Avoiding long abelian powers in words

Matthieu Rosenfeld

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April 7, 2017

Joint work with Michaël Rao
Can you find an infinite binary word that does not contain 2 consecutive identical factors?
First example

Can you find an infinite binary word that does not contain 2 consecutive identical factors?

\[ \varepsilon \]
Can you find an infinite binary word that does not contain 2 consecutive identical factors?

\[
\begin{array}{c}
\varepsilon \\
\downarrow \\
a
\end{array}
\]

⇒ Squares are not avoidable over the binary alphabet.

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\[ \varepsilon \]

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⇒

Squares are not avoidable over the binary alphabet.
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Can you find an infinite binary word that does not contain 2 consecutive identical factors?

\[
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a \\
ba \\
\text{aa} \\
\text{ab} \\
\text{abb}
\end{array}
\]

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Can you find an infinite binary word that does not contain 2 consecutive identical factors?

\[
\begin{align*}
&\varepsilon \\
&a &\text{aa} \\
&\text{ab} &\text{aba} \\
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\end{align*}
\]
Can you find an infinite binary word that does not contain 2 consecutive identical factors?
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⇒ Squares are not avoidable over the binary alphabet.
A \textbf{n-th power} is a word of the form $u^n$.

$aa$, $abcabc$, $aaaa$ are \textbf{squares} ($uu$).

$aaa$, $bababa$, $abcabcabc$ are \textbf{cubes} ($uuu$).
“Usual” repetitions: squares and cubes

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The \textbf{period} is $|u|$.
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The \textbf{period} is \( |u| \).

A word is \textbf{X-free} if it does not contain a factor which is of the form \( X \).

- \( abacabcac \) is \textbf{square-free}.
- \( ababbabaabba \) is \textbf{cube-free}.
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\textbf{Question}

What are the \(k, n \in \mathbb{N}\) such that there is an infinite \(k\)-th power-free word over \(n\) letters?
“Usual” repetitions: squares and cubes

A *n-th power* is a word of the form $u^n$.

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$abacabcac$ is **square-free**.

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**Question**

What are the $k, n \in \mathbb{N}$ such that there is an infinite $k$-th power-free word over $n$ letters?

What about a ternary alphabet?
What about cubes?
Avoiding cubes: The Thue-Morse Sequence

- Thue-Morse morphism: \( f : \{0,1\}^* \rightarrow \{0,1\}^* \), s.t.:
  \[
  f(0) = 01 \\
  f(1) = 10
  \]
Avoiding cubes: The Thue-Morse Sequence

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  $011010011001011010011001101001\ldots = \lim_{n \rightarrow \omega} f^n(0) = f^\omega(0)$

**Theorem (Thue)**

*The Thue-Morse sequence is cube-free.*
Avoidability of repetitions

Theorem (Thue, 1906)

*There is an infinite ternary square-free word.*

Let $h'$:

\[
\begin{align*}
0 & \rightarrow 1 \\
1 & \rightarrow 20 \\
2 & \rightarrow 210.
\end{align*}
\]

$h'^\infty(2) = 210201210120210 \ldots = \text{Ternary Thue-Morse sequence.}$
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**Theorem (Thue, 1906)**

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<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>No</td>
<td>Yes (Thue 1906)</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>3</td>
<td>Yes (Thue 1906)</td>
<td></td>
<td>$\cdots$</td>
</tr>
<tr>
<td>4</td>
<td>$\vdots$</td>
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### Question (Erdős 1961)

Can we avoid long squares over a binary alphabet?

In fact, he “conjectured” that one cannot avoid long squares over a binary alphabet.
Avoidability of long repetitions

Question (Erdős 1961)
Can we avoid long squares over a binary alphabet?

In fact, he “conjectured” that one cannot avoid long squares over a binary alphabet.

Theorem (Entringer, Jackson & Schatz 1974)

There exists an infinite binary word avoiding squares of period at least 3.

Construction:

\[
g : \begin{cases} 
0 & \rightarrow \ 1010 \\
1 & \rightarrow \ 1100 \\
2 & \rightarrow \ 0111.
\end{cases}
\]

Apply \( g \) to a square-free ternary word.
Abelian $n$-th powers

Two words $u$ and $v$ are **abelian equivalent**, denoted $u \sim_a v$ if $v$ is a permutation of $u$, i.e.:

$$\forall \sigma \in A, |u|_\sigma = |v|_\sigma.$$ 

A word $w$ is an **abelian $n$-th power** if it can be written $w = u_1 \ldots u_n$ such that for all $i, j$, $u_i \sim_a u_j$.

