
Statistical modelling of spatial extremes using the SpatialExtremes package

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Rationale for the SpatialExtremes package

“The aim of the SpatialExtremes package is to provide tools for the areal modelling of extreme events. The modelling strategies heavily rely on the **extreme value theory** and in particular **block maxima** techniques—unless explicitly stated.”

As a consequence, most often

- the data used by the package **have to be extreme**—do not pass daily values for instance;
- **the marginal distribution family is fixed**, i.e., the generalized extreme value distribution family, but you have hands on how within this family parameters change in space;
- **the process family is fixed**, i.e., max-stable processes, but you have hands on which type of max-stable processes to use.

▷ 0. Max-stable
process

Spectral
characterization
Extremal functions

1. Data and
descriptive analysis

2. Simple max-stable
processes

3. Trends surfaces

4. General max-stable
processes

5. Conclusion

0. About the inner structure of max-stable processes

Spectral characterization

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▷ Spectral characterization Extremal functions

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$$Z(s) = \max_{i \geq 1} \zeta_i Y_i(s), \quad s \in \mathcal{X},$$

where $\{\zeta_i : i \geq 1\}$ is a Poisson point process on $(0, \infty)$ with intensity measure $d\Lambda(\zeta) = \zeta^{-2} d\zeta$ and Y_i independent copies of a (non-negative) stochastic process such that $\mathbb{E}\{Y(s)_+\} = 1$ for all $s \in \mathcal{X}$.

Spectral characterization

0. Max-stable process

▷ Spectral characterization Extremal functions

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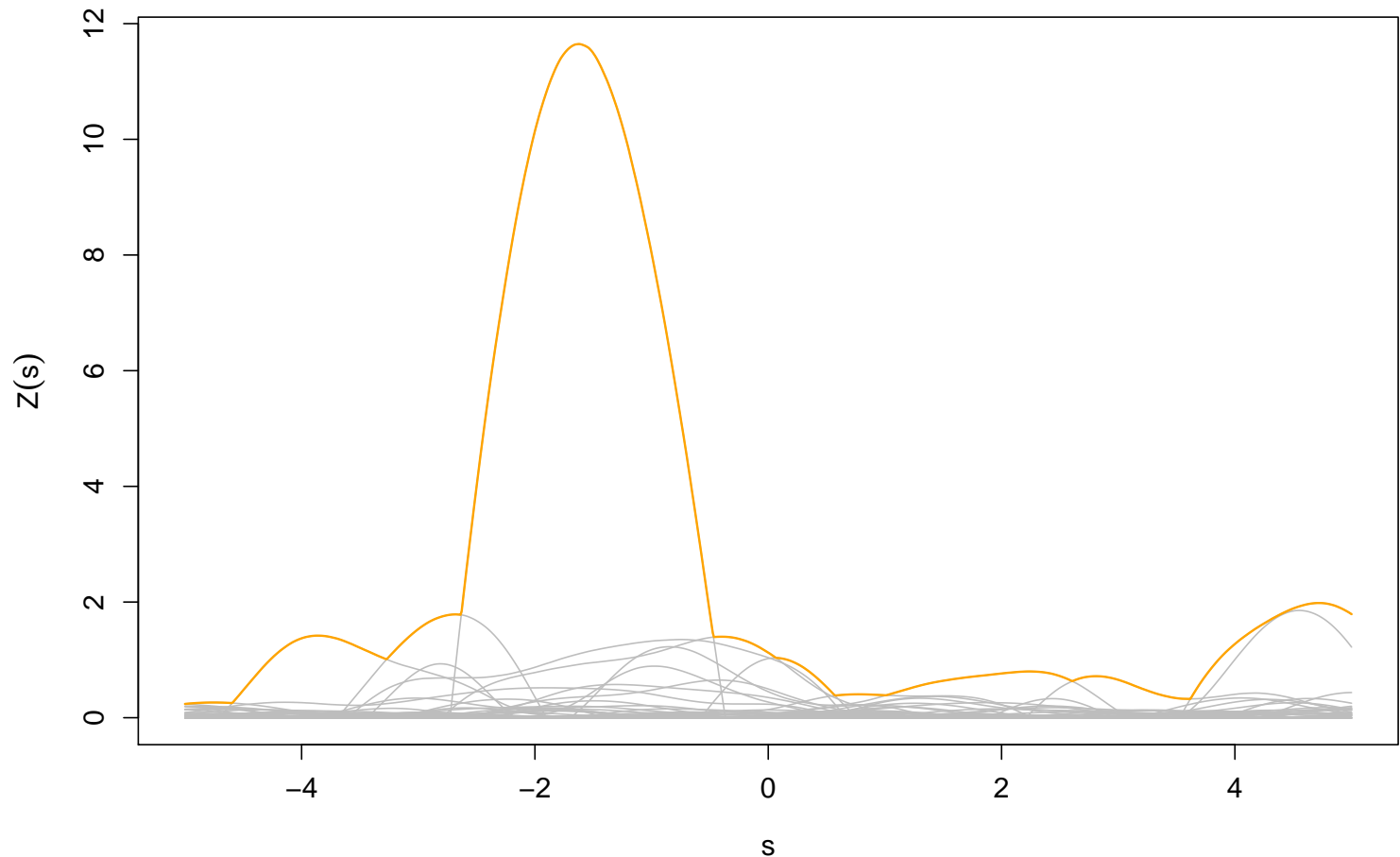
3. Trends surfaces

