Face no more (?)

(An exhaustive search of the convex pentagons which tiles the plane)

Michaël Rao

École Normale Supérieure de Lyon / CNRS, LIP, MC2 team

CRM April 5, 2017

Sketch

We present an exhaustive search of all convex pentagons which tile the plane

Sketch

We present an exhaustive search of all convex pentagons which tile the plane

Let $\mathcal{P}$ be a convex pentagon which tiles the plane.

- Part 1: There exist a tiling by $\mathcal{P}$ such that each vertex type has positive density
- The set of vertex type (i.e. conditions implied by angles) must be "good"
- Part 2: There are only 371 good sets to consider
- Part 3: For each good set : we do an exhaustive search
- Result: we found only the 15 known families (and some special cases).

Part 1: positive density tiling and good sets

Let $\mathcal{P}$ be a convex pentagon

- the vertices are $s_1, \dots s_5$, in clockwise order
- the angles are respectively $\alpha_1 \times \pi, \dots, \alpha_5 \times \pi$

$$\forall 1 \leq i \leq 5, \quad 0 < \alpha_i < 1$$

$$\sum_{i=1}^{5} \alpha_i = (1,1,1,1,1) \cdot \alpha = 3$$

Part 1: positive density tiling and good sets

Let $\mathcal{P}$ be a convex pentagon

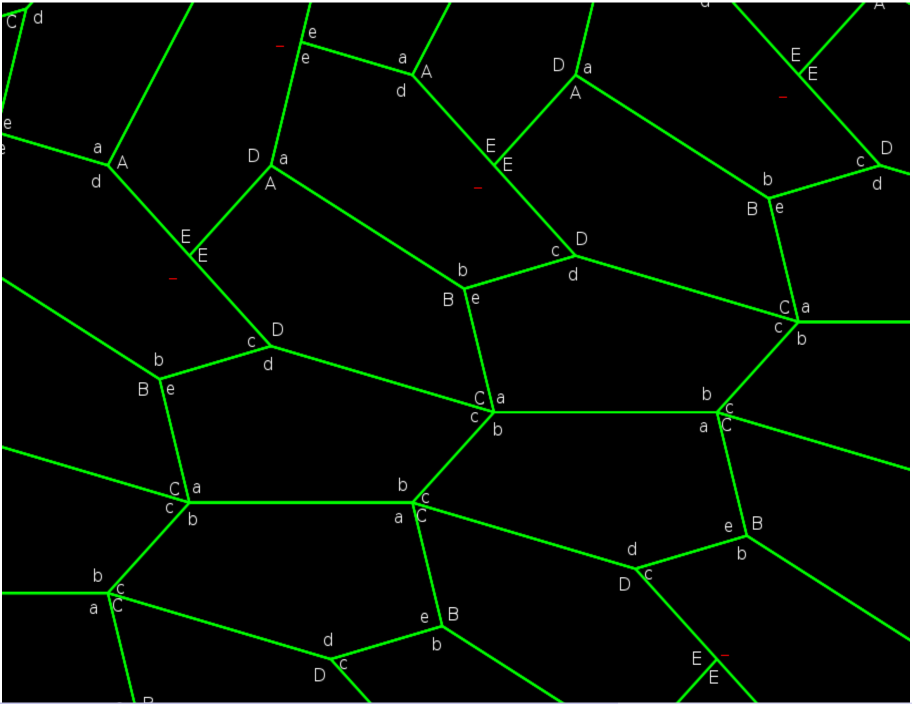
- the vertices are $s_1, \ldots s_5$, in clockwise order
- the angles are respectively $\alpha_1 \times \pi, \ldots, \alpha_5 \times \pi$

$$\forall 1 \leq i \leq 5, \quad 0 < \alpha_i < 1$$

$$\sum_{i=1}^{5} \alpha_i = (1,1,1,1,1) \cdot \alpha = 3$$

Let $\mathcal{T}$ be tiling of the plane by $\mathcal{P}$ (we allow rotation/transtation/mirror)

(Note: no hypothesis on periodicity / transitivity)


C d
e
e a A D a E E
d A
e
a A D a E E D c
d A b d
B e
E D
E c
b d
B e C a
D c c b
b d
B e
C a b c c a C b
C a b c c a C b e B
d b
b c D c a C
e B
d b
c D E E

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

The *corrected vector type* of s, denoted $V^c(s)$, is:
$V(s)$ if s is full, or $2 \times V(s)$ if s is half.

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

The *corrected vector type* of s, denoted $V^c(s)$, is:
$V(s)$ if s is full, or $2 \times V(s)$ if s is half.

For every vertex s, $V^c(s) \cdot \alpha = 2$

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

The *corrected vector type* of s, denoted $V^c(s)$, is:
$V(s)$ if s is full, or $2 \times V(s)$ if s is half.

For every vertex s, $V^c(s) \cdot \alpha = 2$

$\mathcal{W}$: set of vector types of vertices in $\mathcal{T}$
$\mathcal{W}^c$: set of corrected vector types

Vector type

Let s be a vertex of $\mathcal{T}$ (that is a vertex of one pentagon in $\mathcal{T}$)

The *vector type* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

The *corrected vector type* of s, denoted $V^c(s)$, is:
$V(s)$ if s is full, or $2 \times V(s)$ if s is half.

For every vertex s, $V^c(s) \cdot \alpha = 2$

$\mathcal{W}$: set of vector types of vertices in $\mathcal{T}$
$\mathcal{W}^c$: set of corrected vector types
Note: $\mathcal{W}$ and $\mathcal{W}^c$ are finite

An informal toy problem

Suppose that the density of each corrected vector type is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V^c(s) = v}{\text{number of tiles}}$$

An informal toy problem

Suppose that the density of each corrected vector type is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V^c(s) = v}{\text{number of tiles}}$$

$\mathcal{W}^c$ is the following:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

An informal toy problem

Suppose that the density of each corrected vector type is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V^c(s) = v}{\text{number of tiles}}$$

$\mathcal{W}^c$ is the following:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

What are the densities of v_i's ?

An informal toy problem

Suppose that the density of each corrected vector type is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V^c(s) = v}{\text{number of tiles}}$$

$\mathcal{W}^c$ is the following:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

What are the densities of v_i's ? Respectively 1, $\frac{1}{2}$, 0

An informal toy problem

Suppose that the density of each corrected vector type is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V^c(s) = v}{\text{number of tiles}}$$

$\mathcal{W}^c$ is the following:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

What are the densities of v_i's ? Respectively 1, $\frac{1}{2}$, 0

Is v_3 mandatory to tile with $\mathcal{P}$?