Example: $abcabc$, $abcbac$, $cabacabcac$, $chicane caniche$
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**Fact**

Every ternary word of size at least 8 has an abelian square.
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Example: $abcabc, abc bac, cabac abcac, chicane caniche$

**Fact**

Every ternary word of size at least 8 has an abelian square.

**Question (Erdős 1957 & 1961)**

Is there an infinite abelian square-free word over 4 letters?
Erdős’ questions

28. There exists a sequence \( \{s_n\} \) of 0’s, 1’s and 2’s such that no two adjacent blocks from \( \{s_n\} \) are the same. Presumably, the first (unpublished) proof of this was obtained by Rose Peltesohn and J. W. Sutherland.

Let \( N(k) \) be the least number \( N \) with the property that each sequence \( \{s_n\}_{n=1}^{N} \) of numbers taken from the set \( \{1, 2, \ldots, k\} \) contains two adjacent blocks such that each is a rearrangement of the other. My earliest conjecture, that \( N(k) = 2^k - 1 \), has been disproved by Bruijn and myself. It is not even known whether \( N(4) < \infty \).

2) As far as I know R. Peltesohn and Sutherland (unpublished) were the first to construct an infinite sequence formed from the symbols 0, 1, 2 where no two consecutive blocks were identical. It is easy to see that in a sequence of length four formed from the symbols 0 and 1 two consecutive blocks will be identical. I understand that Euwe proved that in an infinite sequence formed from 0 and 1 there will be arbitrarily large identical consecutive blocks, but that there do not have to be three consecutive identical blocks.

Let us now call two consecutive blocks „identical” if each symbol occurs the same number of times in both of them (i.e. we disregard order). I conjectured that in a sequence of length \( 2^k - 1 \) formed from \( k \) symbols there must be two “identical” blocks. This is true for \( k \leq 3 \), but for \( k = 4 \) de Bruijn and I disproved it and perhaps an infinite sequence of four symbols can be formed without consecutive “identical” blocks.

Theorem (Ker¨ anen, 1992)

Any fixed point of the following morphism is abelian square-free:

\[
\begin{align*}
    a & \rightarrow \text{abcacdcbcdcadcdbdabacabadbabcbdbcbacbcdcacbabdabacadcbcdcacdbcbacbcdcacdcbdcdadbdcbca} \\
    b & \rightarrow \text{bcdbdadcdadbadacabcbdbcbacbcdcacdcbdcdadbdcbcabcbdbadcdadbdacdcbdcdadbdadcadabacadcdb} \\
    c & \rightarrow \text{cdacabadabacbabdbcdcacdcbdcdadbdadcadabacadcdbcdcacbadabacabdadcadabacabadbabcbdbadac} \\
    d & \rightarrow \text{dabdbcbabcbdcbcacdadbdadcadabacabadbabcbdbadacdadbdcbabcbdbcabadbabcbdbcbacbcdcacbabd}
\end{align*}
\]
Avoiding abelian squares


Theorem (Ker"anen, 1992)
Any fixed point of the following morphism is abelian square-free:

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\begin{align*}
& a \rightarrow abcacdcbcdcadcdbdabacabadbabcbdbcbacbcdcacbabdabacadcbcdcacdbcbacbcdcacdcbdcdadbdcbca \\
& b \rightarrow bcdbdadcdadbadacabcbdbcbacbcdcacdcbdcdadbdcbcabcbdbadcdadbdacdcbdcdadbdadcadabacadcdb \\
& c \rightarrow cdacabadabacbabdbcdcacdcbdcdadbdadcadabacadcdbcdcacbadabacabdadcadabacabadbabcbdbadac \\
& d \rightarrow dabdbcbabacbdcbcacdadbdadcadabacabadbabcbdbadacdadbdcbabcbdbcabadbabcbdbcbacbcdcacbabd
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  a &\rightarrow abcacdcbcdacdbdabacabadbacbdcbdbacbcacdbca \\
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  c &\rightarrow cdacabdabababdbdbdcalcdcdcbdcdadbcdadbcadbadcabacdbadcadacbadcabacdbadacdb \\
  d &\rightarrow dabdbcbabcbdbacdbcdabcdadbdbdacababadababcdbdbcadbabcabcbdbcbacbcacdbcabab 
\end{cases} \]
Other results