4. General max-stable processes

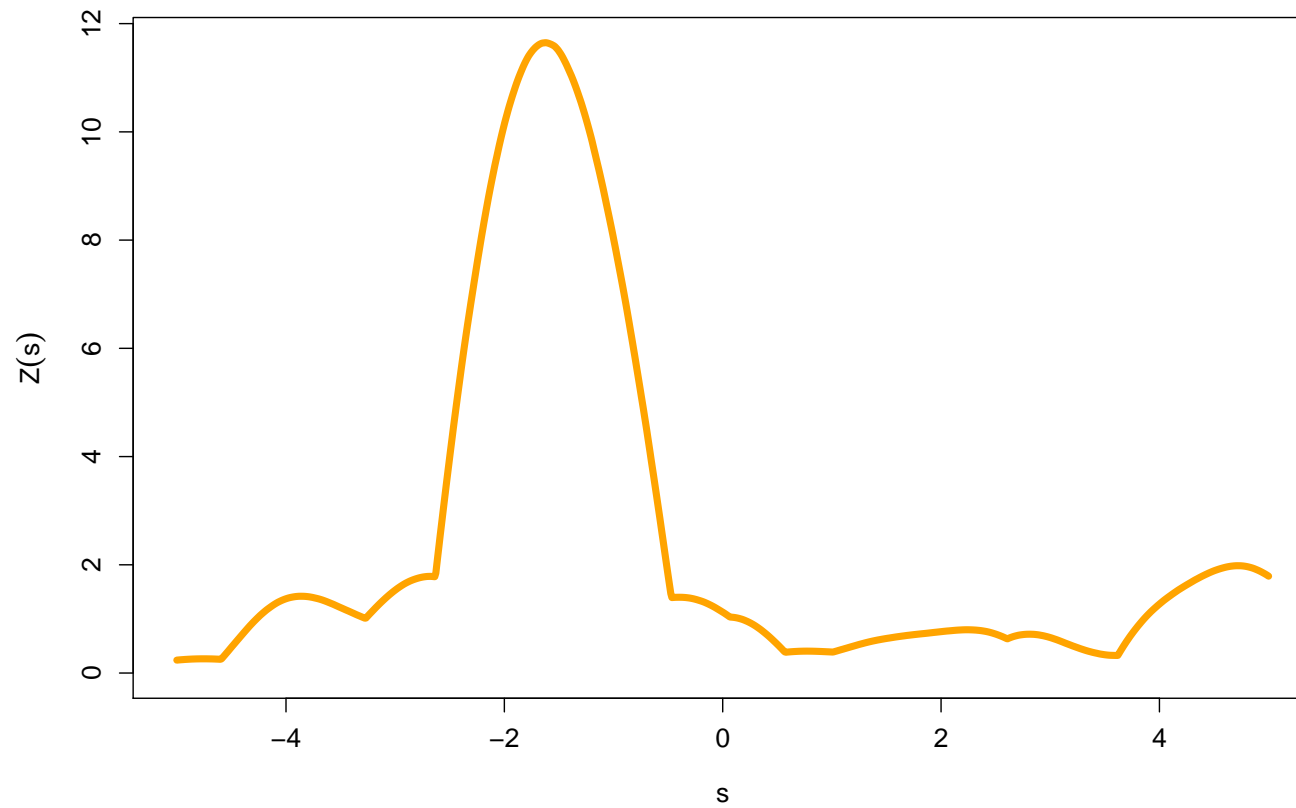
5. Conclusion

$$Z(s) = \max_{\varphi \in \Phi} \varphi(s), \quad s \in \mathcal{X},$$

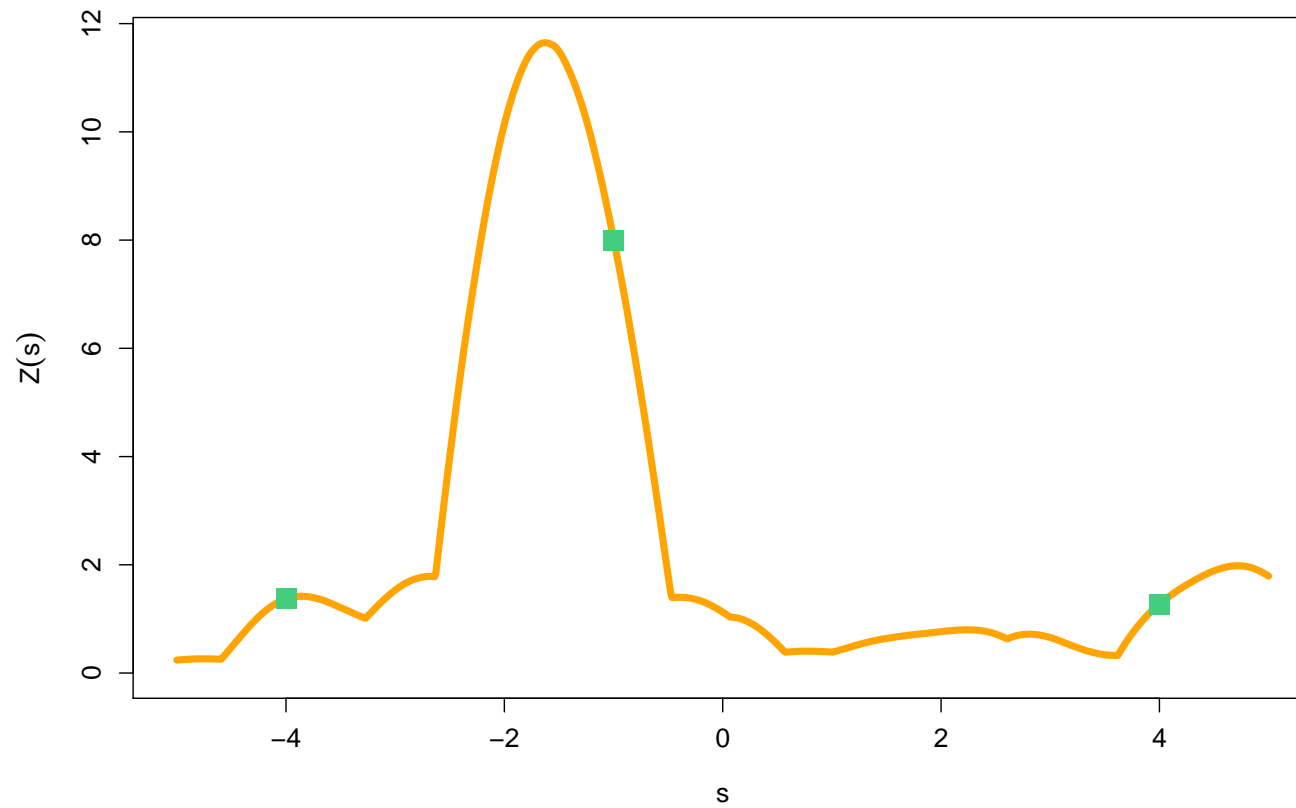
where $\Phi = \{\varphi_i : i \geq 1\}$ is a Poisson point process on \mathbb{C}_0 with an appropriate intensity measure.



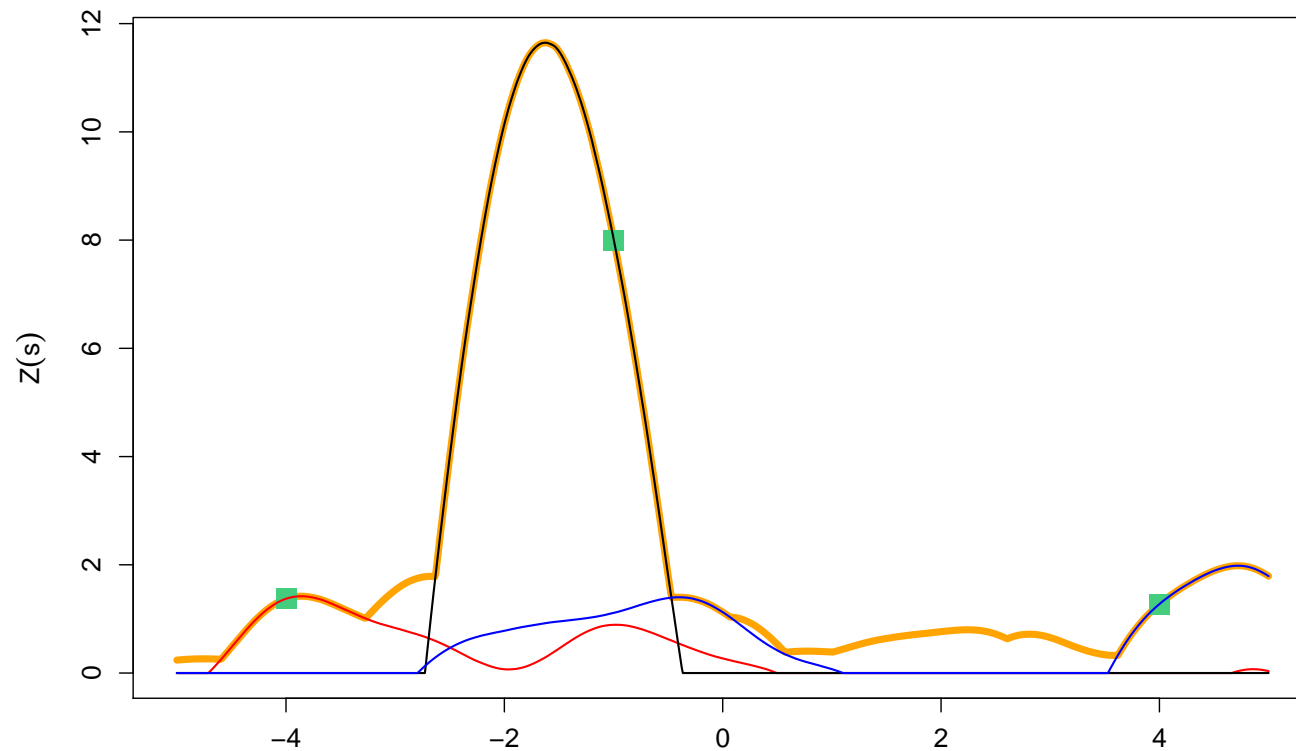
Extremal functions



Extremal functions



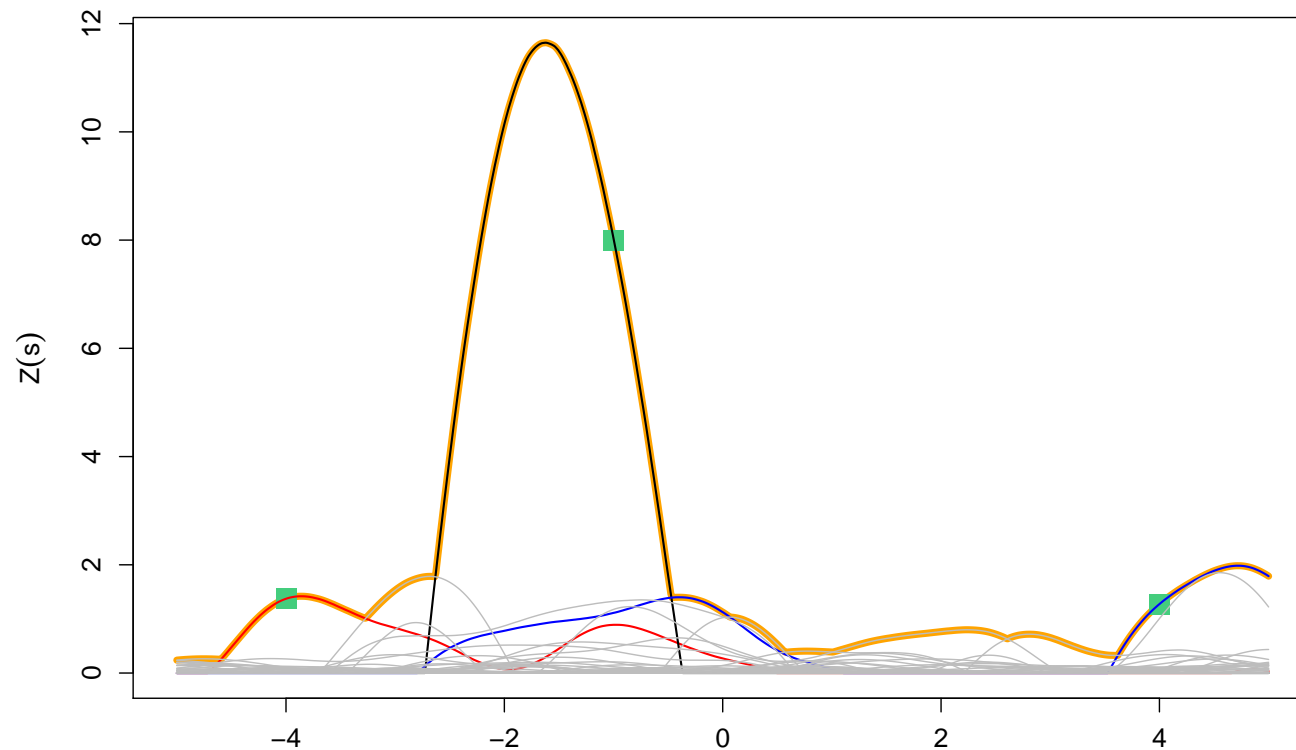
Extremal functions



□ Hidden are the random functions $\Phi^+ \stackrel{s}{=} \{\varphi_1^+, \varphi_2^+, \dots, \varphi_k^+\}$ of Φ such that

$$\varphi_j^+(s_j) = Z(s_j), \quad j = 1, \dots, k, \quad \text{(extremal functions),}$$

Extremal functions



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$$\varphi_j^+(s_j) = Z(s_j), \quad j = 1, \dots, k, \quad \text{(extremal functions),}$$

- and the random functions $\varphi^- \in \Phi \setminus \Phi^+$, i.e., satisfying

$$\varphi^-(s_j) < Z(s_j), \quad j = 1, \dots, k, \quad \text{(sub-extremal functions)}$$

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1. Data and descriptive analysis

Required data

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Before introducing more advanced stuffs, let's talk about data format. It is pretty simple

Observations A numeric matrix such that **each row is one realization of the spatial field**—or if you prefer one column per site;

Coordinates A numeric matrix such that **each row is the coordinates of one site**—or if you prefer the first column is for instance the longitude of all sites, the second one latitude, ...

```
> data
      Valkenburg Ijmuiden De Kooy ...
1971         278      NA    360 ...
1972         334      NA    376 ...
1973         376      NA    365 ...
1974         314      NA    304 ...
1975         278      NA    278 ...
1976         350      NA    345 ...
1977         324      NA    298 ...
1978         298      NA    329 ...
1979         252      NA    298 ...
...

> coord
      lon      lat
Valkenburg 4.419 52.165
Ijmuiden    4.575 52.463
De Kooy     4.785 52.924
Schiphol    4.774 52.301
Vlieland    4.942 53.255
Berkhout    4.979 52.644
Hoorn       5.346 53.393
De Bilt     5.177 52.101
...
```

Additional covariates

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In addition to the storage of observations and coordinates, you might want to use additional covariates. The latter can be of two types

Spatial A numeric matrix such that **each column corresponds to one spatial covariate** such as elevation, urban/rural, ...