Positive density tilings

Let $o \in \mathbb{R}^2$ and $r \geq 0$

$B(o,r)$ is the disk of radius r and center o

$\mathcal{T}_{o,r}$ be the set of tiles $P \in \mathcal{T}$ such that $P \cap B(o,r) \neq \emptyset$

$\mathcal{IV}(\mathcal{T}')$ set of "interior vertices" of a sub-tiling $\mathcal{T}'$

For a vector type v, let :

$$f_{o,r}(v) = \frac{|\{s \in \mathcal{IV}(\mathcal{T}_{o,r}) : V(s) = v\}|}{|\mathcal{T}_{o,r}|}$$

Positive density tilings

Let $o \in \mathbb{R}^2$ and $r \geq 0$

$B(o, r)$ is the disk of radius r and center o

$\mathcal{T}_{o,r}$ be the set of tiles $P \in \mathcal{T}$ such that $P \cap B(o, r) \neq \emptyset$

$\mathcal{IV}(\mathcal{T}')$ set of "interior vertices" of a sub-tiling $\mathcal{T}'$

For a vector type v, let :

$$f_{o,r}(v) = \frac{|\{s \in \mathcal{IV}(\mathcal{T}_{o,r}) : V(s) = v\}|}{|\mathcal{T}_{o,r}|}$$

Definition (Positive density tiling)

$\mathcal{T}$ has *positive density* if for every $v \in \mathcal{W}$ and $o \in \mathbb{R}^2$ we have $\liminf_{r\to\infty} f_{o,r}(v) > 0$

Positive density tilings

Let $o \in \mathbb{R}^2$ and $r \geq 0$

$B(o, r)$ is the disk of radius r and center o

$\mathcal{T}_{o,r}$ be the set of tiles $P \in \mathcal{T}$ such that $P \cap B(o, r) \neq \emptyset$

$\mathcal{IV}(\mathcal{T}')$ set of "interior vertices" of a sub-tiling $\mathcal{T}'$

For a corrected vector type v, let :

$$f''_{o,r}(v) = \frac{|\{s \in \mathcal{IV}(\mathcal{T}_{o,r}) : V^c(s) = v\}|}{|\mathcal{T}_{o,r}|}$$

Definition (Positive density tiling)

$\mathcal{T}$ has *positive density* if for every $v \in \mathcal{W}$ and $o \in \mathbb{R}^2$ we have $\liminf_{r \to \infty} f_{o,r}(v) > 0$

imply also that $\forall v \in \mathcal{W}^c, \liminf_{r \to \infty} f''_{o,r}(v) > 0$

Tiling imply positive density tiling

Lemma

If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.

Tiling imply positive density tiling

Lemma

If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.

- Otherwise, suppose $v \in \mathcal{W}$ with $\liminf_{r\to\infty} f_{o,r}(v) = 0$

Tiling imply positive density tiling

Lemma

If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.

- Otherwise, suppose $v \in \mathcal{W}$ with $\liminf_{r \to \infty} f_{o,r}(v) = 0$
- There are sub-tilings of an arbitrarily large disk without a vertex of vector type v

Tiling imply positive density tiling

Lemma

If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.

- Otherwise, suppose $v \in \mathcal{W}$ with $\liminf_{r\to\infty} f_{o,r}(v) = 0$
- There are sub-tilings of an arbitrarily large disk without a vertex of vector type v
(take a grid of girth $x \in \mathbb{R}$: if there is v in every cell, then one have a contradiction)

Tiling imply positive density tiling

Lemma

If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.

- Otherwise, suppose $v \in \mathcal{W}$ with $\liminf_{r\to\infty} f_{o,r}(v) = 0$
- There are sub-tilings of an arbitrarily large disk without a vertex of vector type v
  (take a grid of girth $x \in \mathbb{R}$: if there is v in every cell, then one have a contradiction)
- By compactness, one can construct a tiling which does not have a vertex of vector type v
- (warning: be careful with "fracture lines")

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

In the toy example:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

In the toy example:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is not good, with $u = (1, 0, -1, 0, 0)$

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

In the toy example:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is not good, with $u = (1, 0, -1, 0, 0)$
- $\{v_1, v_2\}$ is

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

In the toy example:
$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is not good, with $u = (1, 0, -1, 0, 0)$
- $\{v_1, v_2\}$ is good since $2 \times u \cdot v_1 + u \cdot v_2 = 0$

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

$|\mathcal{T}_{o,r}|$ and $|\mathcal{IV}(\mathcal{T}_{o,r})|$ in $\Theta(r^2)$

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^5 u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

$|\mathcal{T}_{o,r}|$ and $|\mathcal{IV}(\mathcal{T}_{o,r})|$ in $\Theta(r^2)$ things on the "border" in $O(r)$

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

$|\mathcal{T}_{o,r}|$ and $|\mathcal{IV}(\mathcal{T}_{o,r})|$ in $\Theta(r^2)$ things on the "border" in $O(r)$

We count the number of angles in $\mathcal{T}_{o,r}$.

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} v \times f'_{o,r}(v) = (1,1,1,1,1)$$

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^5 u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

$|\mathcal{T}_{o,r}|$ and $|\mathcal{IV}(\mathcal{T}_{o,r})|$ in $\Theta(r^2)$ things on the "border" in $O(r)$

We count the number of angles in $\mathcal{T}_{o,r}$.

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} v \times f'_{o,r}(v) = (1,1,1,1,1)$$

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) = 0$$

Positive density imply $\mathcal{W}^c$ is good

Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^5 u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}^c$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}^c$ with $u \cdot v^+ > 0$.

$|\mathcal{T}_{o,r}|$ and $|\mathcal{IV}(\mathcal{T}_{o,r})|$ in $\Theta(r^2)$ things on the "border" in $O(r)$

We count the number of angles in $\mathcal{T}_{o,r}$.

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} v \times f'_{o,r}(v) = (1,1,1,1,1)$$

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) = 0$$

Contradiction since:

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) \geq (u \cdot v^+) \times \liminf_{r\to\infty} f'_{o,r}(v^+) > 0$$

$\mathfrak{P}_{\mathcal{X}}$

Definition ($\mathfrak{P}_{\mathcal{X}}$)

Given a subset $\mathcal{X} \subseteq \mathbb{N}^5$, we define by $\mathfrak{P}_{\mathcal{X}}$ the subset of $\mathbb{R}^5$ such that $\alpha = (\alpha_1, \ldots \alpha_5) \in \mathfrak{P}_{\mathcal{X}}$ if and only if

- for every $i \in \{1, \ldots, 5\}$, $0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $\alpha \cdot v = 2$.

$\mathfrak{P}_{\mathcal{X}}$

Definition ($\mathfrak{P}_{\mathcal{X}}$)

Given a subset $\mathcal{X} \subseteq \mathbb{N}^5$, we define by $\mathfrak{P}_{\mathcal{X}}$ the subset of $\mathbb{R}^5$ such that $\alpha = (\alpha_1, \ldots \alpha_5) \in \mathfrak{P}_{\mathcal{X}}$ if and only if

- for every $i \in \{1, \ldots, 5\}$, $0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $\alpha \cdot v = 2$.