<table>
<thead>
<tr>
<th>n</th>
<th></th>
<th>Σ</th>
<th></th>
<th>2</th>
<th>3</th>
<th>≥ 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (square)</td>
<td>3</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td>∞</td>
</tr>
<tr>
<td>3 (cube)</td>
<td>9</td>
<td>?</td>
<td></td>
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**Figure:** Longest word avoiding abelian $n$-th powers over an alphabet $Σ$. 
Other results

| n     | $|\Sigma|\,$ | 2  | 3  | $\geq 4$ |
|-------|------------|----|----|---------|
| 2 (square) | 3          | 7  | $\infty$ |
| 3 (cube)   | 9          | $\infty$ | $\infty$ |
| $\geq 4$   | ?          | $\infty$ | $\infty$ |

**Figure**: Longest word avoiding abelian $n$-th powers over an alphabet $\Sigma$.

Dekking showed (1978):
- avoidability of abelian cubes on a ternary alphabet,
Other results

| n     | $|\Sigma|$ | 2  | 3  | $\geq 4$ |
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Figure: Longest word avoiding abelian $n$-th powers over an alphabet $\Sigma$.

Dekking showed (1978):
- avoidability of abelian cubes on a ternary alphabet,
- avoidability of abelian 4-th powers on a binary alphabet.
Question (Erdős):
Can we avoid long squares over a binary alphabet? (1961)
Can we avoid abelian squares over 4 letters? (1957)
Avoidability of long abelian repetitions

Question (Erdős):
Can we avoid long squares over a binary alphabet? (1961)
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Avoidability of long abelian repetitions

| n               | $|\Sigma|$ | 2      | 3      | $\geq$ 4 |
|-----------------|-----------|--------|--------|---------|
| 2 (square)      | ?         | ?      | Yes    |
| 3 (cube)        | ?         | Yes    | Yes    |
| $\geq$ 4        | Yes       | Yes    | Yes    |
Avoidability of long abelian repetitions

Question (Erdős):
Can we avoid long squares over a binary alphabet? (1961)
Can we avoid abelian squares over 4 letters? (1957)

Avoidability of long abelian repetitions

| n     | $|\Sigma|$ | 2  | 3  | $\geq 4$ |
|-------|-----------|----|----|--------|
| 2 (square) |           | No | ?  | Yes    |
| 3 (cube)   |           | ?  | Yes| Yes    |
| $\geq 4$  |           | Yes| Yes| Yes    |

Theorem (Entringer, Jackson and Schatz, 1974)

*You cannot avoid long abelian squares on a binary alphabet.*
Avoidability of long abelian repetitions

| n     | \(|\Sigma|\) | 2   | 3   | \(\geq 4\) |
|-------|-------------|-----|-----|------------|
| 2 (square) | No          | ?   | ?   | \(\infty\) |
| 3 (cube)  | ?           | \(\infty\) | \(\infty\) | \(\infty\) |
| \(\geq 4\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) |

**Question 1 (Mäkelä, 2003)**

Can you avoid abelian cubes of the form \(uvw\) where \(|u| \geq 2\), over two letters?

**Question 2 (Mäkelä, 2003)**

Can you avoid abelian squares of the form \(uv\) where \(|u| \geq 2\) over three letters?
Theorem (Rao, R. 2014)

One cannot avoid abelian cubes of the form $uvw$ where $|u| \geq 2$, over two letters.
Negative answer to the first Mäkelä’s question

**Theorem (Rao, R. 2014)**

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Cannot perform exhaustive search. (years of computation)
Negative answer to the first Mäkelä’s question

**Theorem (Rao, R. 2014)**

*One cannot avoid abelian cubes of the form uvw where |u| ≥ 2, over two letters.*

Cannot perform exhaustive search. (years of computation)

**Lemma**

*Any factorial language L with arbitrarily long words without arbitrarily large powers contains arbitrarily long Lyndon words.*
Negative answer to the first Mäkelä’s question

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One cannot avoid abelian cubes of the form uvw where \(|u| \geq 2\), over two letters.

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Lemma

Any factorial language \(L\) with arbitrarily long words without arbitrarily large powers contains arbitrarily long Lyndon words.

Enumeration of all prefixes of Lyndon word from \(L\):
3 hours to find all 2732711352 such words.
The longest Lyndon word avoiding abelian cubes of period at least 2 has length 290.
Definition

A word $w$ is a **Lyndon word** if for every proper suffix $v$ of $w$, $v >_{\text{lex}} w$.

Example

- 0010101 is a Lyndon word.
- 0110101 is not a Lyndon word.
Definition

\( w \) is a *Lyndon word* if for every proper suffix \( v \) of \( w \), \( v >_{lex} w \).