Temporal A numeric matrix such that **each column corresponds to one temporal covariate** such as time, annual mean temperature, ...

> spat.cov		> temp.cov	
	alt		nao
Valkenburg	-0.2	1971	1.87
Ijmuiden	4.4	1972	1.57
De Kooy	0.5	1973	-0.20
Schiphol	-4.4	1974	-0.95
Vlieland	0.9	1975	-0.46
Berkhout	-2.5	1976	2.34
Hoorn	0.5	1977	-0.49
De Bilt	2.0	1978	0.70
...		1979	1.11
		...	

 *It is always a good idea to name your columns and rows.*

Inspecting data

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- As usual, you first have to **scrutinize your data** (weird values, encoding of missing values, check out factors, ...). But you're used to that, aren't you?
- We focus on extremes, so you may wonder
 - are my data **extremes**, i.e., block maxima?
 - is my **block size** relevant?
 - what about **seasonality**? Refine the block or use temporal covariate?
- You might want to **check that the generalized extreme value family is sensible for your data**—the evd package + a few lines of code will do the job for you (homework)
- This will generally be OK, but now you have to go a bit further by analyzing
 - the spatial dependence;
 - and the presence / absence of any spatial trends.

Spatial dependence [Cooley et al., 2006]

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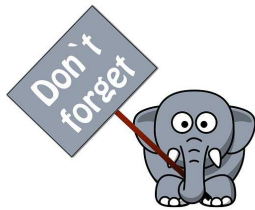
4. General max-stable processes

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- Essentially you want to check if your data exhibit any (spatial) dependence. If not why would you bother with spatial models?
- The most convenient way to do this is through the *F-madogram* and its connection with the *extremal coefficient*:

$$\nu_F(h) = \frac{1}{2} \mathbb{E}[|F\{Z(o)\} - F\{Z(h)\}|], \quad \theta(h) = \frac{1 + 2\nu_F(h)}{1 - 2\nu_F(h)}.$$

- The fmadogram function will estimate (empirically) the pairwise extremal coefficient from the *F*-madogram.



$$\theta(h) = -z \log \Pr\{Z(s) \leq z, Z(s+h) \leq z\}$$

and that $1 \leq \theta(h) \leq 2$ with complete dependence iff $\theta(h) = 1$ and independence iff $\theta(h) = 2$.

Spatial dependence (2) [Dombry et al., 2017]

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- Another recent summary measure of the spatial dependence is the **extremal concurrence probability function**
 $p: h \mapsto p(h) \in [0, 1]$ where

$$p(h) = \Pr\{\exists! \varphi \in \Phi: \varphi(s) = Z(s), \varphi(s+h) = Z(s+h)\},$$

Spatial dependence (2) [Dombry et al., 2017]

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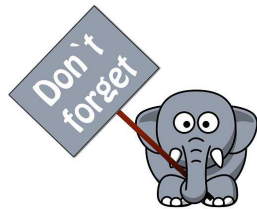
4. General max-stable processes

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$$p(h) = \Pr\{\exists! \varphi \in \Phi: \varphi(s) = Z(s), \varphi(s+h) = Z(s+h)\},$$

i.e., there is a single extremal function at position s and $s+h$.



$$p(h) = \mathbb{E}[\text{sign}\{Z(s) - \tilde{Z}(s)\} \text{sign}\{Z(s+h) - \tilde{Z}(s+h)\}] \\ = \text{Kendall's } \tau,$$

and that $0 \leq p(h) \leq 1$ with complete dependence iff $p(h) = 1$ and independence iff $p(h) = 0$.

Spatial dependence (2) [Dombry et al., 2017]

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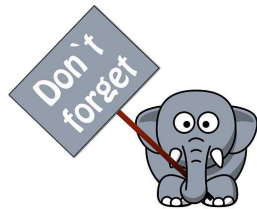
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$p: h \mapsto p(h) \in [0, 1]$ where

$$p(h) := p(s, s+h) = \Pr\{\exists! \varphi \in \Phi: \varphi(s) = Z(s), \varphi(s+h) = Z(s+h)\},$$

i.e., there is a single extremal function at position s and $s+h$.



$$p(h) = \mathbb{E}[\text{sign}\{Z(s) - \tilde{Z}(s)\} \text{sign}\{Z(s+h) - \tilde{Z}(s+h)\}] \\ = \text{Kendall's } \tau,$$

and that $0 \leq p(h) \leq 1$ with complete dependence iff $p(h) = 1$ and independence iff $p(h) = 0$.

👉 Note that here we assume a stationnary dependence structure!

The `fmadogram` function

- Run the file `fmadogram.R`. You should get the figure below.

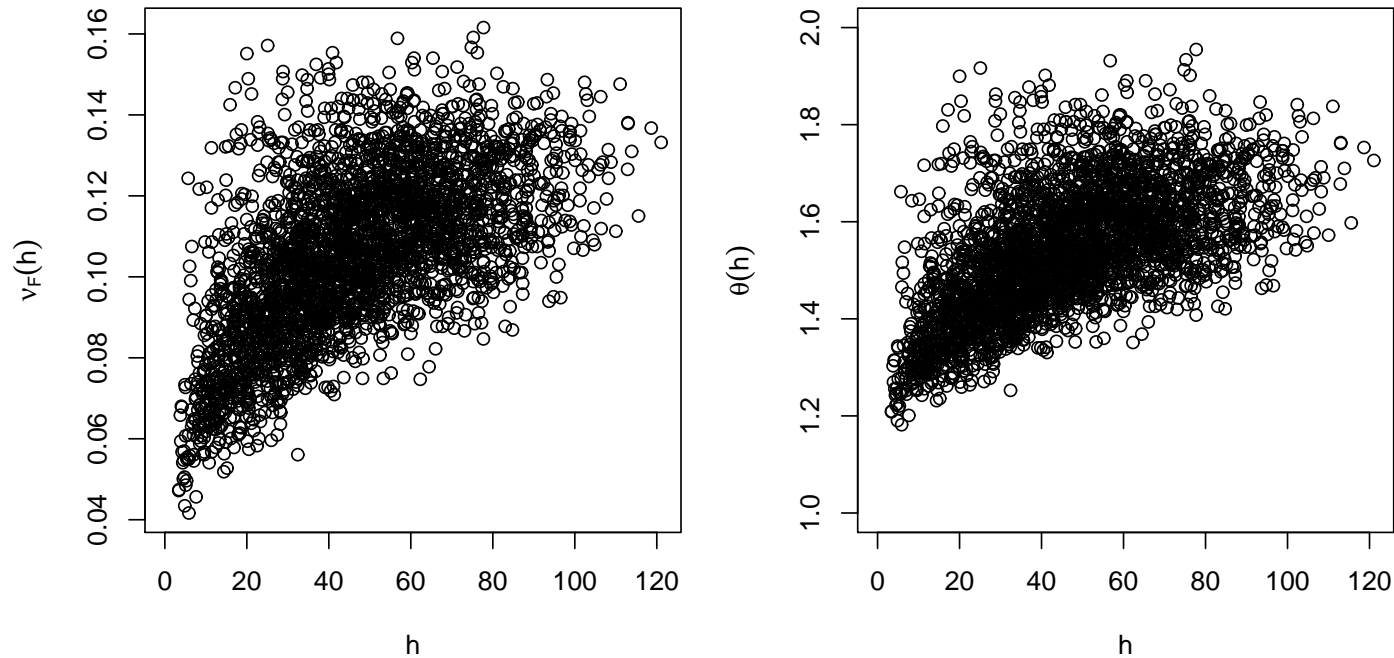


Figure 1: Use of the `fmadogram` function to assess the spatial dependence.

The `fmadogram` function

- Run the file `fmadogram.R`. You should get the figure below.
- You can also use a binned version with `n.bins = 300`...

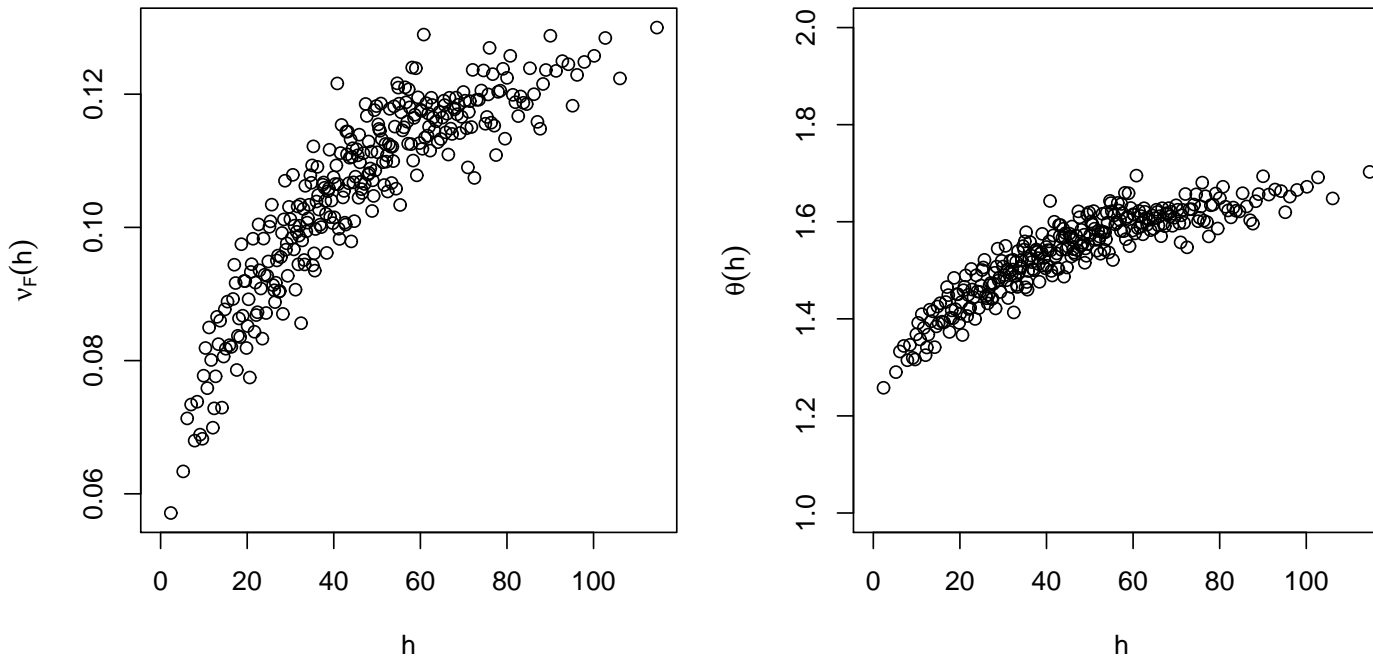


Figure 1: Use of the `fmadogram` function to assess the spatial dependence.