$\mathfrak{P}_{\mathcal{X}}$ is a convex polytope

$\mathfrak{P}_{\mathcal{X}}$

Definition ($\mathfrak{P}_{\mathcal{X}}$)

Given a subset $\mathcal{X} \subseteq \mathbb{N}^5$, we define by $\mathfrak{P}_{\mathcal{X}}$ the subset of $\mathbb{R}^5$ such that $\alpha = (\alpha_1, \dots \alpha_5) \in \mathfrak{P}_{\mathcal{X}}$ if and only if

- for every $i \in \{1, \dots, 5\}$, $0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $\alpha \cdot v = 2$.

$\mathfrak{P}_{\mathcal{X}}$ is a convex polytope

In a tiling by a convex pentagon: $\alpha \in \mathfrak{P}_{\mathcal{W}^c}$, thus $\mathfrak{P}_{\mathcal{W}^c} \cap]0, 1[^5 \neq \emptyset$

$\mathfrak{P}_{\mathcal{X}}$

Definition ($\mathfrak{P}_{\mathcal{X}}$)

Given a subset $\mathcal{X} \subseteq \mathbb{N}^5$, we define by $\mathfrak{P}_{\mathcal{X}}$ the subset of $\mathbb{R}^5$ such that $\alpha = (\alpha_1, \dots \alpha_5) \in \mathfrak{P}_{\mathcal{X}}$ if and only if

- for every $i \in \{1, \dots, 5\}$, $0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $\alpha \cdot v = 2$.

$\mathfrak{P}_{\mathcal{X}}$ is a convex polytope

In a tiling by a convex pentagon: $\alpha \in \mathfrak{P}_{\mathcal{W}^c}$, thus $\mathfrak{P}_{\mathcal{W}^c} \cap]0,1[^5 \neq \emptyset$

What are the good sets $\mathcal{X}$ such that $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$?

Compatible vectors, maximal set

span($\mathcal{X}$): vectors which are linear combination of vectors in $\mathcal{X}$

$$\text{Compat}(\mathcal{X}) = \{w \in \mathbb{N}^5 : \\ (w,2) \in \text{span}(\{(1,1,1,1,1,3)\} \cup \{(v,2) : v \in \mathcal{X}\})\}$$

Compatible vectors, maximal set

span($\mathcal{X}$): vectors which are linear combination of vectors in $\mathcal{X}$

$$\mathrm{Compat}(\mathcal{X}) = \{w \in \mathbb{N}^5 : (w, 2) \in \mathrm{span}(\{(1,1,1,1,1,3)\} \cup \{(v,2) : v \in \mathcal{X}\})\}$$

- One have $\mathfrak{P}_{\mathcal{X}} = \mathfrak{P}_{\mathrm{Compat}(\mathcal{X})}$.

Compatible vectors, maximal set

span($\mathcal{X}$): vectors which are linear combination of vectors in $\mathcal{X}$

$$\mathrm{Compat}(\mathcal{X}) = \{w \in \mathbb{N}^5 : (w, 2) \in \mathrm{span}(\{(1,1,1,1,1,3)\} \cup \{(v,2) : v \in \mathcal{X}\})\}$$

- One have $\mathfrak{P}_{\mathcal{X}} = \mathfrak{P}_{\mathrm{Compat}(\mathcal{X})}$.
- If $\mathcal{X}$ is good then Compat($\mathcal{X}$) is also good

Compatible vectors, maximal set

span($\mathcal{X}$): vectors which are linear combination of vectors in $\mathcal{X}$

$$\mathrm{Compat}(\mathcal{X}) = \{w \in \mathbb{N}^5 : (w, 2) \in \mathrm{span}(\{(1,1,1,1,1,3)\} \cup \{(v, 2) : v \in \mathcal{X}\})\}$$

- One have $\mathfrak{P}_{\mathcal{X}} = \mathfrak{P}_{\mathrm{Compat}(\mathcal{X})}$.
- If $\mathcal{X}$ is good then Compat($\mathcal{X}$) is also good

A set $\mathcal{X}$ is *maximal* if $\mathcal{X} = \mathrm{Compat}(\mathcal{X})$.
W.l.o.g., we can work on maximal good sets

What are the maximal good sets $\mathcal{X}$ such that $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$?

Part 2: Computation of all good sets

In this part, the order of the angles is not important.
(i.e. W.l.o.g. $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$)

Part 2: Computation of all good sets

In this part, the order of the angles is not important.
(i.e. W.l.o.g. $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$)

$\mathfrak{P}_{\mathcal{X}}^{\geq}$

Let $\mathfrak{P}_{\mathcal{X}}^{\geq}$ be the set of vectors $(\alpha_1, \ldots \alpha_5)$ such that

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$,
- $\sum_i \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $v \cdot \alpha = 2$.

Part 2: Computation of all good sets

In this part, the order of the angles is not important.
(i.e. W.l.o.g. $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$)

$\mathfrak{P}_{\mathcal{X}}^{\geq}$

Let $\mathfrak{P}_{\mathcal{X}}^{\geq}$ be the set of vectors $(\alpha_1, \ldots \alpha_5)$ such that

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$,
- $\sum_i \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $v \cdot \alpha = 2$.

What are the good sets $\mathcal{X}$ such that $\mathfrak{P}_{\mathcal{X}}^{\geq} \cap]0, 1[^5 \neq \emptyset$?

Part 2: Computation of all good sets

In this part, the order of the angles is not important.
(i.e. W.l.o.g. $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$)

$\mathfrak{P}^{\geq}_{\mathcal{X}}$

Let $\mathfrak{P}^{\geq}_{\mathcal{X}}$ be the set of vectors $(\alpha_1, \dots \alpha_5)$ such that

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$,
- $\sum_i \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $v \cdot \alpha = 2$.

What are the good sets $\mathcal{X}$ such that $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$?

If $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap [0,1]^5$ is non empty, let $m_{\mathcal{X}} \in [0,1]^5$ be such that $(m_{\mathcal{X}})_i = \min\{\alpha_i : \alpha \in \mathfrak{P}^{\geq}_{\mathcal{X}}\}$.

Part 2: Computation of all good sets

In this part, the order of the angles is not important.
(i.e. W.l.o.g. $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$)

$\mathfrak{P}_{\mathcal{X}}^{\geq}$

Let $\mathfrak{P}_{\mathcal{X}}^{\geq}$ be the set of vectors $(\alpha_1, \dots \alpha_5)$ such that

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$,
- $\sum_i \alpha_i = 3$ and
- for every $v \in \mathcal{X}$, $v \cdot \alpha = 2$.

What are the good sets $\mathcal{X}$ such that $\mathfrak{P}_{\mathcal{X}}^{\geq} \cap]0,1[^5 \neq \emptyset$?

If $\mathfrak{P}_{\mathcal{X}}^{\geq} \cap [0,1]^5$ is non empty, let $m_{\mathcal{X}} \in [0,1]^5$ be such that $(m_{\mathcal{X}})_i = \min\{\alpha_i : \alpha \in \mathfrak{P}_{\mathcal{X}}^{\geq}\}$.