Example

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Theorem (Lyndon, Chen, Fox)

*For any word \( w \) there are Lyndon words \( u_1 > u_2 > \ldots > u_k \) and \( \alpha_1, \ldots, \alpha_k \in \mathbb{N} \) such that \( w = u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_k^{\alpha_k} \).*
Proof of the Lemma

**Lemma**

Any factorial language $L$ with arbitrarily long words without powers larger than $p$ contains arbitrarily long Lyndon words.

For the sake of contradiction assume there are finitely many Lyndon words:

$l_1 > l_2 > l_3 > \ldots > l_k$

Using the Lyndon decomposition and the fact that $L$ is factorial we get:

for any $w \in L$ there are $\alpha_1, \ldots, \alpha_n$ such that $w = l_{\alpha_1}^1 l_{\alpha_2}^2 \ldots l_{\alpha_k}^k$.

But since $L$ does not contain powers larger than $p$, for all $i$, $\alpha_i \leq p$.

The length of $w$ is bounded by $\sum_i |l_i| \cdot p$. Contradiction!
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Avoidability of long abelian repetitions

Question 1 (weak version)
Can we avoid long abelian cubes over two letters?

⇒ Still open... but the technique presented here may be used to

Question 2 (weak version)
Can we avoid long abelian squares over three letters?

⇒ Yes ! One can avoid abelian squares $uv$ with $|u| \geq 6.$
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Tools we need

All the previous positive results were proved by giving a morphic word with the desired property.

**Question**

How to decide if a morphic word avoids (long) abelian power?

Previous work: [Dekking 1979], [Carpi 1993], [Currie & Rampersad 2012]
Two notions of “abelian-power-freeness”

A morphism $h$ is abelian square-free if for every abelian square-free word $w$, $h(w)$ is abelian square-free.

[Dekking 1979] and [Carpi 1993] gave sufficient conditions which imply abelian power-freeness of morphisms.
Two notions of “abelian-power-freeness”

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[Dekking 1979] and [Carpi 1993] gave sufficient conditions which imply abelian power-freeness of morphisms.

Many (interesting) morphisms are not abelian square-free, but have abelian-square-free fixed point.

E.g. $h : 0 \rightarrow 02\ 1 \rightarrow 32\ 2 \rightarrow 1\ 3 \rightarrow 01$. $h^\omega(0)$ is abelian cube-free, but $h(1003) = 32020201$ contains the abelian cube 202020.

We are interested in checking abelian square-freeness of fixed points.
Assume we can compute for any given set $F$ of factor the set $h^{-1}(F)$ such that:

$$\forall w, h(w) \text{ avoids } F \iff w \text{ avoids } h^{-1}(F)$$

For every $n$, $h^n(w)$ avoids $F$ if and only if $w$ avoids $h^{-n}(F)$. 
Assume we can compute for any given set $F$ of factor the set $h^{-1}(F)$ such that:

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For every $n$, $h^n(w)$ avoids $F$ if and only if $w$ avoids $h^{-n}(F)$.

$h^\omega(a)$ avoids $F$ if and only if $a \notin \bigcup_{n \geq 0} h^{-n}(F)$.
Assume we can compute for any given set $F$ of factor the set $h^{-1}(F)$ such that:

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$h^\omega(a)$ avoids $F$ if and only if $a \not\in \bigcup_{n \geq 0} h^{-n}(F)$.

If $F$ is the set of abelian squares $F$ and $\bigcup_{n \geq 0} h^{-n}(F)$ are infinite, but we find finite representations by using templates (see [Currie & Rampersad 2012])
The **Parikh vector** of a word $w$, denoted $\Psi(w)$, is the vector such that for every $a \in \Sigma$, $\Psi(w)[a] = |w|_a$.

Examples: $\Psi(\epsilon) = \vec{0}$, $\Psi(abac) = \Psi(caab) = (2 \ 1 \ 1)^t$

$\Psi(u) = \Psi(v) \iff u$ and $v$ are abelian equivalent.

$uv$ is an abelian square iff $\Psi(u) = \Psi(v)$.
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$\Psi(u) = \Psi(v) \iff u$ and $v$ are abelian equivalent.

$uv$ is an abelian square iff $\Psi(u) = \Psi(v)$

Let $M_h$ be the matrix s.t. $\forall i, j, M_h[i, j] = |h(j)|_i$.