The concprob function

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- Run the file `concprob.R`. You should get the figure below.

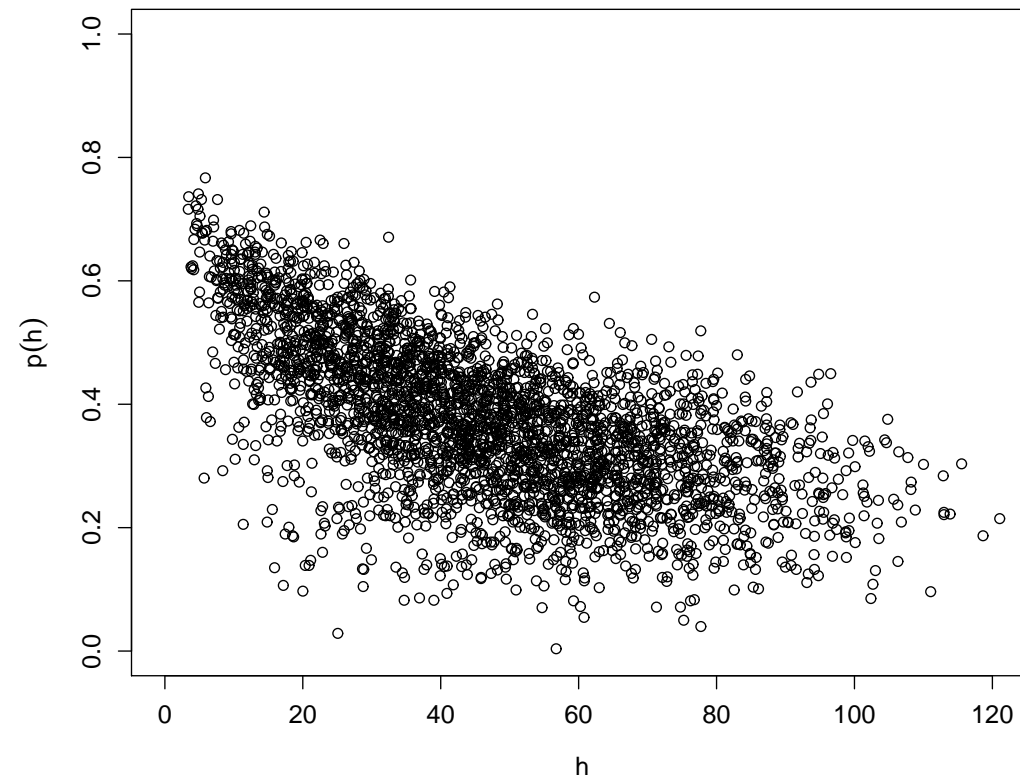


Figure 2: Use of the *concprob* function to assess the spatial dependence.

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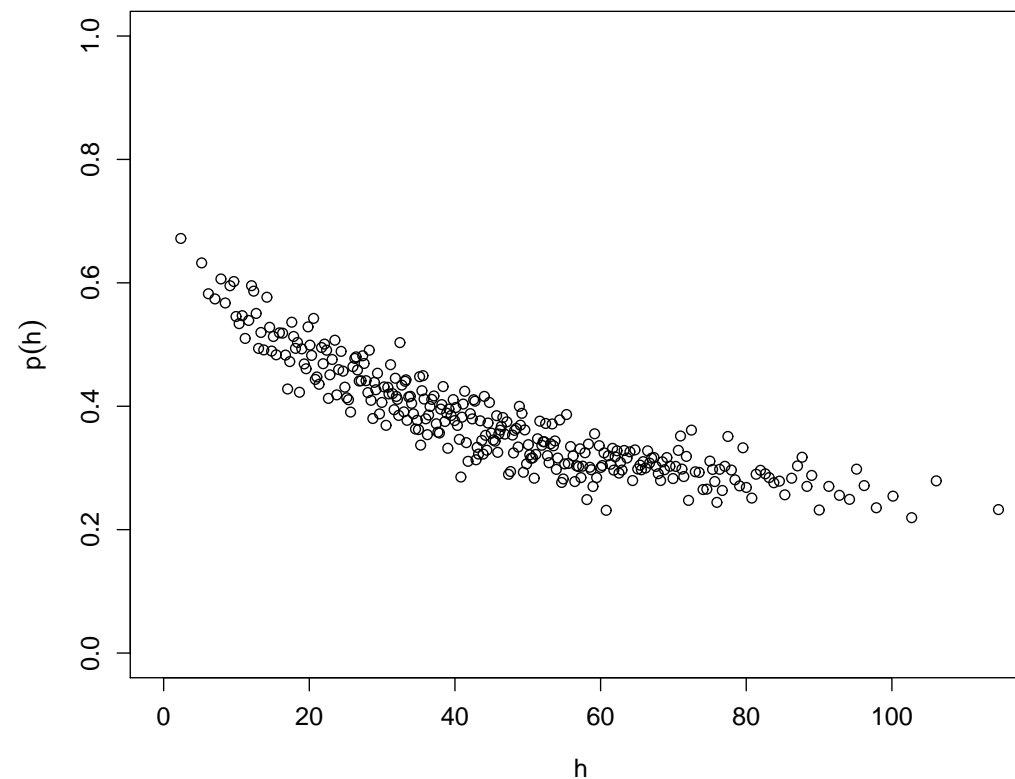
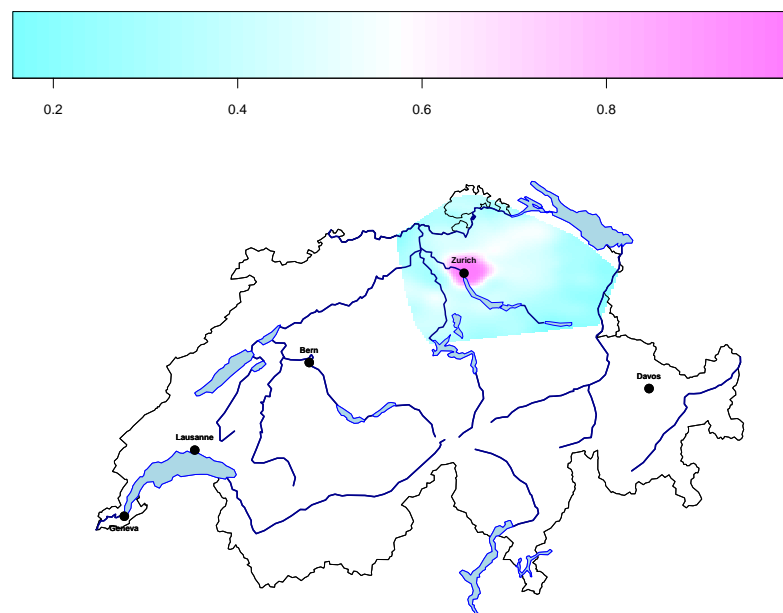


Figure 2: Use of the *concprob* function to assess the spatial dependence.

As an aside (AsAnAside.R) [Dombry et al., 2017]

- We can estimate the spatial distribution of the extremal concurrence probability w.r.t. a given weather station, e.g., plotting

$$\{(s, p(\text{Zurich}, s)) : s \in \mathcal{X}\},$$



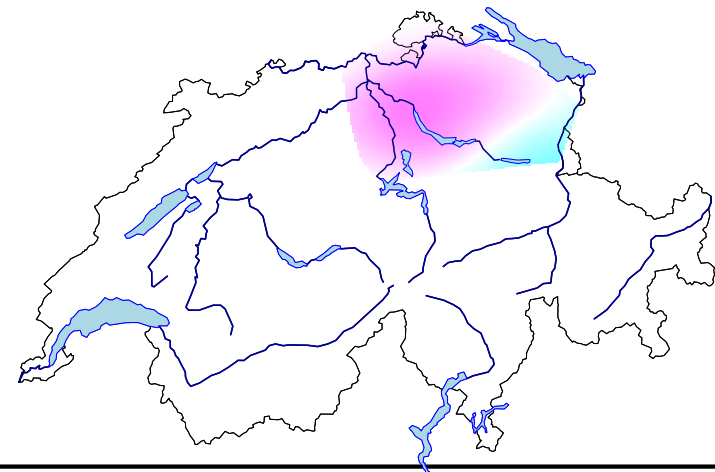
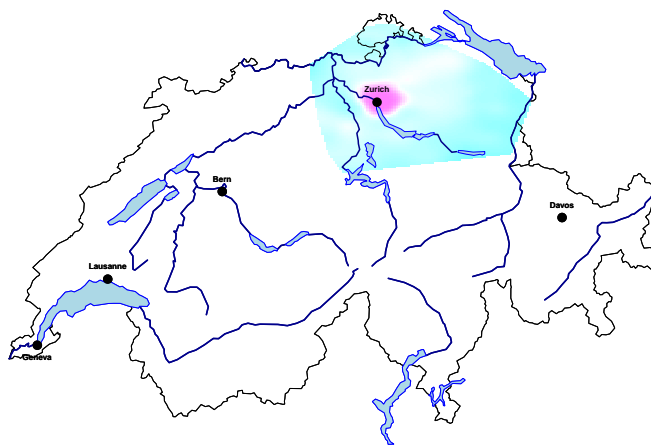
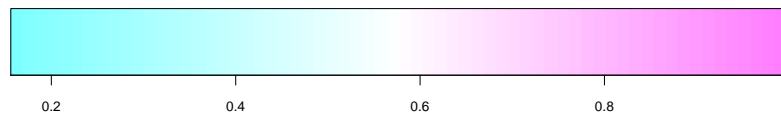
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- We can estimate the spatial distribution of the extremal concurrence probability w.r.t. a given weather station, e.g., plotting

$$\{(s, p(\text{Zurich}, s)) : s \in \mathcal{X}\},$$

- or estimate the expected area of concurrence cells

$$A(s_0) = \mathbb{E} \left\{ \int_{\mathcal{X}} 1_{\{s_0 \text{ and } s \text{ are concurrent}\}} ds \right\} = \int_{\mathcal{X}} p(s_0, s) ds$$



Spatial trends

- We can do a [symbol plot](#) see the file `SpatialTrends.R`.