Note: $(m_{\mathcal{X}})_1 \geq \frac{3}{5}$, $(m_{\mathcal{X}})_2 \geq \frac{1}{2}$, $(m_{\mathcal{X}})_3 \geq \frac{1}{3}$, and $(m_{\mathcal{X}})_i \geq (m_{\mathcal{X}})_{i+1}$

procedure $\textsc{Recurse}(\mathcal{X})$

  $\mathcal{X} \leftarrow \mathrm{Compat}(\mathcal{X})$

  if $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 = \emptyset$ **then return end if**

  if $\mathcal{X}$ is good **then**

    Add $\mathcal{X}$ to the list of good sets

  end if

  Let $u \in \mathbb{R}^5$ such that:

- $u \cdot (1,1,1,1,1) = 0$
- $\forall v \in \mathcal{X}$, $u \cdot v = 0$ and
- $\forall i \in \{4,5\}$, $(m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$

  $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$

  for every $w \in V \setminus \mathcal{X}$ **do**

    $\textsc{Recurse}(\mathcal{X} \cup \{w\})$

  end for

end procedure

procedure $\textsc{Recurse}(\mathcal{X})$

  $\mathcal{X} \leftarrow \mathrm{Compat}(\mathcal{X})$

  if $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 = \emptyset$ **then return end if**

  if $\mathcal{X}$ is good **then**

    Add $\mathcal{X}$ to the list of good sets

  end if

  Let $u \in \mathbb{R}^5$ such that:

- $u \cdot (1,1,1,1,1) = 0$
- $\forall v \in \mathcal{X}$, $u \cdot v = 0$ and
- $\forall i \in \{4,5\}$, $(m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$

  $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$

  for every $w \in V \setminus \mathcal{X}$ **do**

    $\textsc{Recurse}(\mathcal{X} \cup \{w\})$

  end for

end procedure

$\textsc{Recurse}(\mathcal{X})$ computes all max good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{Y}} \cap]0,1[^5 \neq \emptyset$

1: **procedure** $\textsc{Recurse}(\mathcal{X})$
2: $\mathcal{X} \leftarrow \mathrm{Compat}(\mathcal{X})$
3: **if** $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 = \emptyset$ **then return end if**
4: **if** $\mathcal{X}$ is good **then**
5: Add $\mathcal{X}$ to the list of good sets
6: **end if**
7: Let $u \in \mathbb{R}^5$ such that:
- $u \cdot (1,1,1,1,1) = 0$
- $\forall v \in \mathcal{X}$, $u \cdot v = 0$ and
- $\forall i \in \{4,5\}$, $(m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$

8: $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$
9: **for** every $w \in V \setminus \mathcal{X}$ **do**
10: $\textsc{Recurse}(\mathcal{X} \cup \{w\})$
11: **end for**
12: **end procedure**

$\textsc{Recurse}(\mathcal{X})$ computes all max good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{Y}} \cap]0,1[^5 \neq \emptyset$

Line 7: such a u always exists

```
1: procedure RECURSE(𝒳)
2:     𝒳 ← Compat(𝒳)
3:     if 𝔓^≥_𝒳 ∩ ]0,1[^5 = ∅ then return end if
4:     if 𝒳 is good then
5:         Add 𝒳 to the list of good sets
6:     end if
7:     Let u ∈ ℝ^5 such that:
         • u · (1,1,1,1,1) = 0
         • ∀v ∈ 𝒳, u · v = 0 and
         • ∀i ∈ {4,5}, (m_𝒳)_i = 0 ⇒ u_i < 0
8:     V ← {v ∈ ℕ^5 : v · u ≥ 0 and v · m_𝒳 ≤ 2}
9:     for every w ∈ V \ 𝒳 do
10:        RECURSE(𝒳 ∪ {w})
11:    end for
12: end procedure
```

$\textsc{Recurse}(\mathcal{X})$ computes all max good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{Y}} \cap]0,1[^5 \neq \emptyset$

Line 7: such a u always exists   Line 8: V is finite

1: **procedure** $\textsc{Recurse}(\mathcal{X})$

2: $\quad \mathcal{X} \leftarrow \mathsf{Compat}(\mathcal{X})$

3: $\quad$ **if** $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 = \emptyset$ **then return end if**

4: $\quad$ **if** $\mathcal{X}$ is good **then**

5: $\qquad$ Add $\mathcal{X}$ to the list of good sets

6: $\quad$ **end if**

7: $\quad$ Let $u \in \mathbb{R}^5$ such that:

- $u \cdot (1,1,1,1,1) = 0$
- $\forall v \in \mathcal{X}$, $u \cdot v = 0$ and
- $\forall i \in \{4,5\}$, $(m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$

8: $\quad V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$

9: $\quad$ **for** every $w \in V \setminus \mathcal{X}$ **do**

10: $\qquad \textsc{Recurse}(\mathcal{X} \cup \{w\})$

11: $\quad$ **end for**

12: **end procedure**

$\textsc{Recurse}(\mathcal{X})$ computes all max good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{Y}} \cap]0,1[^5 \neq \emptyset$

Line 7: such a u always exists $\qquad$ Line 8: V is finite

$\textsc{Recurse}$ always terminates: finitely many good sets with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$.

Good sets: results

We execute $\textsc{Recurse}(\emptyset)$ and it finds 193 non-empty sets.

Good sets: results

We execute $\textsc{Recurse}(\emptyset)$ and it finds 193 non-empty sets.

- 193 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$

Good sets: results

We execute $\textsc{Recurse}(\emptyset)$ and it finds 193 non-empty sets.

- 193 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$
- Take all permutations: 3495 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$

Good sets: results

We execute $\textsc{Recurse}(\emptyset)$ and it finds 193 non-empty sets.

- 193 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$
- Take all permutations: 3495 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$
- Keep only one represents for each class up to rotation/mirror, one have the 371 sets.

Good sets: results

371 sets to consider:

- 2 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 3
- 26 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 2
- 92 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 1
- 251 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 0

Good sets: results

371 sets to consider:

- 2 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 3
- 26 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 2
- 92 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 1
- 251 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 0

- 90 of "Type 1" (that is $\alpha_i + \alpha_{i+1} = 1$)

Part 3 : Testing a family corresponding to a good set

For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

Part 3 : Testing a family corresponding to a good set

For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

We chose maximal good set $\mathcal{X}$. Let $\mathfrak{P} = \mathfrak{P}_{\mathcal{X}}$.
We do an exhaustive search of all tilings, allowing only corrected vector types in $\mathcal{X}$.

Part 3 : Testing a family corresponding to a good set

For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

We chose maximal good set $\mathcal{X}$. Let $\mathfrak{P} = \mathfrak{P}_{\mathcal{X}}$.
We do an exhaustive search of all tilings, allowing only corrected vector types in $\mathcal{X}$.

We backtrack if the conditions (angles and lengths) imply

Part 3 : Testing a family corresponding to a good set

For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

We chose maximal good set $\mathcal{X}$. Let $\mathfrak{P} = \mathfrak{P}_{\mathcal{X}}$.
We do an exhaustive search of all tilings, allowing only corrected vector types in $\mathcal{X}$.