**Example**

Let $h : \begin{cases} 
0 \rightarrow 02 \\
1 \rightarrow 32 \\
2 \rightarrow 1 \\
3 \rightarrow 01. 
\end{cases}$, then $M_h = \begin{pmatrix} 
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 
\end{pmatrix}$

For any $w$, $\Psi(h(w)) = M_h \Psi(w)$. 
2-Templates

2-template : \( t = [a_1, a_2, a_3, d] \)
- where \( a_i \in \Sigma \cup \{\epsilon\} \)
- and \( d \in \mathbb{Z}|\Sigma| \)

A word \( w \) is a realization of \( t \) if
- \( w = a_1 \, w_1 \, a_2 \, w_2 \, a_3 \)
- \( \forall i \in \{1, \ldots, k - 1\}, \ \Psi(w_2) - \Psi(w_1) = d \)

- \( babab \) and \( ababaaa \) are realizations of \( T = [\epsilon, a, b, \begin{pmatrix} 1 \\ 0 \end{pmatrix}] \)
- realizations of \( [\epsilon, \epsilon, \epsilon, 0] = \text{abelian squares} \)
[\[a'_1, a'_2, a'_3, d']\] is a **parent** of \([a_1, a_2, a_3, d_1]\) by \(h\) if there are \(p_1, s_1, p_2, s_2, p_3, s_3 \in \Sigma^*\) such that:

- \(\forall i, \ h(a'_i) = p_i a_i s_i\)
- \(\forall i, \ d = M_h d' + \Psi(s_2 p_3) - \Psi(s_1 p_2)\)

**Theorem**

\(h(w)\) avoids \(t\)

\(\iff\)

\(h(w)\) avoids small realizations of \(t\), and \(w\) avoids \(\text{Parents}(t)\).
Templates: ancestors

\([a_1', a_2', a_3', d']\) is a parent of \([a_1, a_2, a_3, d_1]\) by \(h\) if there are \(p_1, s_1, p_2, s_2, p_3, s_3 \in \Sigma^*\) such that:

- \(\forall i, \ h(a'_i) = p_i a_i s_i\)
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**Theorem**

\(h(w)\) avoids \(t\) if and only if

\(h(w)\) avoids small realizations of \(t\), and \(w\) avoids Parents(\(t\)).

Let **Ancestors** be the transitive and reflexive closure of Parents.

\(h^\omega(a)\) is \(t\)-free if and only if \(h^\omega(a)\) avoids small realizations of Ancestors(\(t\)).
Templates: ancestors

\[ [a_1', a_2', a_3', d'] \text{ is a parent of } [a_1, a_2, a_3, d_1] \text{ by } h \text{ if there are } p_1, s_1, p_2, s_2, p_3, s_3 \in \Sigma^* \text{ such that:} \]

- \( \forall i, h(a'_i) = p_i a_i s_i \)
- \( \forall i, d = M_h d' + \Psi(s_2 p_3) - \Psi(s_1 p_2) \)

**Theorem**

\( h(w) \) avoids \( t \)
\[ \iff \]
\( h(w) \) avoids small realizations of \( t \), and \( w \) avoids \( \text{Parents}(t) \).

Let \( \text{Ancestors} \) be the transitive and reflexive closure of \( \text{Parents} \).

\( h^\omega(a) \) is \( t \)-free \( \iff \) \( h^\omega(a) \) avoids small realizations of \( \text{Ancestors}(t) \).

- If \( \text{Ancestors}(t) \) is finite, we are done.
Templates: ancestors

$[a_1', a_2', a_3', d']$ is a **parent** of $[a_1, a_2, a_3, d_1]$ by $h$ if there are $p_1, s_1, p_2, s_2, p_3, s_3 \in \Sigma^*$ such that:

- $\forall i, \ h(a_i') = p_i a_i s_i$
- $\forall i, \ d = M h d' + \Psi(s_2 p_3) - \Psi(s_1 p_2)$

**Theorem**

$h(w)$ avoids $t$  
$\iff$  
h(w) avoids small realizations of $t$, and $w$ avoids $\text{Parents}(t)$.

Let **Ancestors** be the transitive and reflexive closure of $\text{Parents}$.

$h^\omega(a)$ is $t$-free $\iff$ $h^\omega(a)$ avoids small realizations of $\text{Ancestors}(t)$.