Spatial trends

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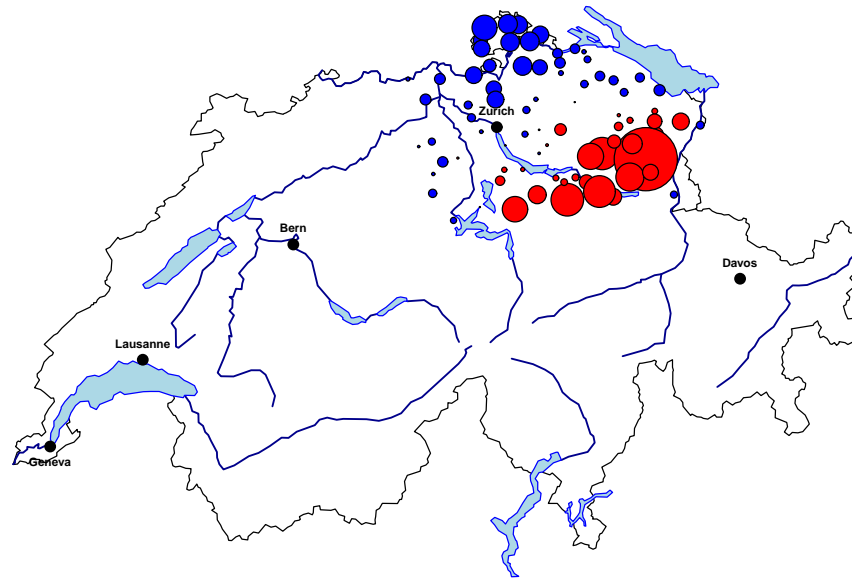


Figure 3: *Symbol plot for the swiss precipitation data.*

Spatial trends

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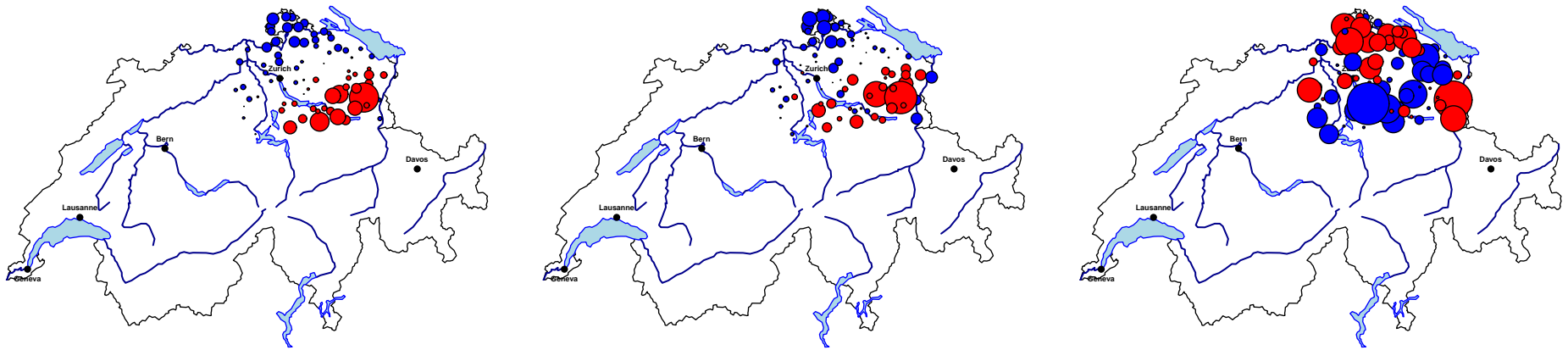


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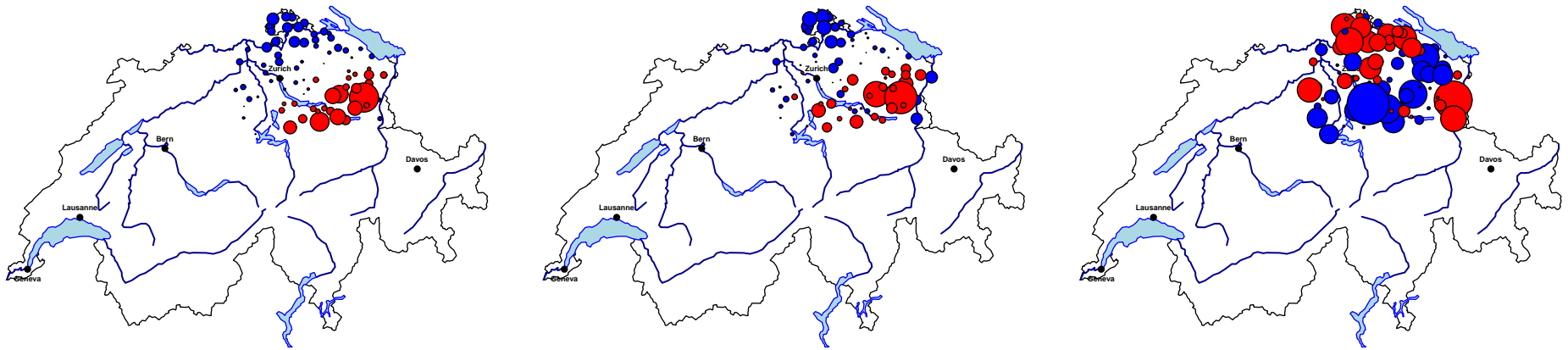



Figure 3: Symbol plot for the swiss precipitation data.

 *When exporting figures into eps/pdf, always pay attention to the aspect ratio.*

What we have learned so far (apart from using SpatialExtremes)

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- The data exhibit some **spatial dependence** and there is still some (weak) dependence at a separation lag of 100km.
- There's a clear **north-west / south-east gradient** in the intensities of rainfall storms.
- In conclusion it makes sense to use max-stable **processes** whose **marginal parameters are not constant across space**.
- More specifically, we have:
 - a clear north-west / south-east gradient for the location and scale parameters;
 - no clear pattern for the shape parameter.

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Least squares

Pairwise likelihood

Model selection

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- In this section we **focus only on the spatial dependence** and so assume that the margins are known and **unit Fréchet**—this is a standard choice in extreme value theory.
- From the spectral characterization

$$Z(s) = \max_{i \geq 1} \zeta_i Y_i(s), \quad s \in \mathcal{X},$$

we can propose several parametric models for spatial extremes. Hence by letting Y to be

Gaussian densities with random displacements we get the **Smith** process;

Gaussian we get the **Schlather** process;

Log-normal (with a drift) we get the **Brown–Resnick** process;

Gaussian but elevated to some power we get the **Extremal- t** process.

Dependence parameters

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Smith Elements of the covariance matrix appearing in the Gaussian densities;

Schlather Parameters of the correlation function;

Brown–Resnick Parameters of the semi-variogram;

Extremal- t Parameters of the correlation function and degrees of freedom.

- ☐ Since the margins are fixed, we only need to get estimates for the dependence parameters.
- ☐ How can we do that?

Least squares (leastSquares.R) [Smith, 1990]

$$\operatorname{argmin}_{\psi \in \Psi} \sum_{1 \leq i < j \leq k} \{\theta(s_j - s_i; \psi) - \hat{\theta}(s_j - s_i)\}^2,$$

where $\theta(\cdot; \psi)$ is the extremal coefficient obtained from the max-stable model with dependence parameters set to ψ and $\hat{\theta}(\cdot)$ is any empirical estimates of the extremal coefficient, e.g., F -madogram based.

```
> MO
  Estimator: Least Squares
    Model: Schlather
    Weighted: TRUE
  Objective Value: 3592.429
Covariance Family: Whittle-Matern
```

```
Estimates
Marginal Parameters:
Assuming unit Frechet.
```

```
Dependence Parameters:
range  smooth
54.3239 0.4026
```

```
Optimization Information
Convergence: successful
Function Evaluations: 61
```

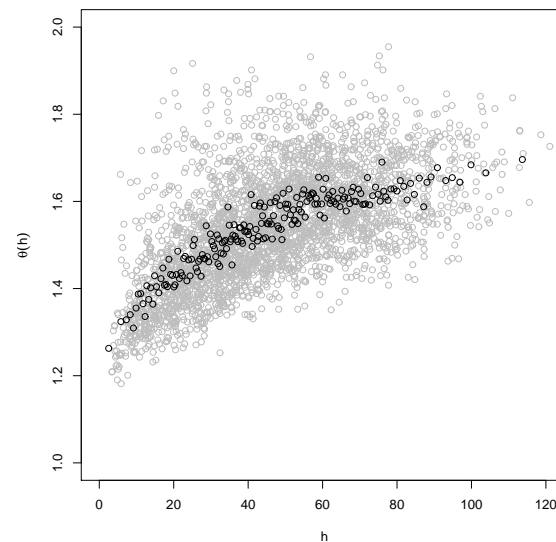


Figure 4: *Fitting simple max-stable processes from least squares.*

Least squares (leastSquares.R) [Smith, 1990]

$$\operatorname{argmin}_{\psi \in \Psi} \sum_{1 \leq i < j \leq k} \{\theta(s_j - s_i; \psi) - \hat{\theta}(s_j - s_i)\}^2,$$

