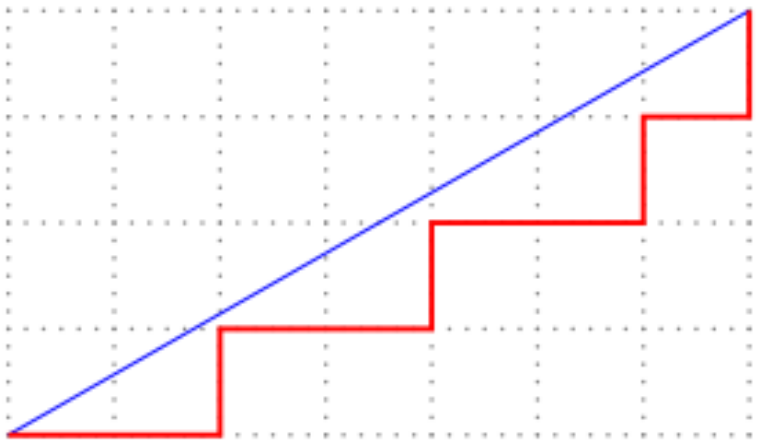

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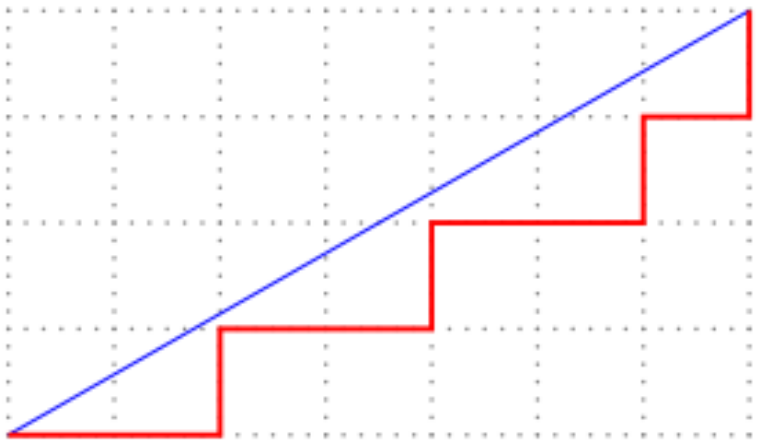

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1. The closest path to the line segment.
2. There are no points of $\mathbb{Z} \times \mathbb{Z}$ between the path and line segment.

Christoffel word

Horizontal step $\longrightarrow$ 0.

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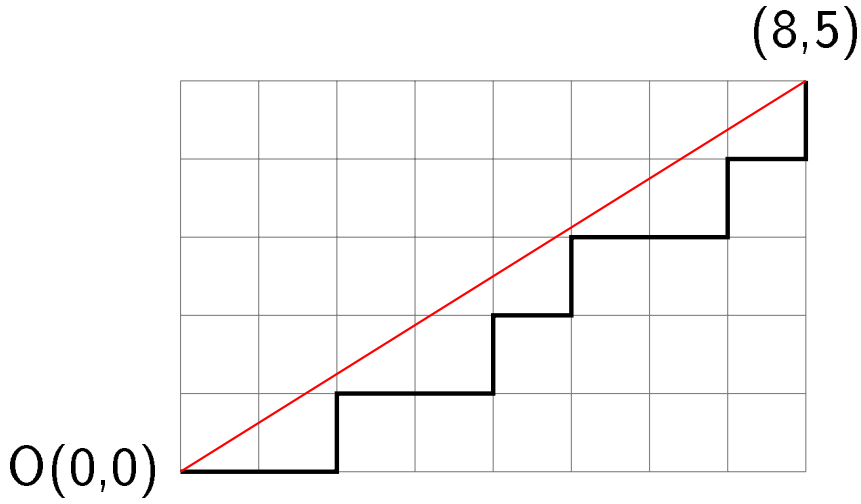
Proposition (De Luca 1997)

Christoffel words are of the form: $0w_1 1$, *where* w_1 *is a palindrome.*

Example

The line segment from $O(0,0)$ to $(8,5)$ has the following Christoffel word:



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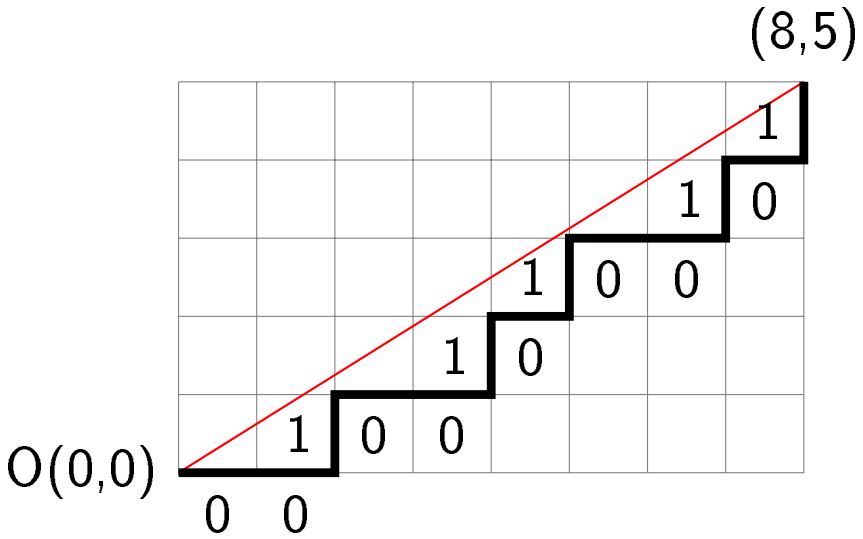

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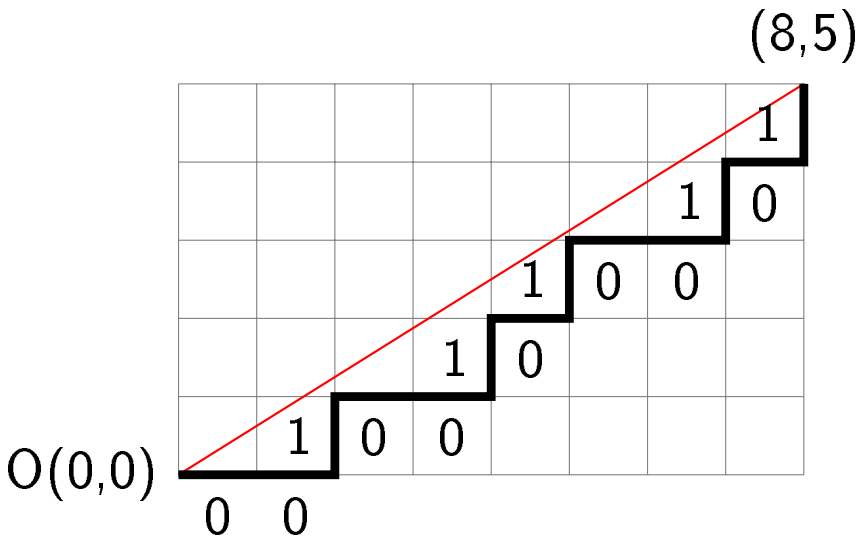
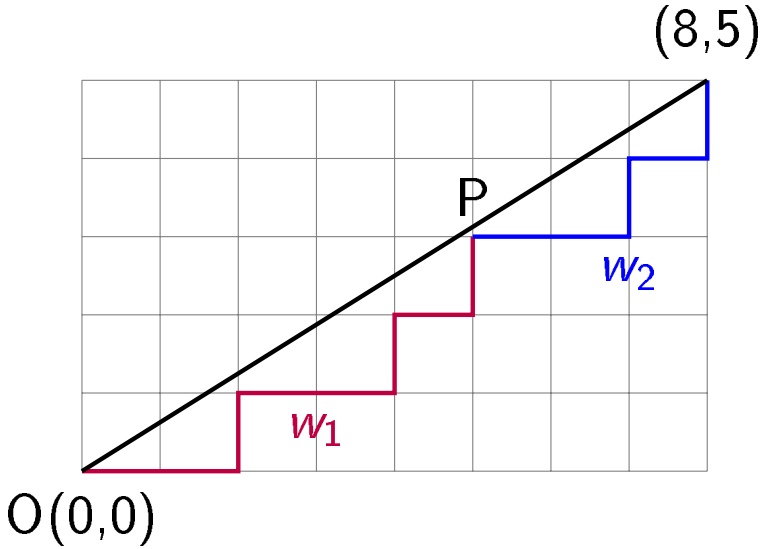


$C(\frac{5}{8}) = 00100101001010 = 0\boxed{01001010010}1 = 0v1$, where v is a palindrome.