We backtrack if the conditions (angles and lengths) imply

- we are in a known case: known family (Types 1 to 15 in Table 1), or a special case of a known family (Types 16 to 19)

Part 3 : Testing a family corresponding to a good set

For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

We chose maximal good set $\mathcal{X}$. Let $\mathfrak{P} = \mathfrak{P}_{\mathcal{X}}$.
We do an exhaustive search of all tilings, allowing only corrected vector types in $\mathcal{X}$.

We backtrack if the conditions (angles and lengths) imply

- we are in a known case: known family (Types 1 to 15 in Table 1), or a special case of a known family (Types 16 to 19)
- or no convex pentagon exists with these conditions

Backtracking: general idea

The object on which we work and backtrack is a pair:

- a planar graph which represent the partial tiling ("Tiling graph")
- a set of conditions we know on the lengths of the pentagon: a linear program (LP) Q on $\ell_1 \ldots \ell_5$

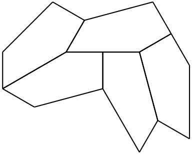
Tiling graph

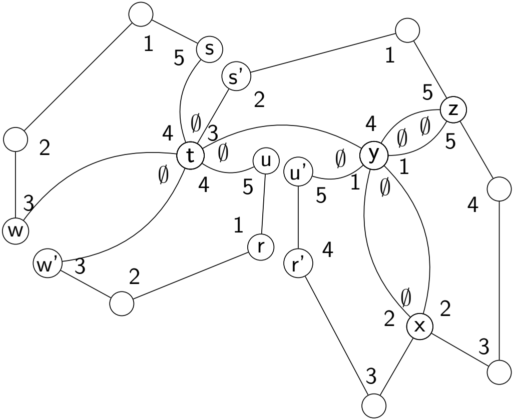
Tiling graph : planar graph with labels on angles and edges

Two types of faces: normal and special

- normal: corresponds to a pentagon in the tiling. The degree is 5, and the angles are marked from 1 to 5 (in CW or CCW)
- special: corresponds to frontier between tiles, or an unknown area of the plane. Angles are marked with $\emptyset$, π or ?

A special face is *complete* if there no “?”





Example of a tiling graph (Type 15). Unmarked angles are labeled "?"

Invariants on the tiling graph

The (forthcoming) operations on the tiling graph keep the following conditions:

- the graph is planar
- there is at most one ? angle around a vertex
- there is exactly one non-complete special face
- if a special face is complete, it has exactly two $\emptyset$ angles

Completing vertices and staying in $\mathcal{X}$

For a vertex s, let $v_s \in \mathbb{N}^5$ s.t. there are v_i angles labeled i adjacent to s

For every vertex s:

- If there is no $v \in \mathcal{X}$ such that $v_s \leq v$, then we backtrack
- If $v_s \in \mathcal{X}$, then we *complete* the vertex s : if there is an angle ? adjacent to s, then we relabel it into $\emptyset$

Length suppositions and completing faces

A *run* on a special face is a succession of consecutive $\emptyset$ and π angles.

Length suppositions and completing faces

A *run* on a special face is a succession of consecutive $\emptyset$ and π angles.
Each run corresponds to aligned points in the tiling.

Length suppositions and completing faces

A *run* on a special face is a succession of consecutive $\emptyset$ and π angles. Each run corresponds to aligned points in the tiling.

If there is a pair of vertices (s, s') on a same run, and Q implies that s and s' are the same point in the tiling, then we merge s and s'

Length suppositions and completing faces

A *run* on a special face is a succession of consecutive $\emptyset$ and π angles. Each run corresponds to aligned points in the tiling.

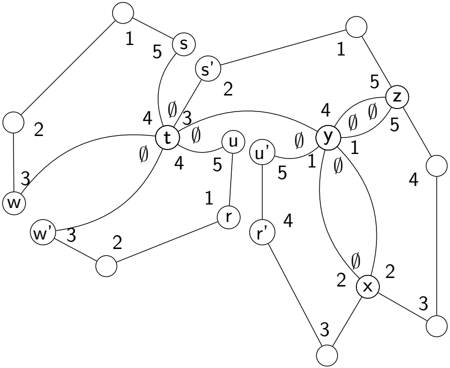
If there is a pair of vertices (s, s') on a same run, and Q implies that s and s' are the same point in the tiling, then we merge s and s'

If Q does not permit to decide among the following 3 possibilities :

- s and s' are the same point in the tiling,
- s is on the right of s',
- s is on the left of s,

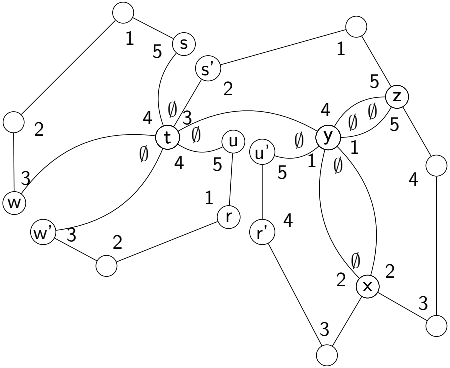
then we branch on the 3 possibilities: we add the corresponding condition in Q and recurse

Branching on length suppositions: example



Q :

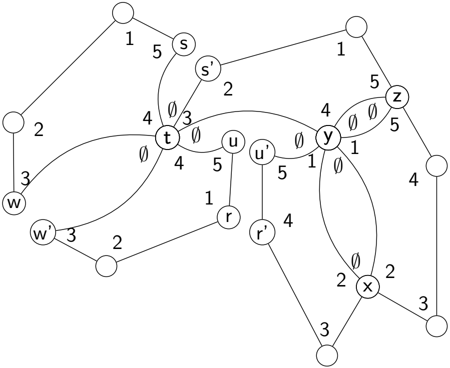
Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

(y,z) is a complete face. So we (already) have $l_4 = l_5$ in the LP Q

Branching on length suppositions: example

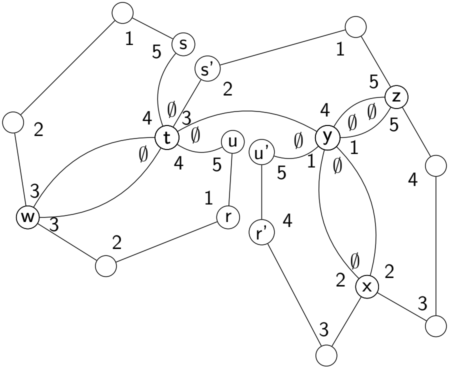


$Q : l_4 - l_5 = 0$

(y,z) is a complete face. So we (already) have $l_4 = l_5$ in the LP Q

(w,t,w') is a run. the length wt is ℓ_3, and the length tw' is ℓ_3. so we merge w and w',