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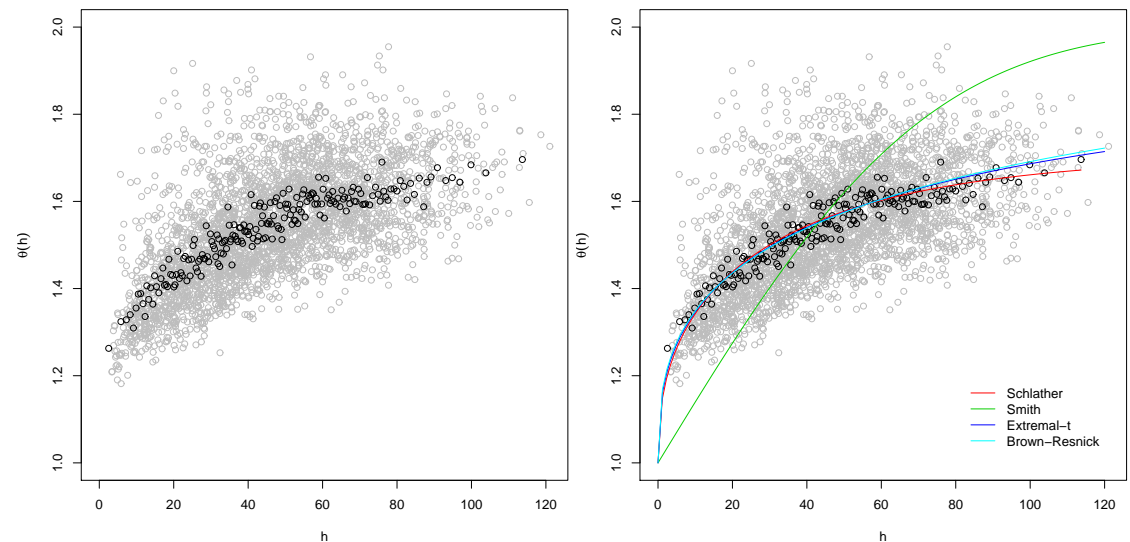


Figure 4: *Fitting simple max-stable processes from least squares.*

Pairwise likelihood (pairwiseLlik.R) [Padoan et al., 2010]

$$\arg \max_{\psi \in \Psi} \sum_{\ell=1}^n \sum_{1 \leq i < j \leq k} \log f\{z_{\ell}(s_i), z_{\ell}(s_j); \psi\},$$

where $f(\cdot, \cdot; \psi)$ is the bivariate density of the considered max-stable model.

```
Estimator: MPLE
Model: Schlather
Weighted: FALSE
Pair. Deviance: 1136863
TIC: 1137456
Covariance Family: Whittle-Matern
```

Estimates

```
Marginal Parameters:
Assuming unit Frechet.
```

```
Dependence Parameters:
range    smooth
50.1976  0.3713
```

Standard Errors

```
range    smooth
20.7085  0.0789
```

Asymptotic Variance Covariance

```
range    smooth
range    428.841018  -1.570081
smooth   -1.570081   0.006225
```

Optimization Information

```
Convergence: successful
Function Evaluations: 67
```

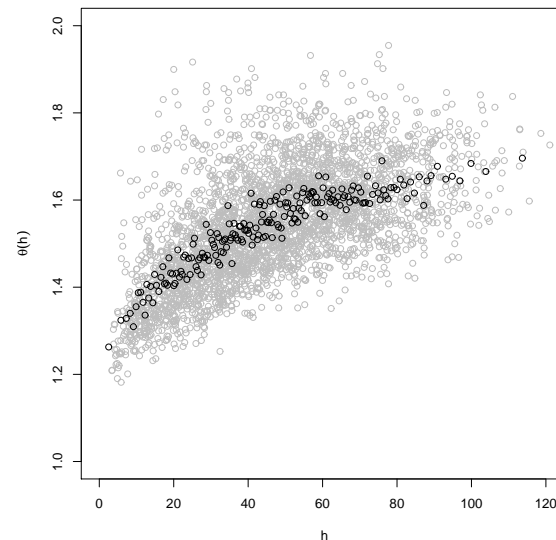


Figure 5: *Fitting simple max-stable processes maximizing pairwise likelihood.*

Pairwise likelihood (pairwiseLlik.R) [Padoan et al., 2010]

$$\arg\max_{\psi \in \Psi} \sum_{\ell=1}^n \sum_{1 \leq i < j \leq k} \log f\{z_{\ell}(s_i), z_{\ell}(s_j); \psi\},$$

where $f(\cdot, \cdot; \psi)$ is the bivariate density of the considered max-stable model.

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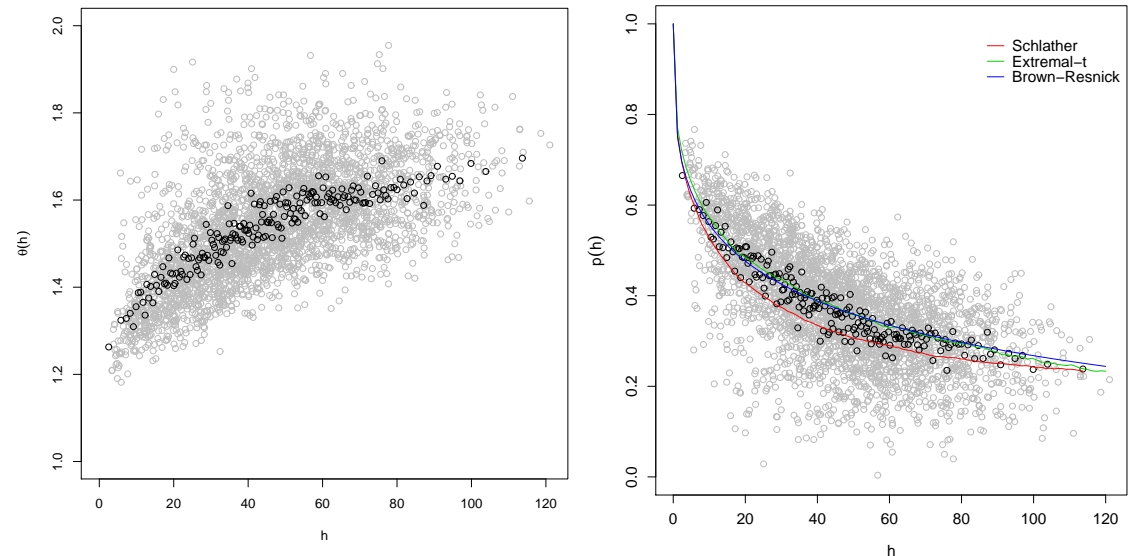


Figure 5: *Fitting simple max-stable processes maximizing pairwise likelihood.*

Model Selection [Varin and Vidoni, 2005]

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- The advantage of the pairwise likelihood estimator over the least squares one is that you can do **model selection**.
- For instance one can use the TIC, **Takeuchi Information Criterion** or sometimes known as CLIC, Composite Likelihood Information Criterion,

$$\text{TIC} = 2\ell_{\text{pairwise}}(\hat{\psi}) - 2\text{tr}\{J(\hat{\psi})H^{-1}(\hat{\psi})\},$$

$$H(\hat{\psi}) = \mathbb{E}\{\nabla^2 \ell_{\text{pairwise}}(Y; \hat{\psi})\}, J(\hat{\psi}) = \text{Var}\{\nabla \ell_{\text{pairwise}}(Y; \hat{\psi})\}.$$

- From our previous fitted models, we get

```
> TIC(M0,M1,M2)
      M1      M2      M0
1133660 1134823 1137449
```

Simulating simple max-stable processes (simulation.R) [Schlather, 2002; Dombry et al., 2016]

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- Once you have fitted a suitable model, you usually want to simulate from it.
- Simulation from max-stable models is rather complex, recall that

$$Z(s) = \max_{i \geq 1} \zeta_i Y_i(s), \quad s \in \mathcal{X}.$$

```
sim <- rmaxstab(n.obs, cbind(x, y), "twhitmat", DoF = 4,
+             nugget = 0, range = 3, smooth = 1)
> sim
```

	[,1]	[,2]	[,3]	[,4]	[,5]
[1,]	3.8048914	0.4767980	6.3613989	1.4548317	1.0433912
[2,]	1.2200332	0.6711422	0.8078701	2.0928629	0.7537061
[3,]	0.5466466	2.0498561	4.8852572	2.3497976	0.6857268
[4,]	...				

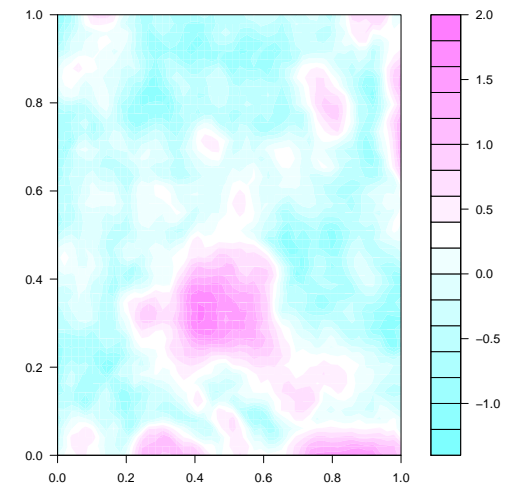


Figure 6: One simulation on a 50 x 50 grid from the extremal- t model. (log scale)

What we have learned so far (apart from using SpatialExtremes)

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

Max-stable models

Least squares

Pairwise likelihood

Model selection

Simulation

▷ Debrief #2

3. Trends surfaces

4. General max-stable processes

5. Conclusion

- The Smith model is clearly not a sensible model for our data—because of its linear behaviour near the origin;
- Schlather, Brown–Resnick and Extremal- t seems relevant;
- According to the TIC, the Extremal- t should be preferred.