- If $\text{Ancestors}(t)$ is finite, we are done.
- If every eigenvalue of $h$ has norm $> 1$, $\text{Ancestors}(t)$ is finite [Currie & Rampersad 2012].
Main Theorem

We prove:

**Theorem (Rao, R. 2015)**

*For any primitive morphism $h$ whose matrix has no eigenvalue of norm 1 and any template $t$, whether $h^\omega(a)$ realizes $t$ is decidable.*
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*For any primitive morphism $h$ whose matrix has no eigenvalue of norm 1 and any template $t$, whether $h^\omega(a)$ realizes $t$ is decidable.*

**Corollary**

Under the same conditions one can decide if $h^\omega(a)$ avoids abelian $n$-th powers.
Main Theorem

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**Theorem (Rao, R. 2015)**

For any primitive morphism $h$ whose matrix has no eigenvalue of norm 1 and any template $t$, whether $h^\omega(a)$ realizes $t$ is decidable.

**Corollary**

Under the same conditions one can decide if $h^\omega(a)$ avoids abelian $n$-th powers.

**Idea (similar to [Cassaigne, Currie, Schaeffer & Shallit 2011])**:  
- We are only interested in *realizable* ancestors.  
- If no eigenvalue has norm 1, the set of *realizable* ancestors is finite.
Suppose that $M_h$ has no eigenvalue of norm 1.

Then $\mathbb{C}^{|\Sigma|}$ is the direct sum of $E_c$ and $E_e$, where:

- $E_c$ is the contracting eigen-subspace, i.e. eigenvectors with eigenvalues of norm $< 1$;
- $E_e$ is the expanding eigen-subspace, i.e. eigenvectors with eigenvalues of norm $> 1$. 
Contracting and expanding eigenspace

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**Bounds for eigenvalues $< 1$**

There is $c \in \mathbb{R}$ s.t.:

Parikh vector of factors of $h^\omega(a)$ is at distance at most $c$ of $E_e$.

$\Rightarrow$ vector in a realizable template is at distance at most $2c$ of $E_e$. 
Contracting and expanding eigenspace

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**Bounds for eigenvalues $< 1$**

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Parikh vector of factors of $h^\omega(a)$ is at distance at most $c$ of $E_e$  
$\Rightarrow$ vector in a realizable template is at distance at most $2c$ of $E_e$.

**Bounds for eigenvalues $> 1$**

Every vector in an ancestor of $t$ is at bounded distance of $E_c$. 
Effect of eigen values

\[ g : \begin{cases} 
    a \rightarrow aba \\
    b \rightarrow bca \\
    c \rightarrow ba 
\end{cases} \]

\[ g' : \begin{cases} 
    a \rightarrow cbac \\
    b \rightarrow bbc \\
    c \rightarrow baa 
\end{cases} \]

\[ eg = \{2.87, 0.65, -0.53\} \]

\[ eg' = \{3.26, -1.60, 1.33\} \]
Algorithm (Sketch)

To decide if $h^\omega(a)$ is $t$-free:

1. Compute the maximum distance $c$ from $E_e$ to a Parikh vector of a factor of $h^\omega(a)$. 

2. Compute a finite set $S$ that contains all the ancestors of $t$ at distance at most $2c$ of $E_e$.

3. Check if a small factor realizes a template of $S$. ($h^\omega$ is $t$-free $\iff h^\omega$ has no small realization of $S$.)
To decide if $h^\omega(a)$ is $t$-free:

1. Compute the maximum distance $c$ from $E_e$ to a Parikh vector of a factor of $h^\omega(a)$.
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$h^\omega(a)$ is $t$-free $\iff$ $h^\omega(a)$ has no small realization of $S$. 

Matthieu Rosenfeld
Avoiding long abelian powers in words
To decide if $h^\omega(a)$ is $t$-free:

1. Compute the maximum distance $c$ from $E_e$ to a Parikh vector of a factor of $h^\omega(a)$.
2. Compute a finite set $S$ that contains all the ancestors of $t$ at distance at most $2c$ of $E_e$.
3. Check if a small factor realizes a template of $S$. ($h^\omega$ is $t$-free $\iff$ $h^\omega$ has no small realization of $S$.)
If $M_h$ is non invertible : compute the parents using the Smith decomposition. Since $\text{Ker}(M_h) \cap E_e = \{0\}$, there are finitely many realizable parents.
If $M_h$ is non invertible : compute the parents using the Smith decomposition. Since $\text{Ker}(M_h) \cap E_e = \{0\}$, there are finitely many realizable parents.

Important to get sharp bounds on the distance to $E_e$. 