Standard factorization



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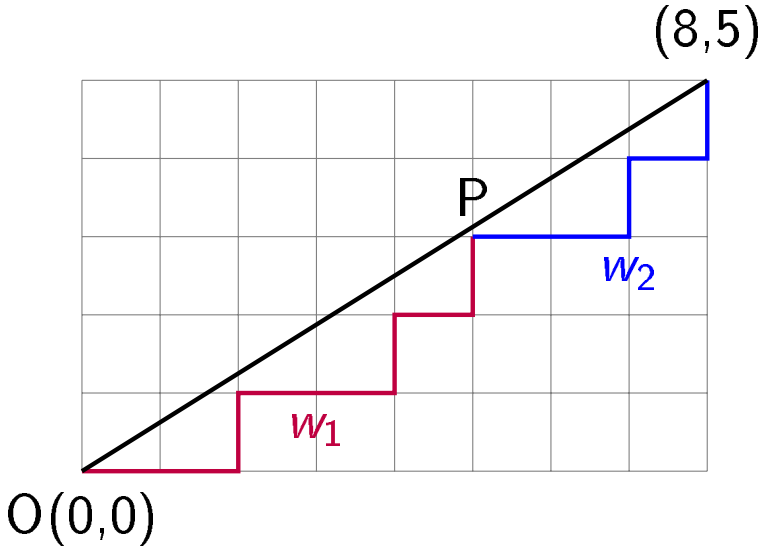

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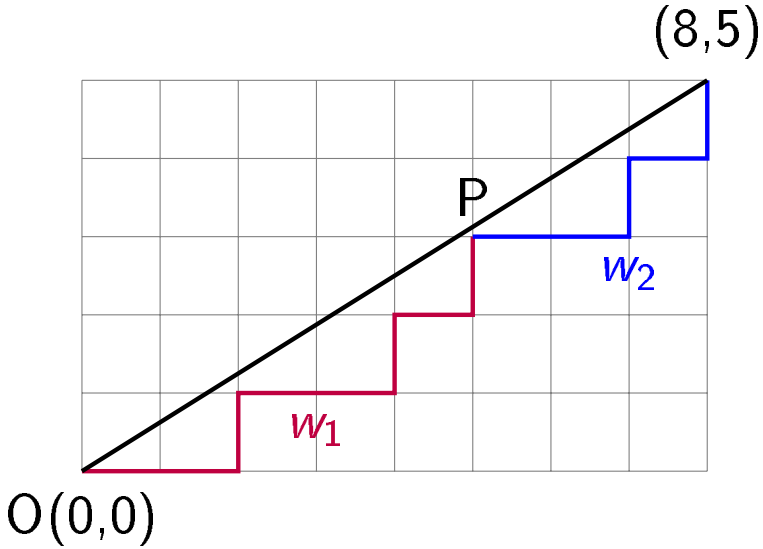

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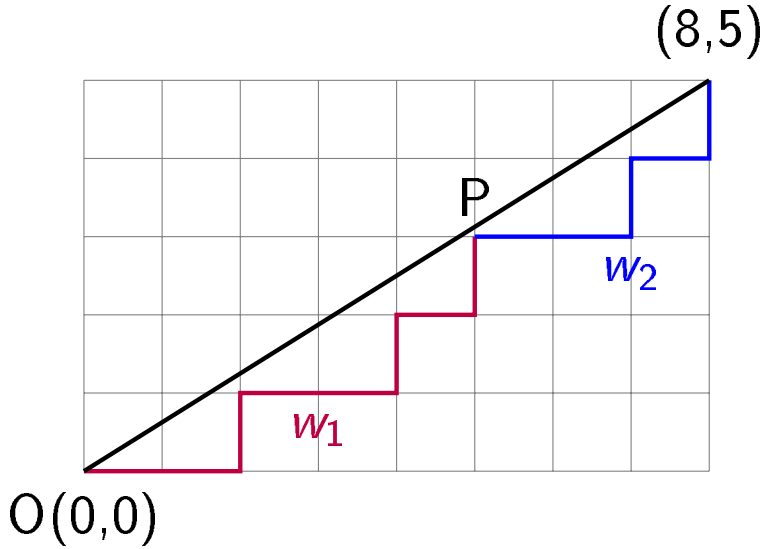

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Theorem (Borel, Laubie (BL1993))

A nontrivial Christoffel word w has a unique factorization $w = (w_1, w_2)$ with w_1 and w_2 Christoffel words.

Symmetric standard factorization

Definition

If $w = (w_1, w_2)$, where $w_1 = 0w_1'1$ and $w_2 = 0w_2'1$ then $w = 0(p, q, p)_{ss}1$, where:

$$p = \begin{cases} w_1' & \text{if } |w_1| < |w_2|, \\ w_2' & \text{if } |w_1| > |w_2|. \end{cases} \tag{1}$$

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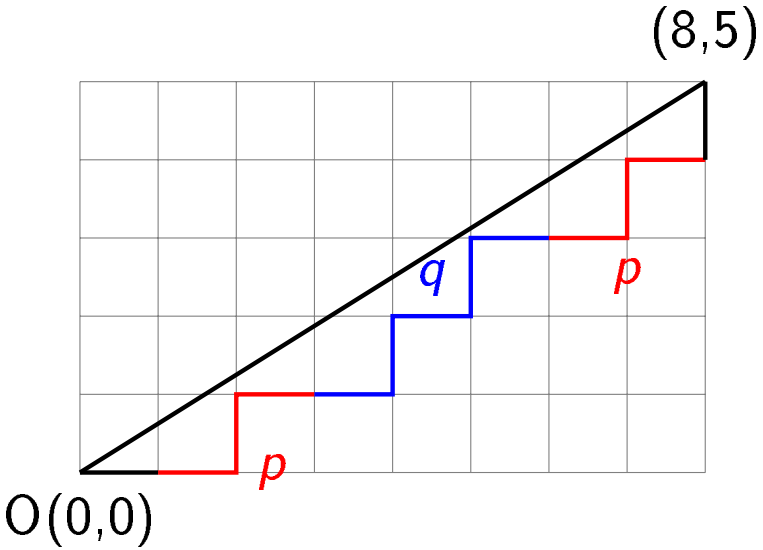

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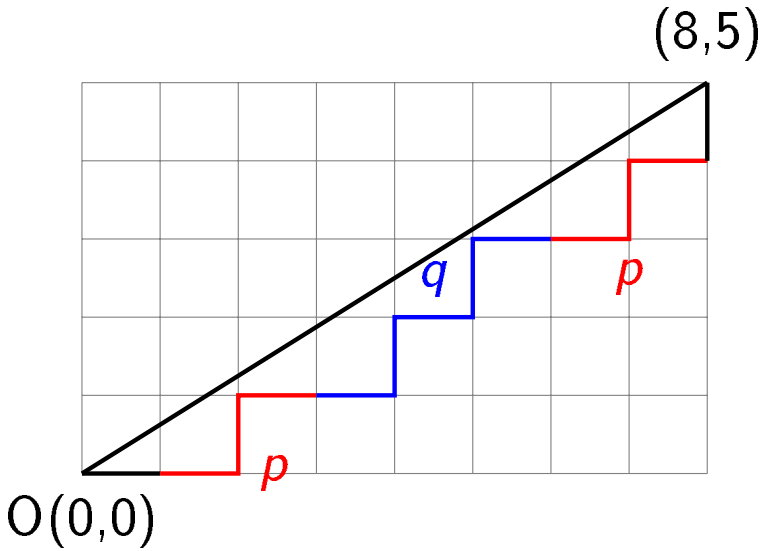

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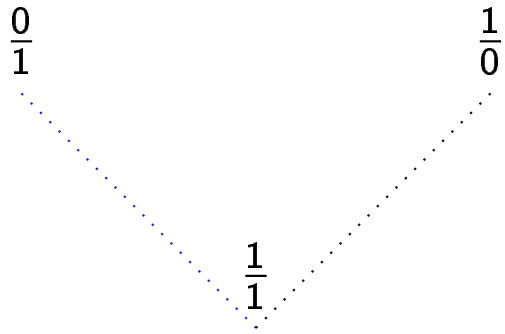