Branching on length suppositions: example

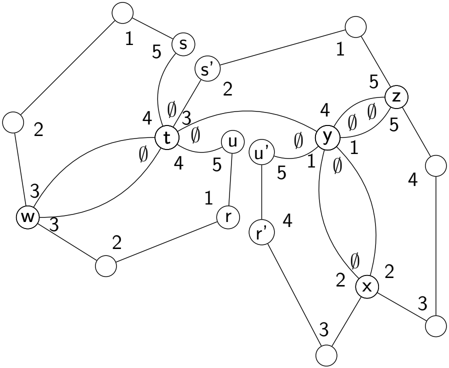


$Q : l_4 - l_5 = 0$

(y,z) is a complete face. So we (already) have $l_4 = l_5$ in the LP Q

(w,t,w') is a run. the length wt is ℓ_3, and the length tw' is ℓ_3. so we merge w and w',

Branching on length suppositions: example

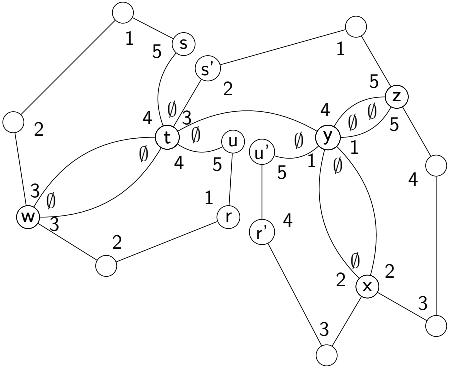


$Q : l_4 - l_5 = 0$

(y,z) is a complete face. So we (already) have $l_4 = l_5$ in the LP Q

(w,t,w') is a run. the length wt is ℓ_3, and the length tw' is ℓ_3. so we merge w and w', and mark angle (t, w, t) as $\emptyset$

Branching on length suppositions: example

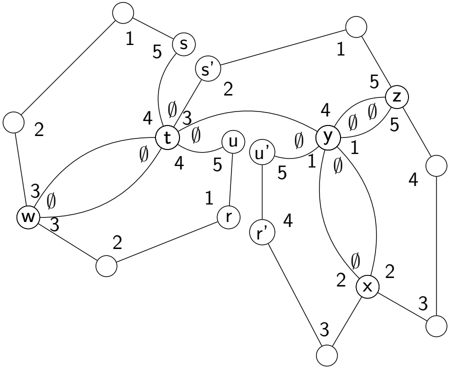


$Q : l_4 - l_5 = 0$

(y,z) is a complete face. So we (already) have $l_4 = l_5$ in the LP Q

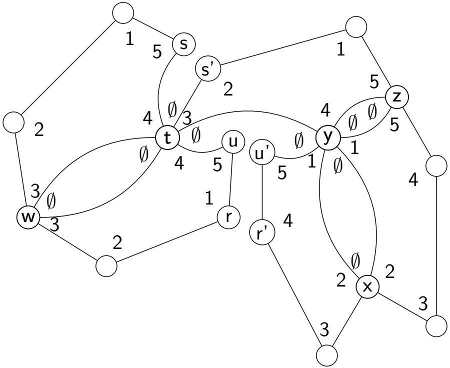
(w,t,w') is a run. the length wt is ℓ_3, and the length tw' is ℓ_3. so we merge w and w', and mark angle (t, w, t) as $\emptyset$

Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

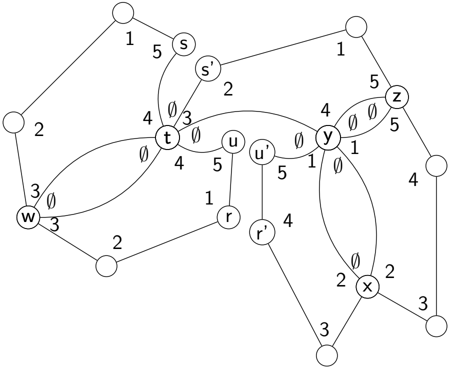
Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

(u, t, y, u') is also a run.

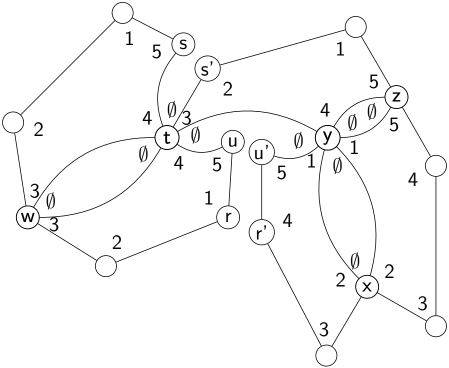
Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

(u, t, y, u') is also a run.
Is u and u' the same vertex ?

Branching on length suppositions: example

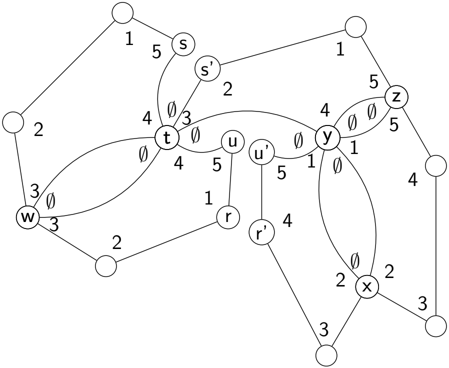


$Q : l_4 - l_5 = 0$

(u, t, y, u') is also a run.

Is u and u' the same vertex ? Is $\ell_3 = \ell_4 + \ell_5$?

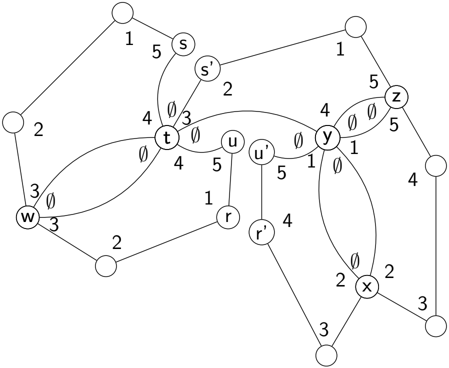
Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

(u, t, y, u') is also a run.
Is u and u' the same vertex ? Is $\ell_3 = \ell_4 + \ell_5$?
We don't know. We branch.