Technical details
Applications

\[ h_6 : \begin{cases} 
  a &\rightarrow& ace \\
  b &\rightarrow& adf \\
  c &\rightarrow& bdf \\
  d &\rightarrow& bdc \\
  e &\rightarrow& afe \\
  f &\rightarrow& bce. 
\end{cases} \]
Applications

\[ h_6 : \begin{cases} 
  a \rightarrow ace \\
  b \rightarrow adf \\
  c \rightarrow bdf \\
  d \rightarrow bdc \\
  e \rightarrow afe \\
  f \rightarrow bce. 
\end{cases} \]

**Theorem**

\( h^\omega_6(a) \) is abelian-square free.
Applications

$$h_6 : \begin{cases} 
    a \rightarrow ace \\
    b \rightarrow adf \\
    c \rightarrow bdf \\
    d \rightarrow bdc \\
    e \rightarrow afe \\
    f \rightarrow bce. 
\end{cases}$$

Theorem

$h_6^\omega(a)$ is abelian-square free.

$$M = \begin{bmatrix}
    1 & 1 & 0 & 0 & 1 & 0 \\
    0 & 0 & 1 & 1 & 0 & 1 \\
    1 & 0 & 0 & 1 & 0 & 1 \\
    0 & 1 & 1 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 1 & 1 \\
    0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix} \quad \text{and} \quad
J = \begin{bmatrix}
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 3 & 0 & 0 \\
    0 & 0 & 0 & 0 & \sqrt{3} & 0 \\
    0 & 0 & 0 & 0 & 0 & -\sqrt{3}
\end{bmatrix}$$

$[\epsilon, \epsilon, \epsilon, \overrightarrow{0}]$ has 48459 different realizable ancestors.
Theorem (Rao, R. 2015)

Let $h$ and $g$ be two morphisms. If $M_h$ has no eigenvalue of absolute value 1 and $E_e(M_h) \cap \ker M_g$ it is possible to decide if $g(h^\omega(0))$ is long abelian $k$-th power-free.
What else can we do?

**Theorem (Rao, R. 2015)**

Let $h$ and $g$ be two morphisms. If $M_h$ has no eigenvalue of absolute value 1 and $E_e(M_h) \cap \ker M_g$ it is possible to decide if $g(h^\omega(0))$ is long abelian $k$-th power-free.

Let $h_6 : \begin{cases} a \rightarrow ace \\ b \rightarrow adf \\ c \rightarrow bdf \\ d \rightarrow bdc \\ e \rightarrow afe \\ f \rightarrow bce \end{cases}$ and $\varphi : \begin{cases} a \rightarrow bbbaabaaac \\ b \rightarrow bccacccbc \\ c \rightarrow cccbbbcbcc \\ d \rightarrow ccccccccaa \\ e \rightarrow bbbbbcabaa \\ f \rightarrow aaaaaabaa.$

**Theorem (Rao, R. 2015)**

The sequence obtained by applying $\varphi$ to the fixed-point of $h_6$, $\varphi(h_6^\omega(a))$, does avoids abelian squares of period more than 5.
Avoid additive powers

$uv$ is an **additive square** if $|u| = |v|$ and $\sum(u) = \sum(v)$.

$uvw$ is an **additive cube** if $|u| = |v| = |w|$ and $\sum(u) = \sum(v) = \sum(w)$. 
Avoid additive powers

$u \cdot v$ is an **additive square** if $|u| = |v|$ and $\sum(u) = \sum(v)$.

$u \cdot v \cdot w$ is an **additive cube** if $|u| = |v| = |w|$ and $\sum(u) = \sum(v) = \sum(w)$.

**Question (Justin 1972, Pirillo & Varricchio 1992)**

*Are additive squares avoidable on a finite subset of $\mathbb{Z}$?*
Avoid additive powers

$uv$ is an additive square if $|u| = |v|$ and $\sum(u) = \sum(v)$.

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**Question (Justin 1972, Pirillo & Varricchio 1992)**

Are additive squares avoidable on a finite subset of $\mathbb{Z}$?

**Theorem (Cassaigne, Currie, Schaeffer & Shallit 2011)**

The fixed point of $g : 0 \rightarrow 03$, $1 \rightarrow 43$, $3 \rightarrow 1$, $4 \rightarrow 01$ is additive cube-free.