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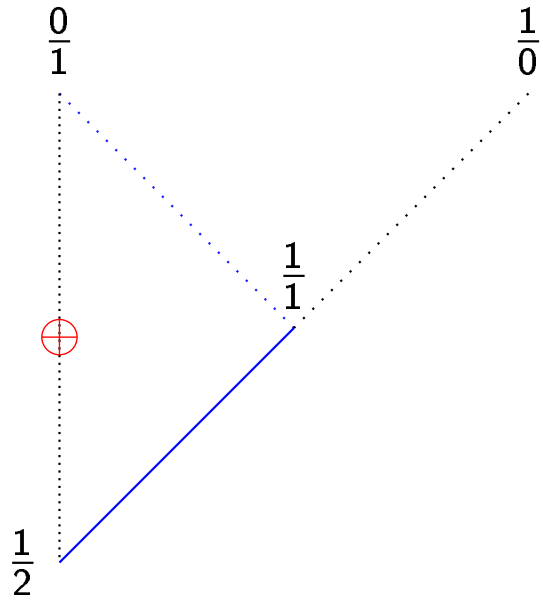

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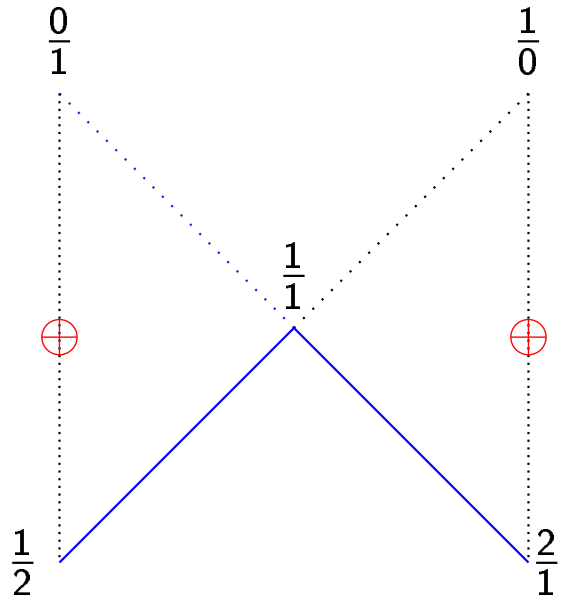

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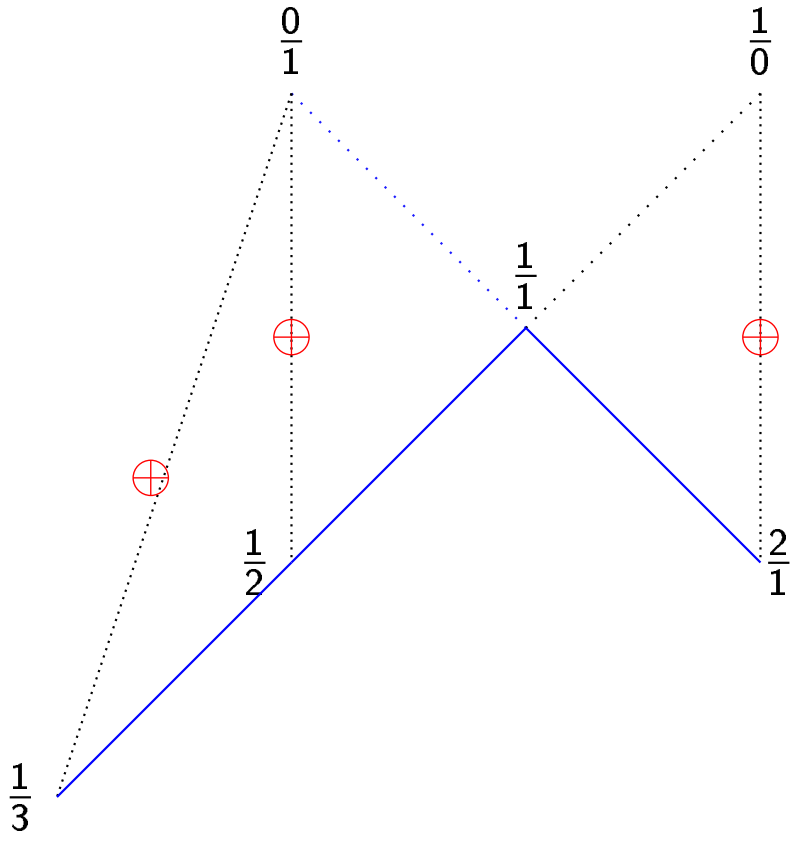

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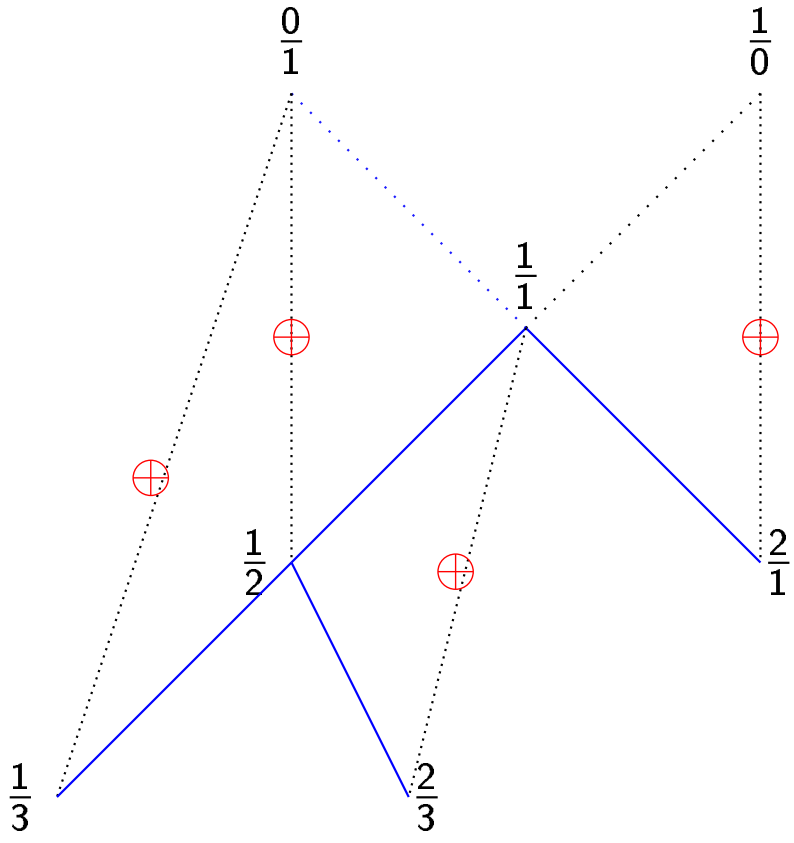

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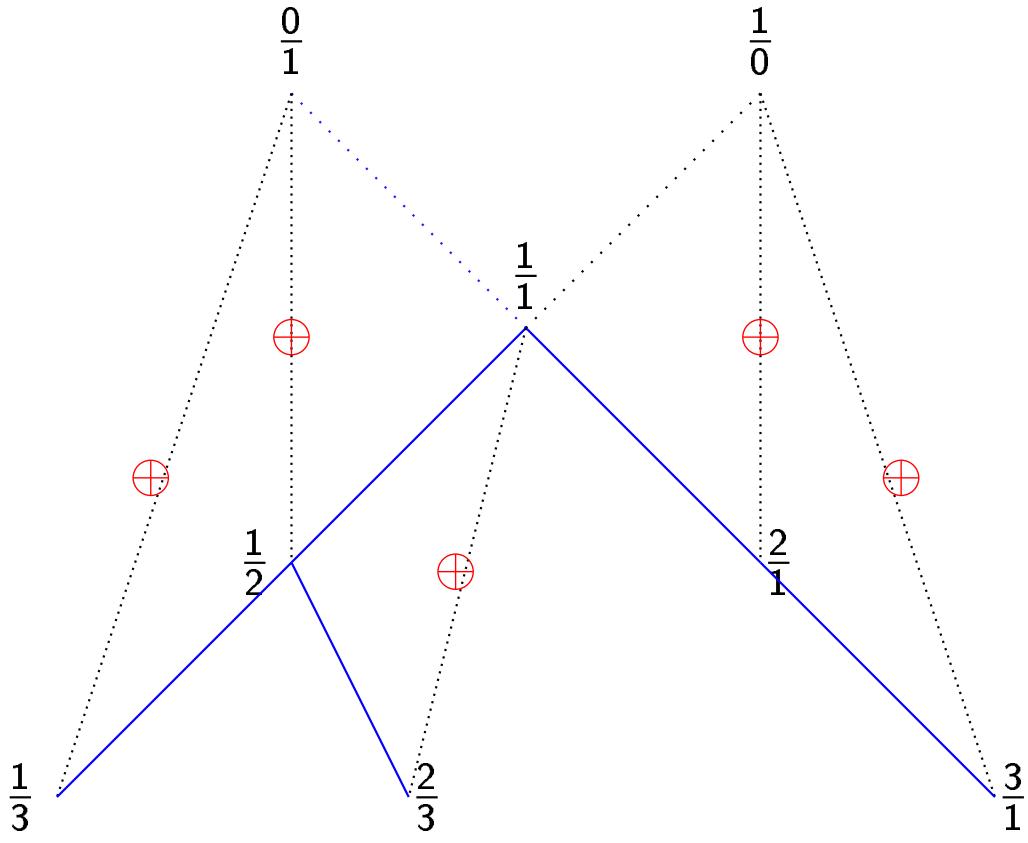