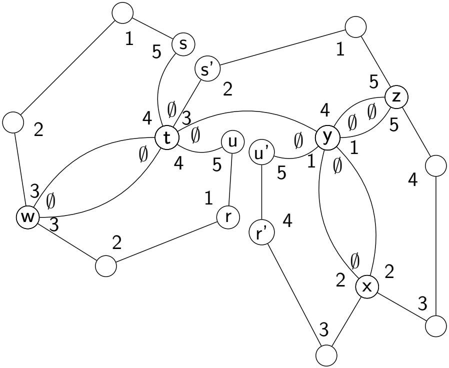
Branching on length suppositions: example



$Q : l_4 - l_5 = 0$

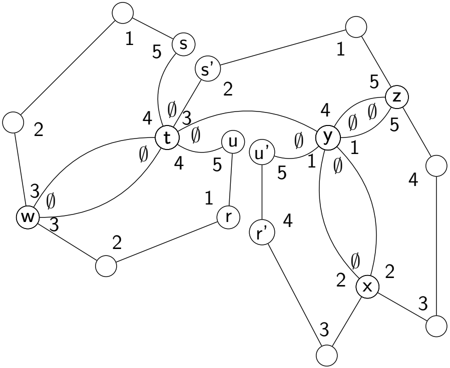
(u, t, y, u') is also a run.
Is u and u' the same vertex ? Is $\ell_3 = \ell_4 + \ell_5$?
We don't know. We branch.
first case : add $\ell_3 > \ell_4 + \ell_5$ to Q and branch
second case : add $\ell_3 = \ell_4 + \ell_5$ to Q and branch

Branching on length suppositions: example



(case 2) $Q: l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

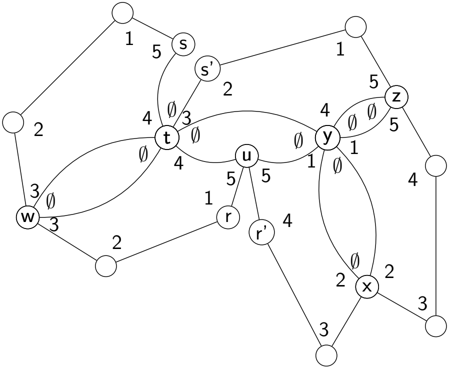
Branching on length suppositions: example



(case 2) $Q : l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

(u, t, y, u') is a run, and we know that u and u' have the same position: we merge u and u',

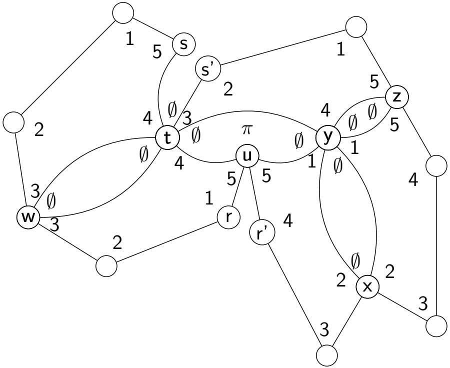
Branching on length suppositions: example



(case 2) $Q : l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

(u, t, y, u') is a run, and we know that u and u' have the same position: we merge u and u',

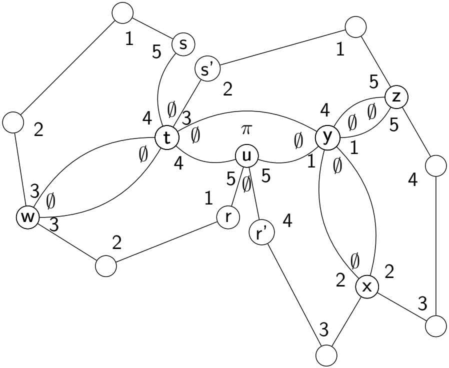
Branching on length suppositions: example



(case 2) $Q : l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

(u, t, y, u') is a run, and we know that u and u' have the same position: we merge u and u', and the angle (t, u, y) is labeled π.

Branching on length suppositions: example

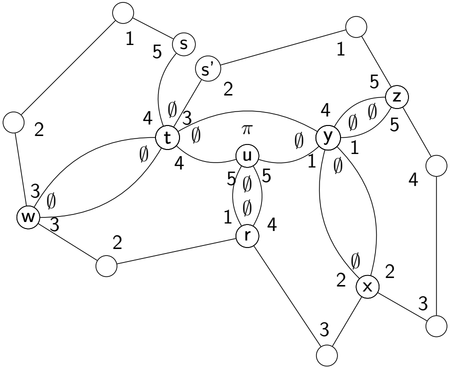


(case 2) $Q : l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

(u, t, y, u') is a run, and we know that u and u' have the same position: we merge u and u', and the angle (t, u, y) is labeled π.

u is now complete: the angle r, u, r' is labeled $\emptyset$

Branching on length suppositions: example



(case 2) $Q : l_4 - l_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0$

(u, t, y, u') is a run, and we know that u and u' have the same position: we merge u and u', and the angle (t, u, y) is labeled π.

u is now complete: the angle r, u, r' is labeled $\emptyset$

in the run (r, u, r'), r and r' have the same position: we merge...

Branching on a new tile

If we are not in any case of completing or branching on length supposition, then we add a new normal face to the tiling graph.

Branching on a new tile

If we are not in any case of completing or branching on length supposition, then we add a new normal face to the tiling graph.

We take a non-complete vertex w in the graph. We known that, if the tiling graph corresponds to a sub-tiling $\mathcal{T}'$ of a tiling $\mathcal{T}$ by $\mathcal{P}$, there is a tile $P \in \mathcal{T} \setminus \mathcal{T}'$ such that w is a vertex of P, and P shares a line segment with $\mathcal{T}'$.

Then we branch on on all theses possibilities of face addition.

(warning : be careful with “half” vertices)

Existence of the pentagon

Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$.
One have:

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \tag{1}$$

Existence of the pentagon

Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$.
One have:

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \tag{1}$$

We backtrack if there is no convex pentagon exists with the properties, that is if the following condition is not fulfilled:

$$\exists \ell \in \mathfrak{Q} \cap]0,1[^5, \exists \alpha \in \mathfrak{P} \cap]0,1[^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \tag{2}$$

Existence of the pentagon

Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$.
One have:

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \qquad (1)$$

We backtrack if there is no convex pentagon exists with the properties, that is if the following condition is not fulfilled:

$$\exists \ell \in \mathfrak{Q} \cap]0,1[^5, \exists \alpha \in \mathfrak{P} \cap]0,1[^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \qquad (2)$$

If $\dim(\mathfrak{P}) = 0$ then $\alpha \in \mathbb{Q}^5$, and easy to decide: we compute on $\mathbb{Q}[\cos(\pi/q)]$ for a $q \in \mathbb{N}$.

Existence of the pentagon

Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$.
One have:

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \tag{1}$$

We backtrack if there is no convex pentagon exists with the properties, that is if the following condition is not fulfilled:

$$\exists \ell \in \mathfrak{Q} \cap]0,1[^5, \exists \alpha \in \mathfrak{P} \cap]0,1[^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \tag{2}$$

If $\dim(\mathfrak{P}) = 0$ then $\alpha \in \mathbb{Q}^5$, and easy to decide: we compute on $\mathbb{Q}[\cos(\pi/q)]$ for a $q \in \mathbb{N}$.

If $\dim(\mathfrak{P}) > 0$: we backtrack if we have a certificate (computations in $\mathbb{Q}$) that there a no solution. Problem: this cannot detect "degenerate case". So we manually add some degenerate case. (Types 20 to 24 in Table 1).