$g^\omega(0) = 03143011034343031011011031430343430343430 \ldots$
Theorem (Rao, R. 2015)

For any morphism $h : \Sigma^* \mapsto \Sigma^*$ with no eigenvalue of absolute value 1, and some other condition, it is possible to decide if $h^\omega(0)$ is additive $k$-th power-free.
Avoiding additive squares over $\mathbb{Z}^2$

Theorem (Rao, R. 2015)

For any morphism $h : \Sigma^* \mapsto \Sigma^*$ with no eigenvalue of absolute value 1, and some other condition, it is possible to decide if $h^\omega(0)$ is additive $k$-th power-free.

Let $h_6 : \begin{cases} 
  a \rightarrow ace \\
  b \rightarrow adf \\
  c \rightarrow bdf \\
  d \rightarrow bdc \\
  e \rightarrow afe \\
  f \rightarrow bce 
\end{cases}$ and $\varphi : \begin{cases} 
  a \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
  b \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
  c \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
  d \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
  e \rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
  f \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} 
\end{cases}$

Theorem (Rao, R. 2015)

$\varphi(h_6^\omega(a))$, does not contain any additive square.
Avoiding P.A. in walks over $\mathbb{Z}^3$

$$
\begin{pmatrix}
0 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

No two consecutive factors of same size and same sum
Avoiding P.A. in walks over $\mathbb{Z}^3$

$$
\begin{array}{cccccccccccccccc}
0 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

No two consecutive factors of same size and same sum

$$
\begin{array}{cccccccccccccccc}
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No two consecutive factors of same sum
Avoiding P.A. in walks over $\mathbb{Z}^3$

\[
\begin{pmatrix}
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0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

No two consecutive factors of same size and same sum

\[
\begin{pmatrix}
0 & 2 & 4 & 5 & 5 & 6 & 6 & 7 & 9 & 9 & 10 & 11 & 11 & 13 & 14 \\
0 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{pmatrix}
\]

No arithmetic progression of order 3.
**Theorem (Cassaigne, Currie, Schaeffer & Shallit 2011)**

*Additive cubes are avoidable on \{0, 1, 3, 4\}.*

On other “small” alphabets?

---

```
\begin{align*}
  h_4 &: \begin{cases}
    0 &\rightarrow 001 \\
    1 &\rightarrow 041 \\
    2 &\rightarrow 41 \\
    4 &\rightarrow 442 \\
  \end{cases} \\
  h'_4 &: \begin{cases}
    0 &\rightarrow 03 \\
    2 &\rightarrow 53 \\
    3 &\rightarrow 2 \\
    5 &\rightarrow 02 \\
  \end{cases}
\end{align*}
```
Additive cubes over \( \mathbb{Z} \)

**Theorem (Cassaigne, Currie, Schaeffer & Shallit 2011)**

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Additive cubes over $\mathbb{Z}$

**Theorem (Cassaigne, Currie, Schaeffer & Shallit 2011)**

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2 &\rightarrow 53 \\
3 &\rightarrow 2 \\
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\end{cases}
\end{align*}
\]

**Theorem (Rao, R. 2015)**

$h_4^\omega(0)$ and $h'_4(0)$ do not contain any additive cube.
Additive cubes over $\mathbb{Z}$

**Theorem (Cassaigne, Currie, Schaeffer & Shallit 2011)**

*Additive cubes are avoidable on $\{0, 1, 3, 4\}$.*

On other “small” alphabets?

$$h_4 : \begin{cases} 
0 \rightarrow 001 \\
1 \rightarrow 041 \\
2 \rightarrow 41 \\
4 \rightarrow 442 
\end{cases} \quad h'_4 : \begin{cases} 
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2 \rightarrow 53 \\
3 \rightarrow 2 \\
5 \rightarrow 02. 
\end{cases}$$

**Theorem (Rao, R. 2015)**

$h_4^\omega(0)$ and $h'_4^\omega(0)$ do not contain any additive cube.

**Question**

Are additive cubes avoidable on $\{0, 1, 2, 3\}$?
Open questions

On existence of words:

- Can we avoid long abelian cubes over two letters?
- What is the minimal $k$ such that one can avoid abelian squares of period at least $k$ over three letters? ($2 \leq k \leq 6$)
- Are additive cubes avoidable over $\{0, 1, 2, 3\}$, $\{0, 1, 4\}$ or $\{0, 2, 5\}$?
Open questions

On existence of words:

- Can we avoid long abelian cubes over two letters?
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On morphisms:

- Find a simpler morphic word on 4 letter which avoids abelian-squares?
Open questions

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On morphisms:
- Find a simpler morphic word on 4 letter which avoids abelian-squares?

On algorithms:
- How to decide if we allow eigenvalues of norm 1?
Thanks!