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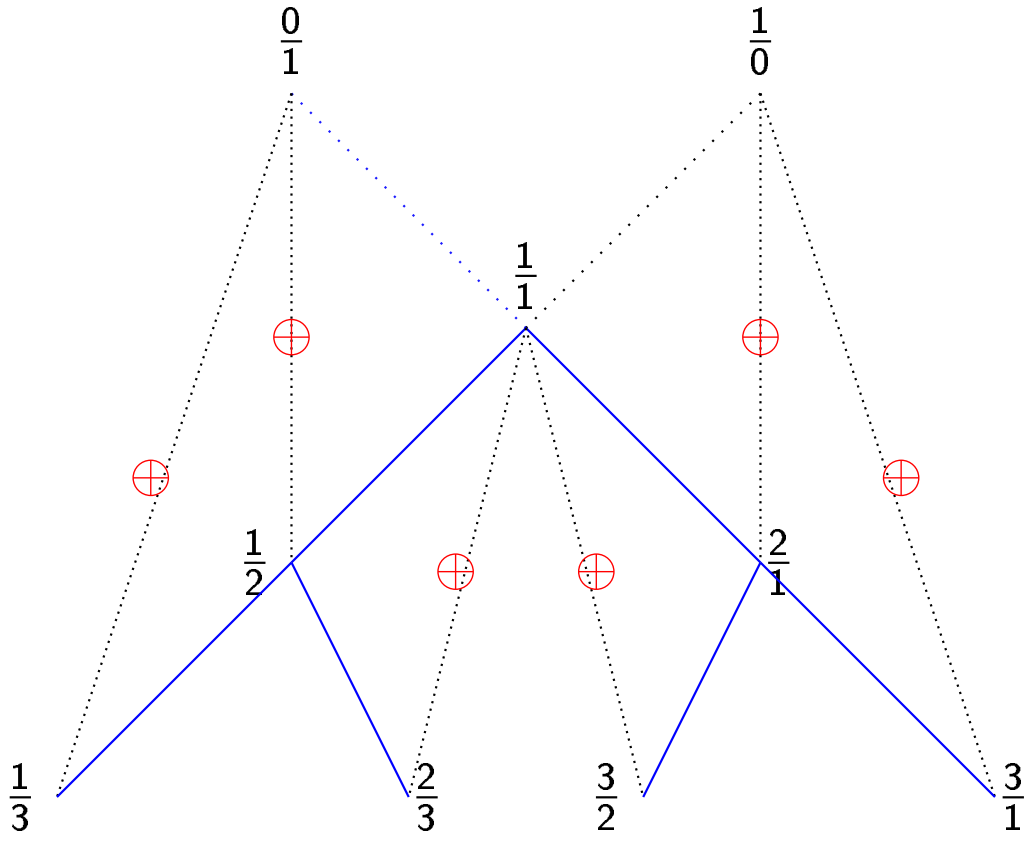

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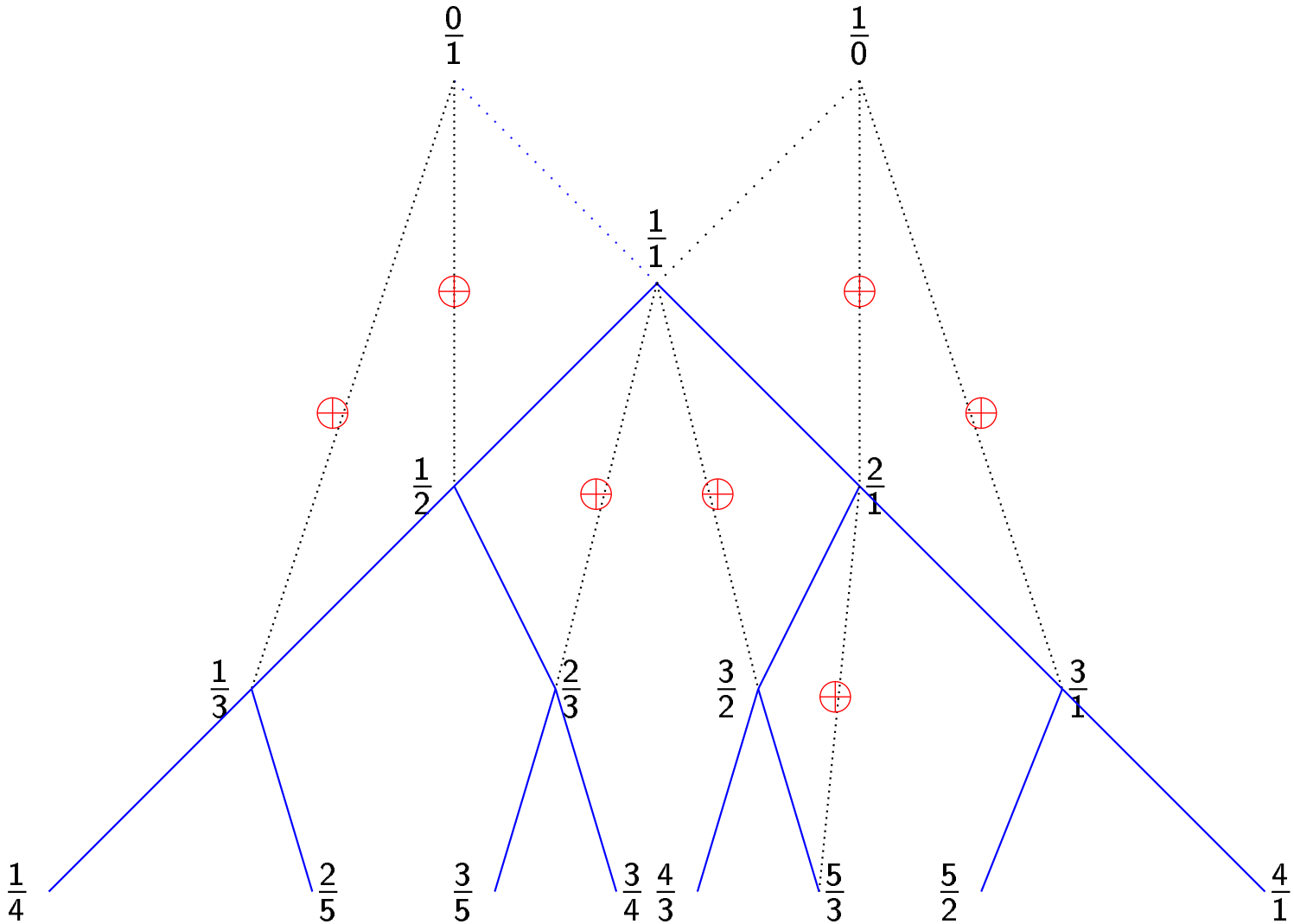

Continued fractions

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The continued fraction of a rational number $\frac{a}{b}$ is given by: $\frac{a}{b} = [a_0, \ldots, a_n]$, s.t:

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Example

We write $\frac{5}{8} = [0, 1, 1, 1, 2]$ since:

$$\frac{5}{8} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$$

More relations

Lemma (Smith 1876)

Let $\frac{a}{b} = [a_0, a_1, \ldots, a_z]$, *the Directive sequence of* a/b *is given by:*
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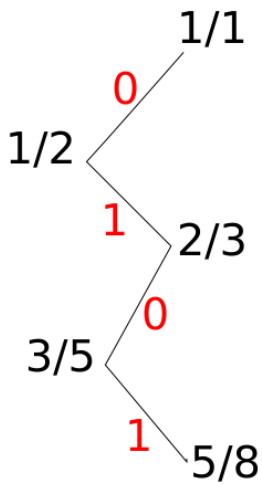