Conditions for which we backtrack (Table 1)

Type 1 (i=1)	$a+b+c=2\pi$		Type 2 (i=2)	$a+b+d=2\pi$	$C=E$
Type 3 (i=31)	$3e=2\pi$ $d+2e=2\pi$ $b+2e=2\pi$	$C+E=D$ $A=B$	Type 4 (i=6)	$a+b+d=2\pi$ $2e=\pi$	$D=E$ $B=C$
Type 5 (i=4)	$3e=2\pi$ $a+b+d=2\pi$	$D=E$ $B=C$	Type 6 (i=13)	$d+2e=2\pi$ $a+c+d=2\pi$	$C=D=E$ $A=B$
Type 7 (i=17)	$d+2e=2\pi$ $a+2c=2\pi$	$A=C=D=E$	Type 8 (i=14)	$d+2e=2\pi$ $2b+c=2\pi$	$A=B=C=D$
Type 9 (i=15)	$d+2e=2\pi$ $2a+c=2\pi$	$A=B=C=D$	Type 10 (i=69)	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$A+C=D=E$
Type 11 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=C+2E$	Type 12 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A+C=B=2E$
Type 13 (i=63)	$b+2d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2B=2C$	Type 14 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=2C=2E$
Type 15 (i=303)	$c+2d=2\pi$ $2b+e=2\pi$ $2a+d=2\pi$ $2e=\pi$	$B=D=E$ $C=2B$	Type 16 (i=72) $\subset$ T10	$b+c+e=2\pi$ $2b+d=2\pi$ $a+2c=2\pi$	$2A=D=E$ $A=C$
Type 17 (i=25) $\subset$ T2	$c+2e=2\pi$ $2b+d=2\pi$	$A=B=C=D=E$	Type 18 (i=73) $\subset$ T2	$d+2e=2\pi$ $c+2e=2\pi$ $b+d+e=2\pi$	$D=E$ $A=B$
Type 19 (i=23) $\subset$ T1	$c+2e=2\pi$ $b+2d=2\pi$	$A=B=C=D$	Type 20 (i=2) degen.	$a+b+d=2\pi$	$A=C+D$ $B=E$
Type 21 (i=12) degen.	$d+2e=2\pi$ $2a+b=2\pi$	$A=B$ $C=D$	Type 22 (i=27) degen.	$c+2e=2\pi$ $a+2d=2\pi$	$A=B=C=E$
Type 23 (i=64) degen.	$2b+d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2C=2D$	Type 24 (i=69) degen.	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$2D=A+C$ $2E=A+C$

Part 3: Results

For every family, the exhaustive search is finite

That is: if a pentagon does not respect condition of Type i for a $i \in \{1, \ldots 24\}$, then it cannot tile the plane.

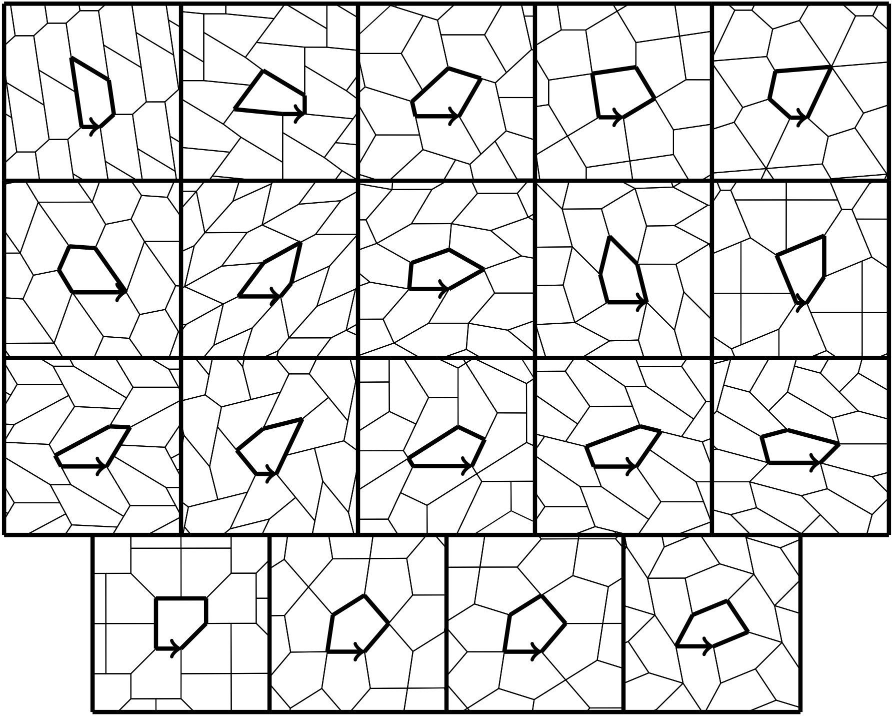
- Types 1 to 15 are the already known families
- Types 16 to 19 are special cases of known families
- Types 20 to 24 are "degenerate" ($\dim(\mathfrak{P}) > 0$): there are no convex pentagon which respects these conditions

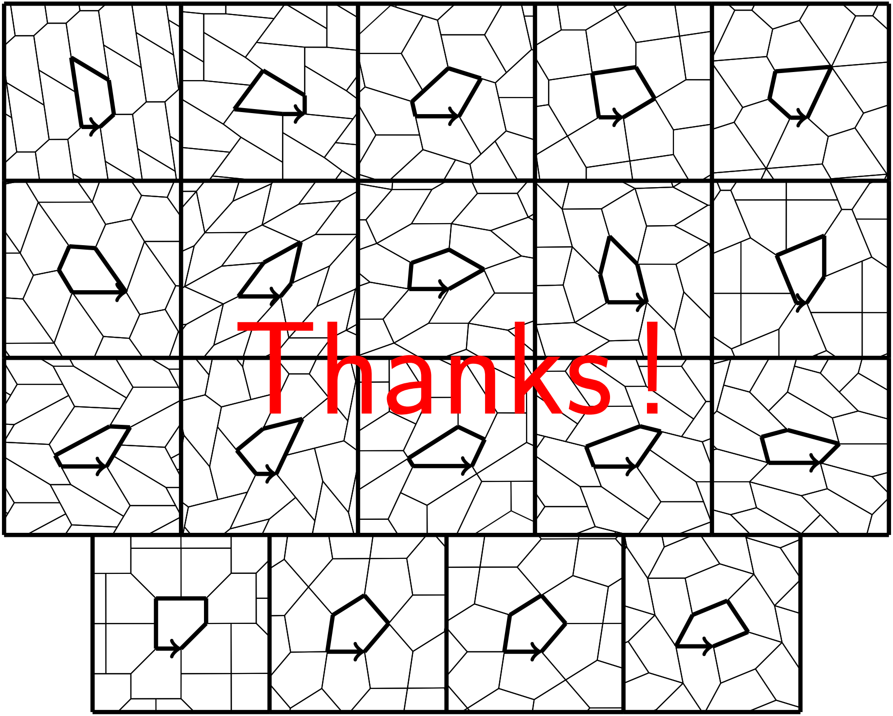
Future / TODO

- Recheck the proof
- Recheck the code ($\sim$5000 lines in C++)
- And/or reproduce the exhaustive search
- Formal proof ? (Coq or similar software)

Simplifications:

- More direct proof for the positive density ?
- Direct proof for the finiteness of goods sets ?




Thanks !