0. Max-stable process

1. Data and
descriptive analysis

2. Simple max-stable
processes

▷ 3. Trends surfaces

Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable
processes

5. Conclusion

3. Trends surfaces

From generalized extreme value margins to unit Fréchet ones

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable processes

5. Conclusion

- Alright! We are able to handle the spatial dependence, but **we assume that our data have unit Fréchet margins**. This is not realistic at all!
- Fortunately, if $Y \sim \text{GEV}(\mu, \sigma, \xi)$ then

$$Z = \left(1 + \xi \frac{Y - \mu}{\sigma}\right)^{1/\xi} \sim \text{Unit Fréchet} = \text{GEV}(1, 1, 1).$$

From generalized extreme value margins to unit Fréchet ones

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable processes

5. Conclusion

- Alright! We are able to handle the spatial dependence, but **we assume that our data have unit Fréchet margins**. This is not realistic at all!
- Fortunately, if $Y \sim \text{GEV}(\mu, \sigma, \xi)$ then

$$Z = \left(1 + \xi \frac{Y - \mu}{\sigma}\right)^{1/\xi} \sim \text{Unit Fréchet} = \text{GEV}(1, 1, 1).$$

- And since we are extreme value and spatial guys

$$Z(s) = \left\{1 + \xi(s) \frac{Y(s) - \mu(s)}{\sigma(s)}\right\}^{1/\xi(s)}, \quad s \in \mathcal{X},$$

is a simple max-stable process.

- Hence we can use the maximum pairwise likelihood estimator as before—up to an **additional Jacobian term**.

Omitting the spatial dependence

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

▷ Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable processes

5. Conclusion

- With simple max-stable models, we omitted the marginal parameters.
- Here we will **omit the spatial dependence for a while** and consider locations as being **mutually independent**, i.e., use independence likelihood

$$\arg \max_{\psi \in \Psi} \sum_{i=1}^k \ell_{\text{GEV}}\{y(s_i); \psi\}.$$

- This is a kind of “*spatial GEV*” where ψ is a vector of marginal parameters.

Defining trend surfaces

0. Max-stable process

1. Data and
descriptive analysis

2. Simple max-stable
processes

3. Trends surfaces

▷ Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable
processes

5. Conclusion

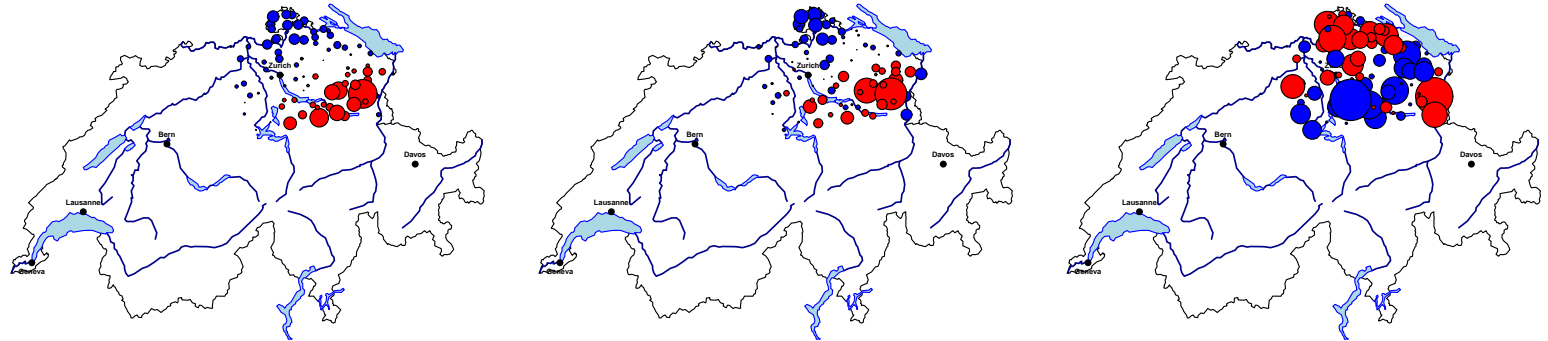


Figure 7: *Symbol plot for the swiss precipitation data.*

This suggests that

$$\mu(s) = \beta_{0,\mu} + \beta_{1,\mu}\text{lon}(s) + \beta_{2,\mu}\text{lat}(s) + \beta_{3,\mu}\text{lon}(s) \times \text{lat}(s),$$

$$\sigma(s) = \beta_{0,\sigma} + \beta_{1,\sigma}\text{lon}(s) + \beta_{2,\sigma}\text{lat}(s) + \beta_{3,\sigma}\text{lon}(s) \times \text{lat}(s),$$

$$\xi(s) = \beta_{0,\xi},$$

or equivalently with the R language

```
loc.form <- scale.form <- y ~ lon * lat; shape.form <- y ~ 1
```

Fitting the *spatial GEV* model (spatialGEV.R) [Davison et al., 2012]

0. Max-stable process

Model: Spatial GEV model
Deviance: 29303.81
TIC: 29499.38

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

▷ Spatial GEV

Prediction #1

Model selection #2

Debrief #3

4. General max-stable processes

5. Conclusion

Location Parameters:

locCoeff1	locCoeff2	locCoeff3	locCoeff4
27.132	1.846	-3.656	-1.080

Scale Parameters:

scaleCoeff1	scaleCoeff2	scaleCoeff3	scaleCoeff4
9.7850	0.7023	-1.0858	-0.5531

Shape Parameters:

shapeCoeff1
0.1572

Standard Errors

locCoeff1	locCoeff2	locCoeff3	locCoeff4	scaleCoeff1	scaleCoeff2
1.13326	0.34864	0.45216	0.38361	0.76484	0.28446

scaleCoeff3	scaleCoeff4	shapeCoeff1
0.31267	0.27566	0.05878

Asymptotic Variance Covariance

	locCoeff1	locCoeff2	locCoeff3	locCoeff4	scaleCoeff1
locCoeff1	1.2842711	0.1131400	-0.1740921	-0.0729564	0.6570988
locCoeff2	0.1131400	0.1215498	-0.0623759	0.0149596	0.0521630
locCoeff3	-0.1740921	-0.0623759	0.2044448	0.0576622	-0.1086629
locCoeff4	-0.0729564	0.0149596	0.0576622	0.1471593	-0.0346376
scaleCoeff1	0.6570988	0.0521630	-0.1086629	-0.0346376	0.5849729
...					

Optimization Information

Convergence: successful
Function Evaluations: 2135

Get predictions (predictionSpatialGEV.R)

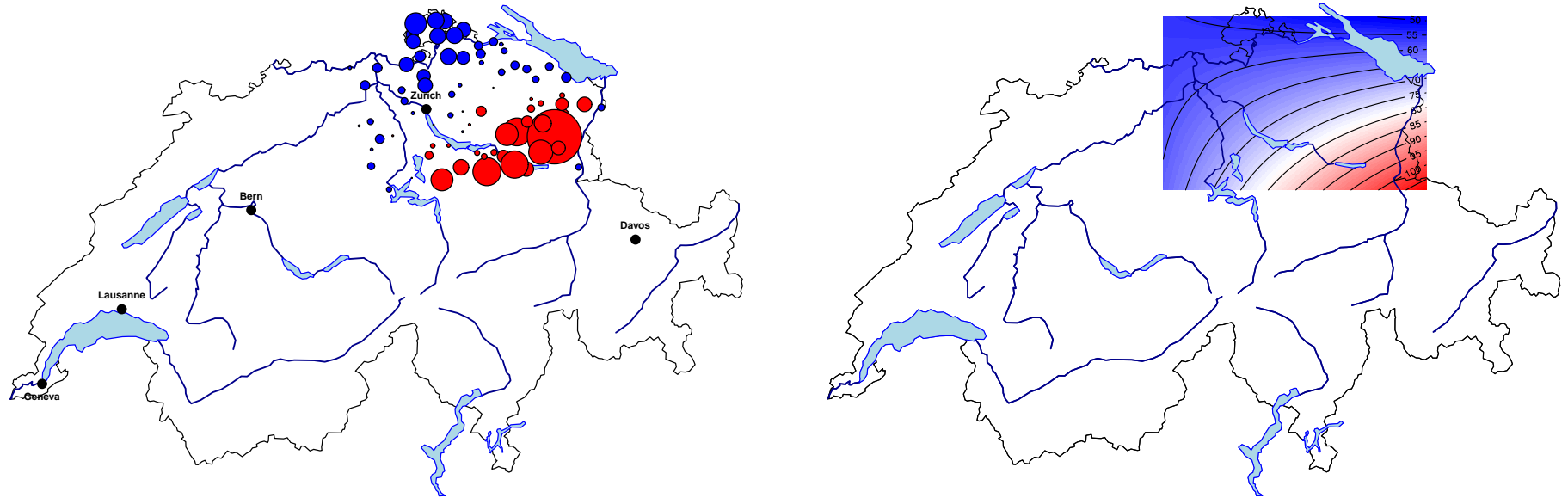


Figure 8: Left: symbol plot. Right: Prediction of the pointwise 25-year return levels from a fitted spatial GEV model.

Get predictions (predictionSpatialGEV.R)

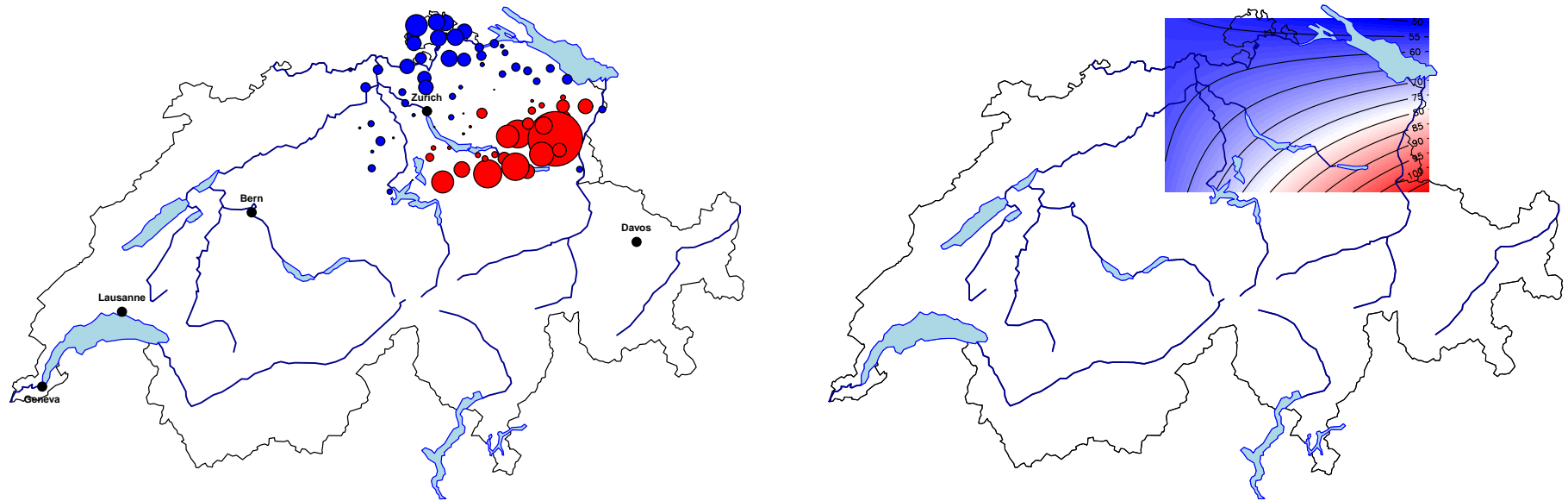


Figure 8: Left: symbol plot. Right: Prediction of the pointwise 25-year return levels from a fitted spatial GEV model.