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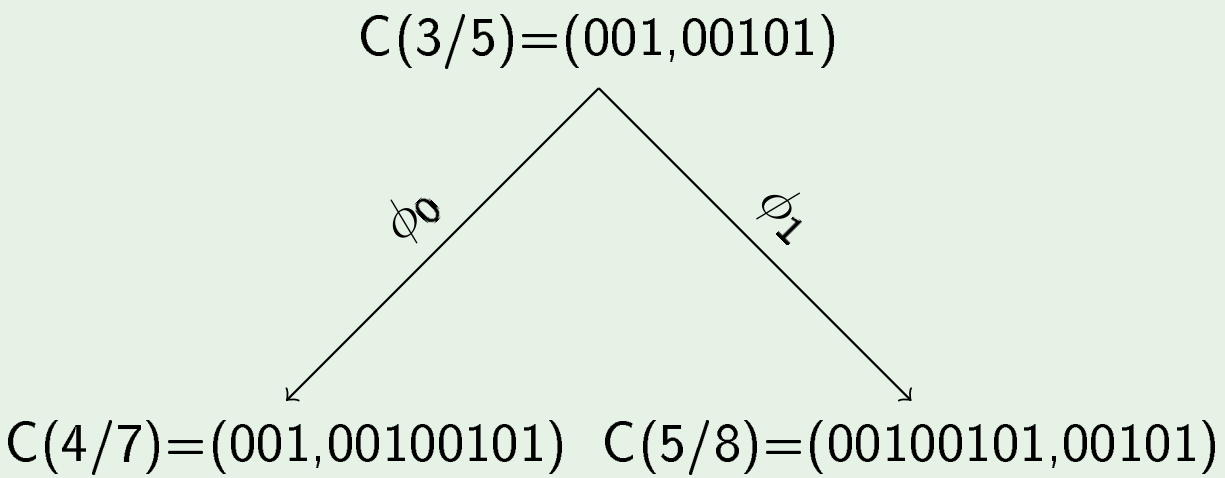
Special morphisms

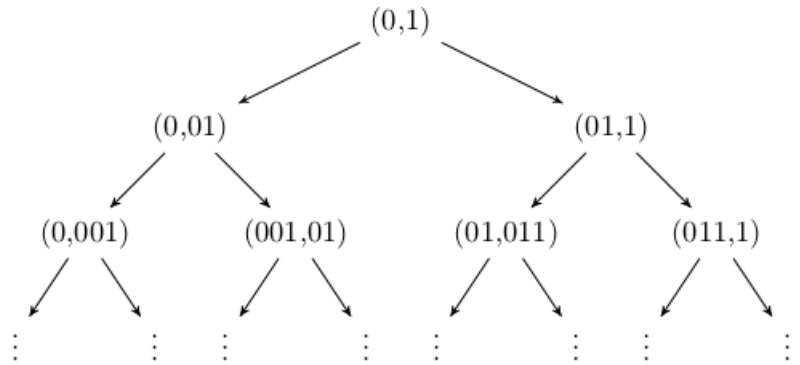
Let ϕ_0; ϕ_1 be two morphisms from A^* to A^* s.t:
$\phi_0(w_1, w_2) = (w_1, w_1 w_2)$ and $\phi_2(w_1, w_2) = (w_1 w_2, w_2)$.

Example

Let the Christoffel word of slope 3/5;



Isomorphism between the two trees



First 3 levels of the Christoffel tree

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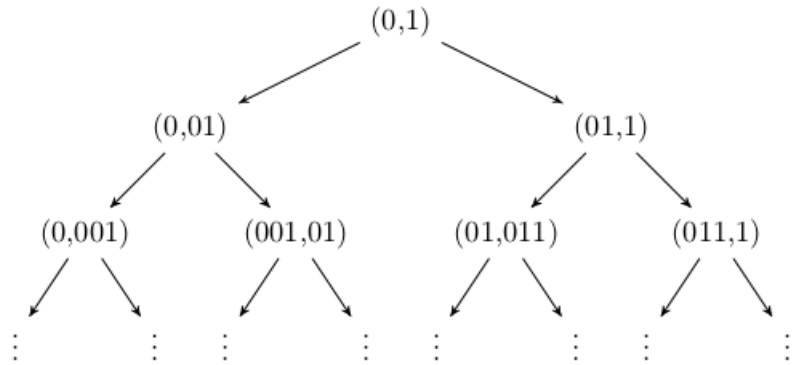
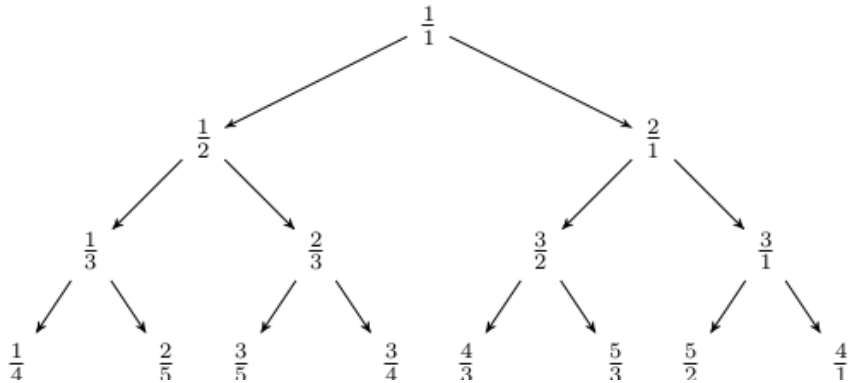


First 3 levels of the Christoffel tree



First 4 levels of the Stern-Brocot tree

Proposition (de Luca 1997)

Let $w = uv$ be a word s.t v is the longest suffix of w that is a palindrome. Then the Palindromic closure $w^+ = w\tilde{u}$ is the unique shortest palindrome having w as a prefix.

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Definition (de Luca 1997)

The iterated palindromic closure of w is:

$$Pal : \{0,1\}^* \longrightarrow \{0,1\}^*$$

recursively such that: $Pal(\epsilon) = \epsilon$ and if $w = vz \in \{0,1\}^*$ for some $z \in \{0,1\}$, then $Pal(w) = Pal(vz) = (Pal(v)z)^+$.

Theorem (BL(93),DL(97),BDLR(08))

The Christoffel word is exactly $C(a/b) = 0PAL(\Delta(a/b))1$, *where* $\Delta(a/b)$ *is the directive sequence of* a/b.

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Let $\frac{5}{8} = [0, 1, 1, 1, 2]$,
we have: $\Delta(5/8) = 0101$.
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$Pal(010) = \underline{01}0\underline{0}10$;
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The Christoffel word of slope 5/8 is:
$C(\frac{5}{8}) = 0.Pal(\Delta(5/8)).1 = 0.01001010010.1 = 0010010100101$.

Definition (Paquin,Reuteneuer 2011)

The set $D(a,b) = \{i \mid w[i] = 1\}$ of the indices of the letter 1 in $C(\frac{a}{b})$ is:

$$D(a,b) = \{i\alpha + 1 \mid i = 1 \dots a\};$$

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Example

Let the rational number $\frac{5}{8}$ with $n = 13$;
By calculating: $5\alpha \equiv_{13} 12$, we get $\alpha = 5$ and $D(5,8) = \{3,6,8,11,13\}$.
$C(\frac{5}{8}) = 0010010100101$.

Balanced property

Definition

For all two factors w' and w'' of w, such that $|w'| = |w''|$;
w is $\delta-$balanced if for all $a \in A$, we have:

$$||w'|_a - |w''|_a| \leq \delta.$$

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Theorem (Berstel, de Luca BdL1997)

A word w is a Christoffel word iff it is a $1-$balanced word.

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Definition (Balanced Matrix)

$B_w[i,j] = S_w[i,j] - \min(S_w[i])$, $\forall\ 1 \leq i \leq n-1$ and $1 \leq j \leq n$.

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- $B_{C(\frac{a}{b})}$ is denoted $B_{\frac{a}{b}}$.
- By construction $B_w[1] = w$.

Examples

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Let $\frac{a}{b} = \frac{5}{2}$; $n = 7$; $C(\frac{5}{2}) = 0110111$. We have: $\max(B_{\frac{5}{2}}) = 1$.

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$$S_{\frac{5}{2}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 4 & 4 & 3 & 4 & 4 & 3 \\ 4 & 5 & 4 & 4 & 5 & 4 & 4 \end{pmatrix}$$

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Example

Let $w = 0011 \in A^*$. w^ω is a $2-$balanced word and $\max(B_w) = 2$.

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Example

Let $w = 0011 \in A^*$. w^ω is a 2−balanced word and $\max(B_w) = 2$.

$$S_w = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \quad , \quad B_w = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

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Definition

A word is an *auto-complementary* word if it is complementary to itself.

Lemma

For a Christoffel word w of slope $\frac{a}{b}$ with $n = a + b$:

$$B_{\frac{a}{b}}[k] \equiv_{n-k} \overline{B_{\frac{a}{b}}[n-k]}, \ \forall\ 1 \leq k \leq n-1.$$

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$$B_{\frac{a}{b}}[k] \equiv_{n-k} \overline{B_{\frac{a}{b}}[n-k]},\ \forall\ 1 \leq k \leq n-1.$$

Let $\frac{a}{b} = \frac{3}{7}$ with $n = 10$, the balanced matrix $B_{\frac{3}{7}}$ is given by:

$$B_{\frac{3}{7}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Note that $B_{\frac{3}{7}}[5]$ is an auto-complementary row.

Second construction of the balanced matrix

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Theorem

We let $D_1 = D(a,b)$ *and for each* i *from* 2 *to* $n-1$*:*

$$D_i = \begin{cases} D_{i-1} \cup (D_1 - (i-1)) & \text{if } i \notin D_1 \\ D_{i-1} \cap (D_1 - (i-1)) & \text{if } i \in D_1 \end{cases}$$

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Lemma

For the balanced matrix $B_{\frac{a}{b}}$*, we have:* $card(D_i) = i.a \bmod n$.

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Definition

We define the second order of balance as $\delta^2(w) = \max(U_w)$.

Properties of MUM

1. $U_{C(\frac{a}{b})}$ is denoted by: $U_{\frac{a}{b}}$.

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3. $U_{\frac{a}{b}} = U_{\frac{b}{a}}$.

Example for $\frac{a}{b} = \frac{5}{8}$

To compute $U_{\frac{5}{8}}[2,4]$:

$$B_{\frac{5}{8}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

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$B_{\frac{5}{8}}[2] = [0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1]$
$U_{\frac{5}{8}}[2,4] = \max(B_{B_{\frac{5}{8}}[2]}[4]) = \max([0,1,1,1,2,1,1,1,1,2,1,1,1]) = 2$

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$$U_{\frac{5}{8}} = \left(\begin{array}{cccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \textcircled{2} & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

$\delta^2(C(\frac{5}{8})) = 2.$

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$$U_{\frac{a}{b}} = \left(\begin{array}{c|c|c} \alpha & \cdot & \cdot \\ \hline \gamma & \beta & \cdot \\ \hline \cdot & \cdot & \cdot \end{array} \right),$$

Γ operator

For $w = C(\frac{a}{b}) = O(p, q, p)_{ss}1$ and $n = a + b$, we have:

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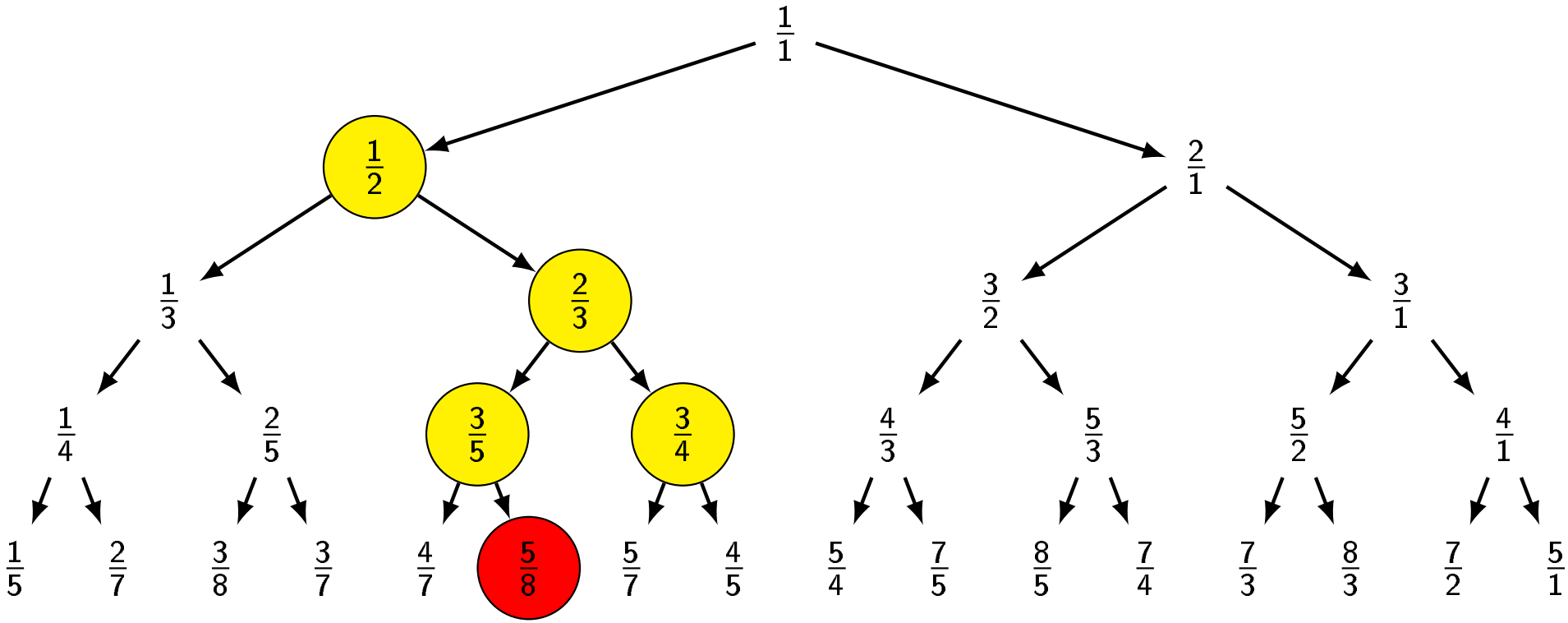
Theorem

The word $\Gamma(\frac{a}{b})$ *is*

$$\Gamma\left(\frac{a}{b}\right) = \begin{cases} 0^{b-2} & \text{if } a = 1 \\ 1^{a-2} & \text{if } b = 1 \\ q' & \text{where } q = aq'a \text{ for } a \in A, \text{ otherwise.} \end{cases}$$

Example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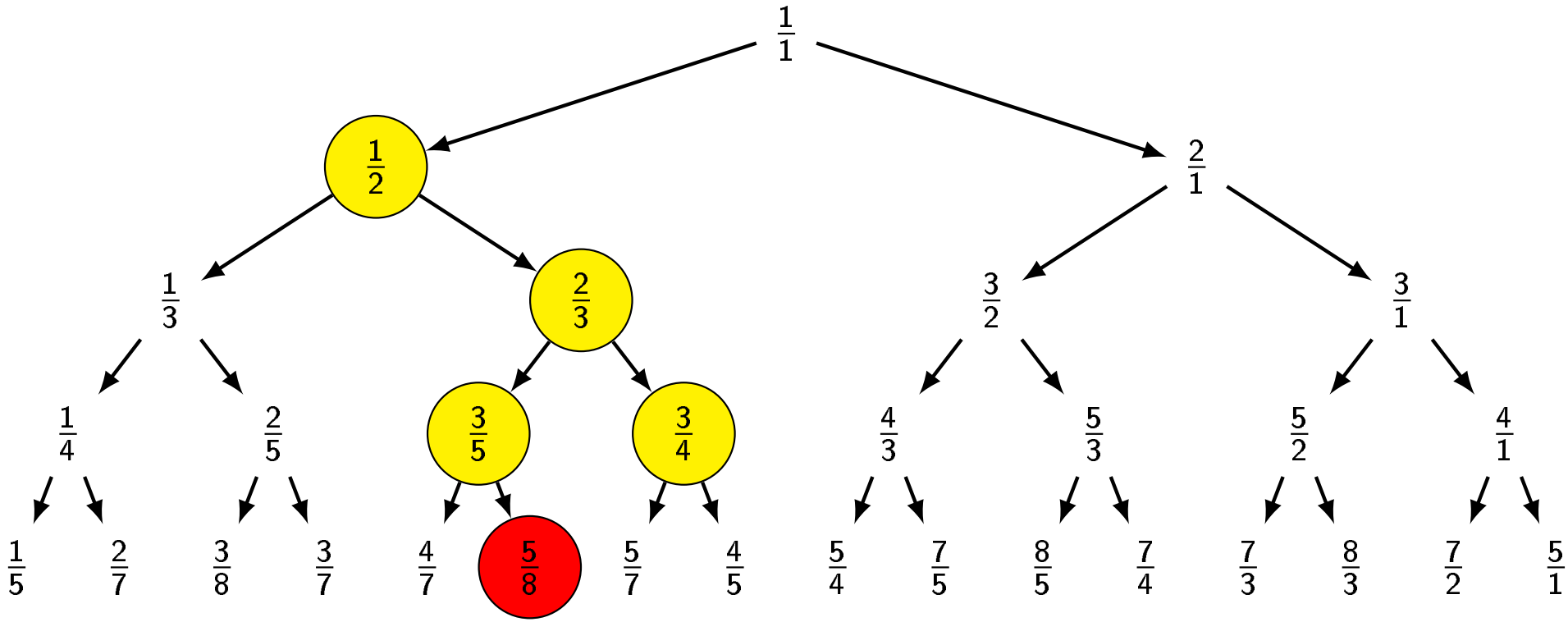

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1. $\frac{3}{5} = [0, 1, 1, 2]$,

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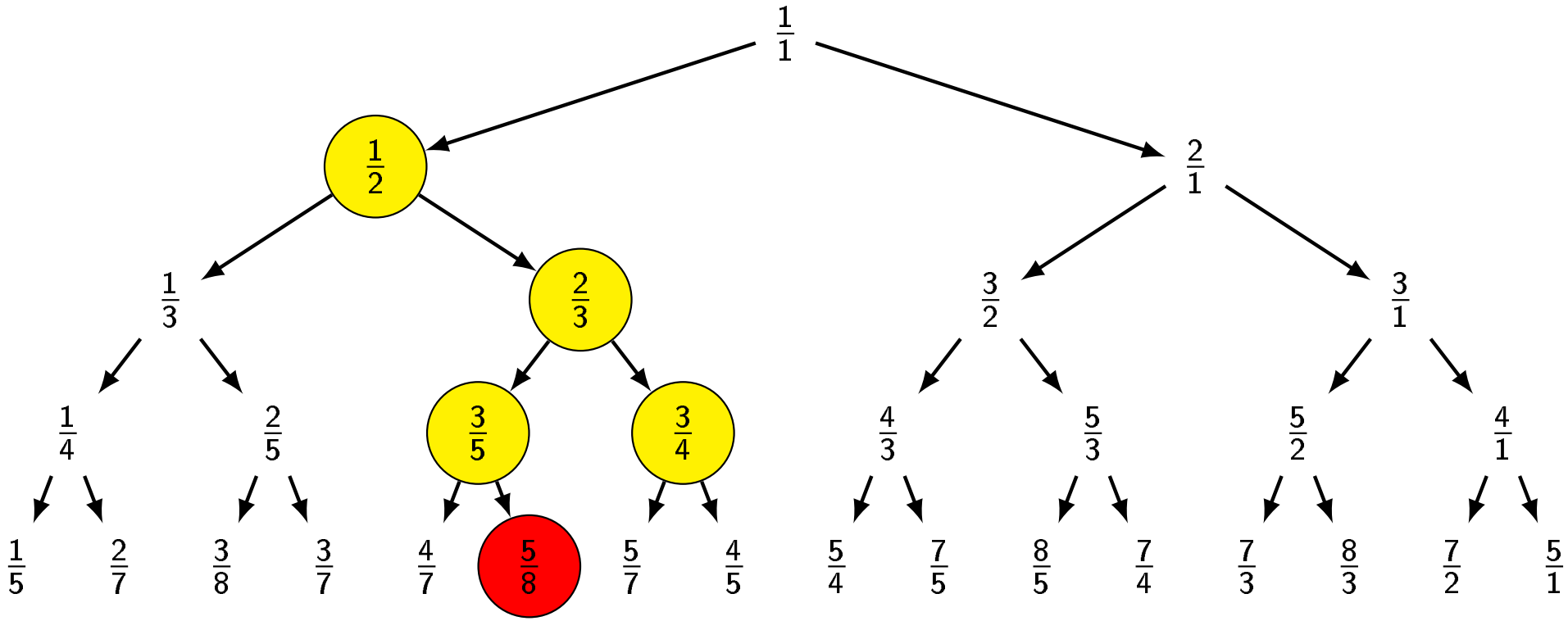

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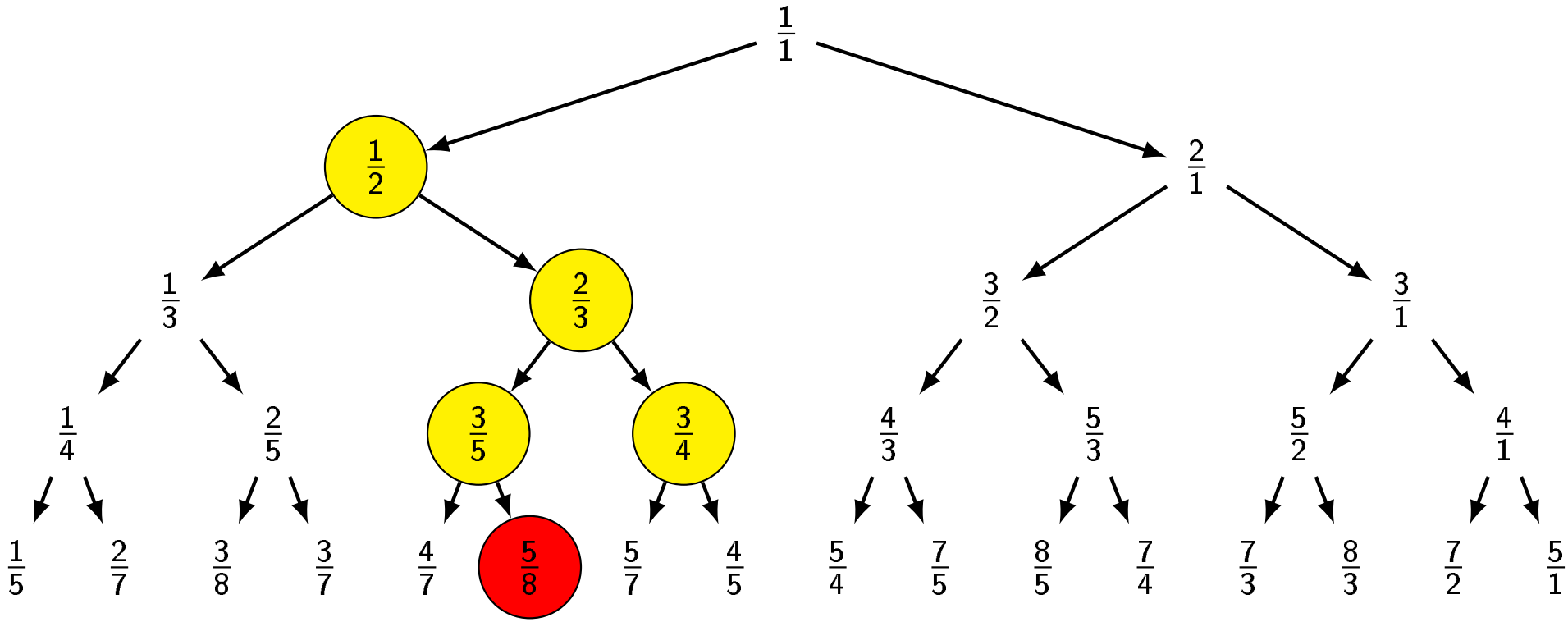

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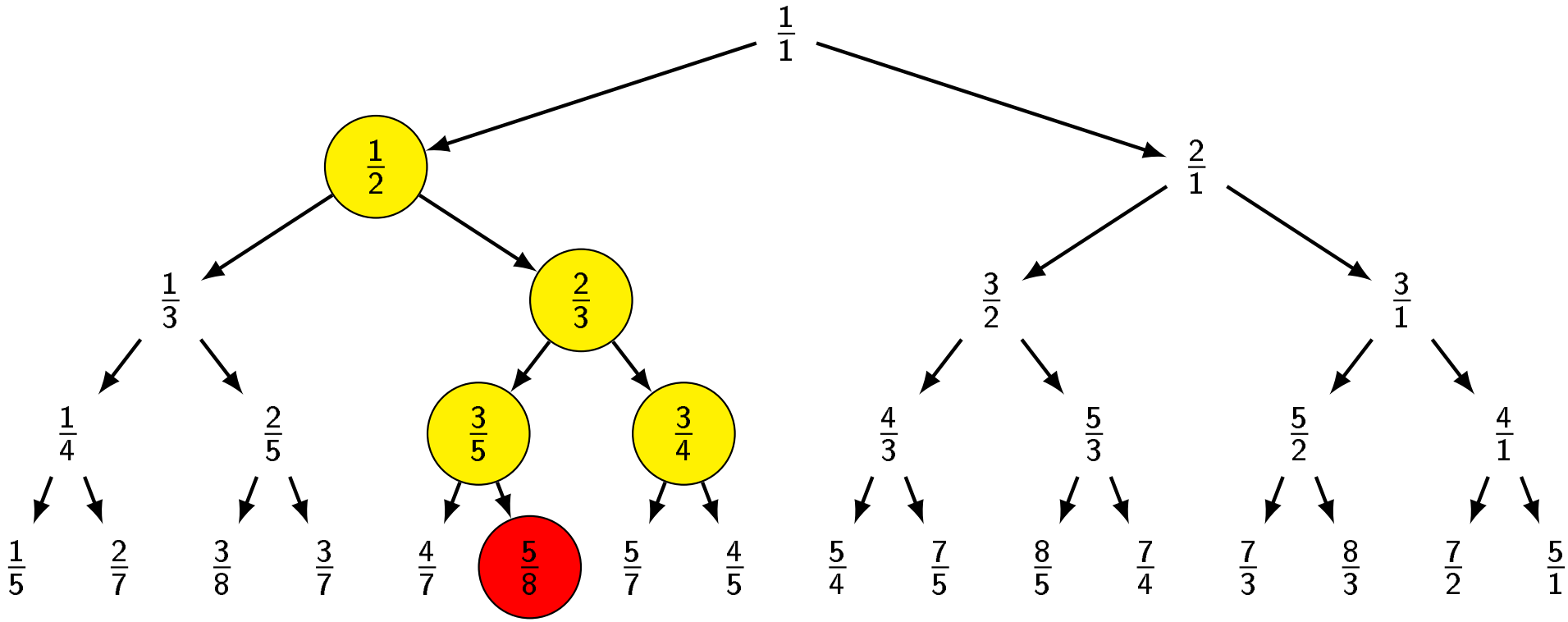

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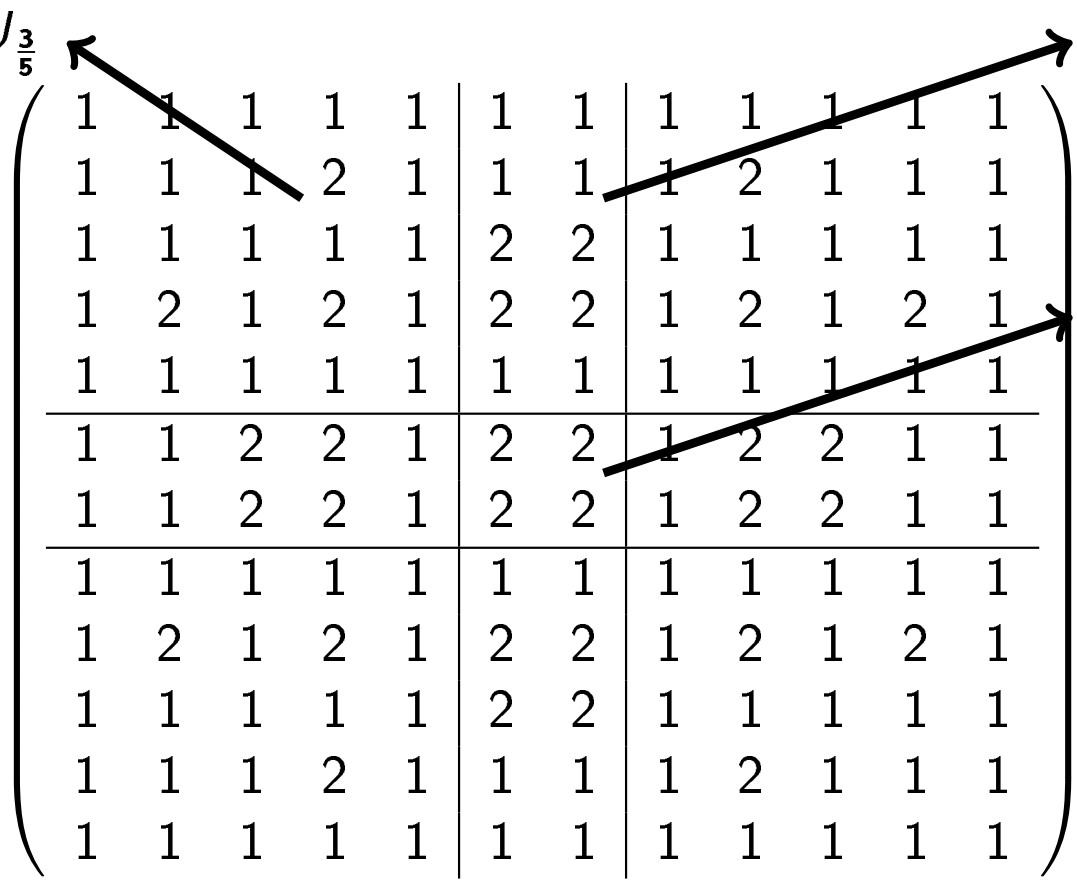
$$\textit{The } \gamma - \textit{block} : U(\frac{3}{4}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Top left part of $U_{\frac{3}{5}}$



$$U_{\frac{5}{8}} = \left(\begin{array}{ccccc|cc|ccccc} 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&2&1&1&1&1&2&1&1&1\\ 1&1&1&1&1&2&2&1&1&1&1&1\\ 1&2&1&2&1&2&2&1&2&1&2&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ \hline 1&1&2&2&1&2&2&1&2&2&1&1\\ 1&1&2&2&1&2&2&1&2&2&1&1\\ \hline 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&2&1&2&1&2&2&1&2&1&2&1\\ 1&1&1&1&1&2&2&1&1&1&1&1\\ 1&1&1&2&1&1&1&1&2&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1 \end{array}\right).$$

Maximal order

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For all level k on the Stern-Brocot tree we have: $\delta_k^2 \geq \lceil \frac{k}{3} \rceil$.

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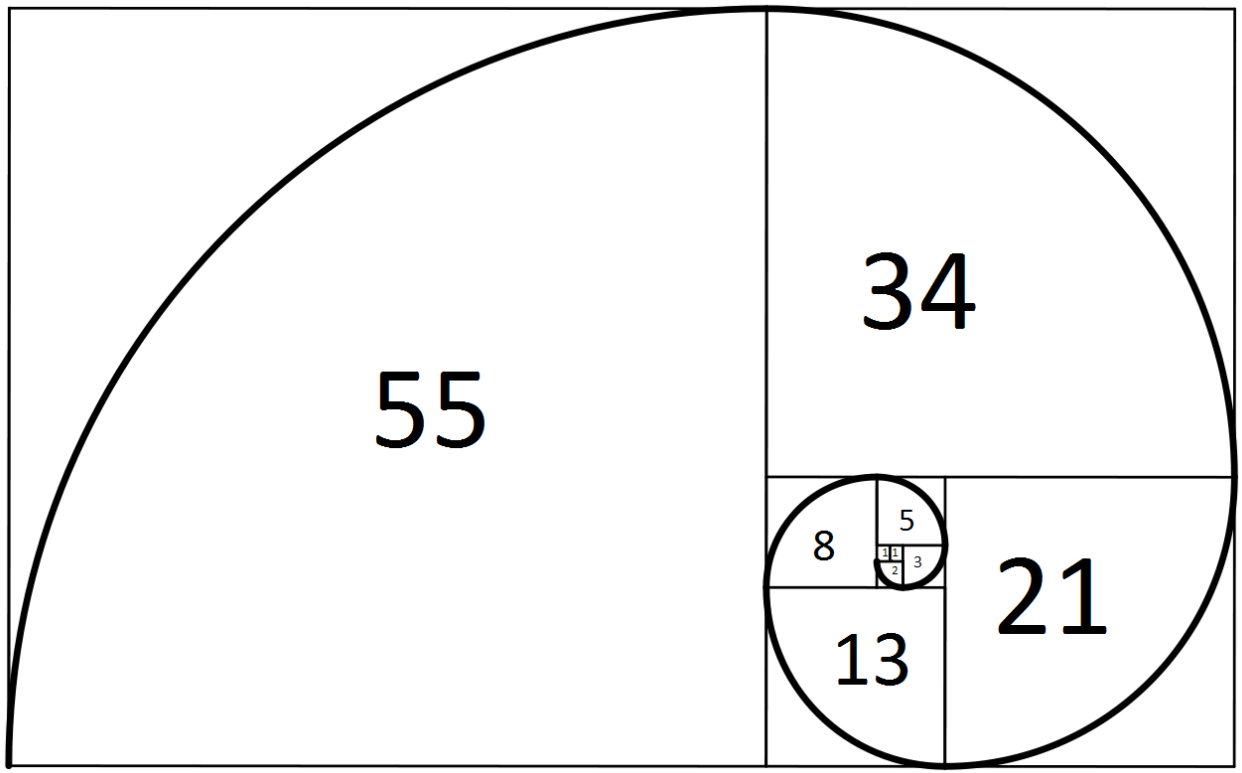

$$F_{n+2} = F_{n+1} + F_n;\ \forall\ n \geq 0.$$

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Lemma

The zigzag path who converges to the golden ratio φ minimizes the growth of δ^2 on the stern-Brocot tree.

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Thank you !!