□ But don't we forget something???

Get predictions (predictionSpatialGEV.R)

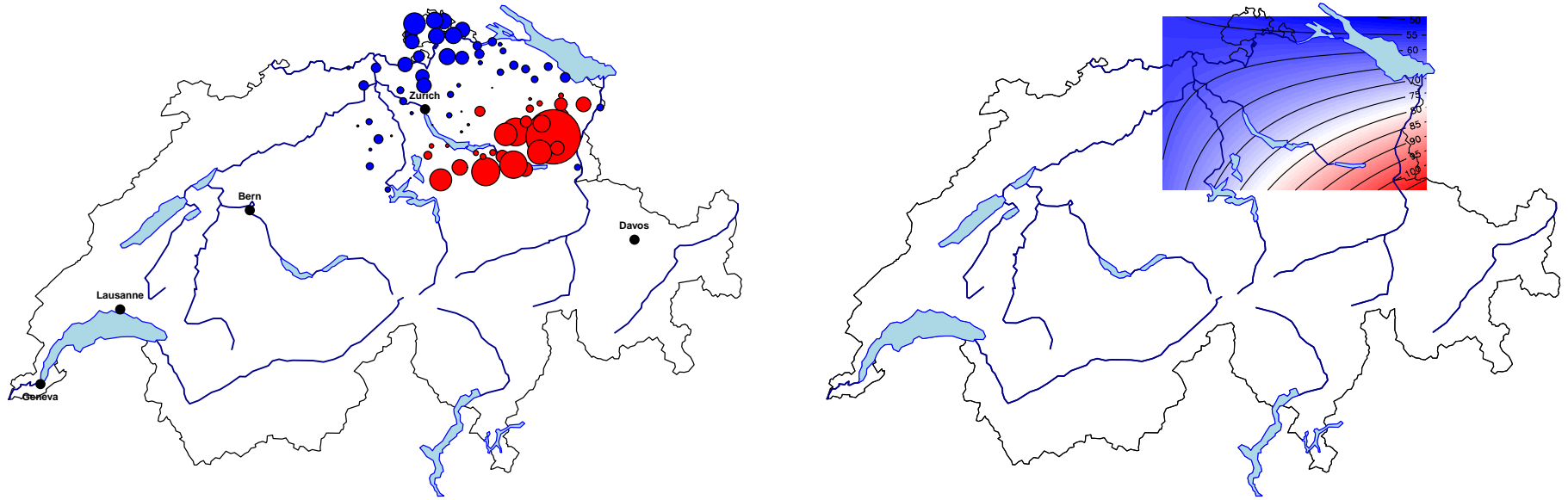


Figure 8: Left: symbol plot. Right: Prediction of the pointwise 25-year return levels from a fitted spatial GEV model.

- ☐ But don't we forget something???
- ☐ Model selection?

Model selection #2 (modelSelection.R) [Chandler and Bate, 2007; Kent, 1982]

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

Spatial GEV

Prediction #1

Model selection

▷ #2

Debrief #3

4. General max-stable processes

5. Conclusion

- Typically here we would like to test if a given covariate is required or not
- Hence we're dealing with nested model for which **composite likelihood ratio test** are especially suited


$$2\{\ell_{\text{composite}}(\hat{\psi}) - \ell_{\text{composite}}(\hat{\phi}_{\lambda_0}, \lambda_0)\} \longrightarrow \sum_{j=1}^p \lambda_j X_i, \quad n \rightarrow \infty.$$

Eigenvalue(s): 2.7 1.95

Analysis of Variance Table

	MDf	Deviance	Df	Chisq	Pr(> sum lambda Chisq)
M2	7	29328			
M0	9	29306	2	22.265	0.008273 **

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

 *Always check that your models are nested. The code won't do that for you!*

What we have learned so far (apart from using SpatialExtremes)

0. Max-stable process

1. Data and descriptive analysis

2. Simple max-stable processes

3. Trends surfaces

Spatial GEV

Prediction #1

Model selection #2

▷ Debrief #3

4. General max-stable processes

5. Conclusion

- Based on the spatial GEV model, we identify what seems to be relevant trend surfaces for the marginal parameters:

$$\mu(s) = \beta_{0,\mu} + \beta_{1,\mu}\text{lon}(s) + \beta_{2,\mu}\text{lat}(s) + \beta_{3,\mu}\text{lon}(s)\text{lat}(s),$$

$$\sigma(s) = \beta_{0,\sigma} + \beta_{1,\sigma}\text{lon}(s) + \beta_{2,\sigma}\text{lat}(s),$$

$$\xi(s) = \beta_{0,\xi},$$

0. Max-stable process

1. Data and
descriptive analysis

2. Simple max-stable
processes

3. Trends surfaces

4. General
max-stable
▷ processes

Fitting
Model checking
[Davison et al., 2012]
Predictions
(`simulationFinal.R`)

5. Conclusion

4. General max-stable processes

Fitting a max-stable process with trend surfaces

- Now it's time to combine everything, i.e., [trend surfaces + dependence](#).
- The syntax won't be a big surprise

```
M0 <- fitmaxstab(rain, coord[,1:2], "twhitmat", nugget = 0, loc.form, scale.form, shape.form)
```

```
      Estimator: MPLE
      Model: Extremal-t
      Weighted: FALSE
Pair. Deviance: 2237562
      TIC: 2249206
Covariance Family: Whittle-Matern
```

Estimates

```
      Marginal Parameters:
      Location Parameters:
locCoeff1  locCoeff2  locCoeff3  locCoeff4
27.136295   0.060145  -0.164755  -0.001117
      Scale Parameters:
scaleCoeff1  scaleCoeff2  scaleCoeff3
   9.88857    0.02869    -0.04581
      Shape Parameters:
shapeCoeff1
   0.1727
      Dependence Parameters:
   range   smooth    DoF
225.9452   0.3645   4.1566
...
```

Model checking [Davison et al., 2012]

- When you want to check your fitted max-stable model, you usually want to check if

- observations at each single location are well modelled: **return level plot**;
- the dependence is well captured: **extremal coefficient function**.

- This can be done using a single line of code

```
> plot(M0)
```

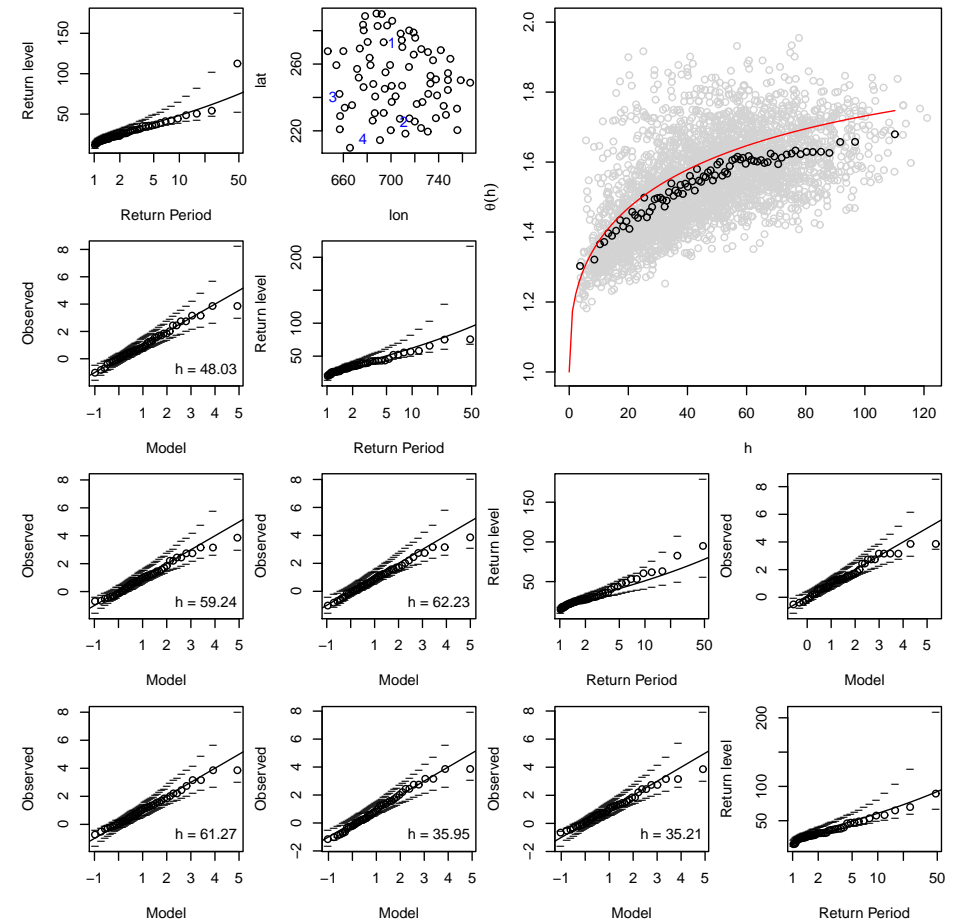


Figure 9: Model checking for a fitted max-stable process having trend surfaces.

Predictions (simulationFinal.R)

- ❑ Prediction works as for the *spatial GEV model* thanks to the `predict` function.
- ❑ But beware these predictions are pointwise—no spatial dependence at all!!!
- ❑ If you want to do take into account spatial dependence then you need to simulate from your fitted model.

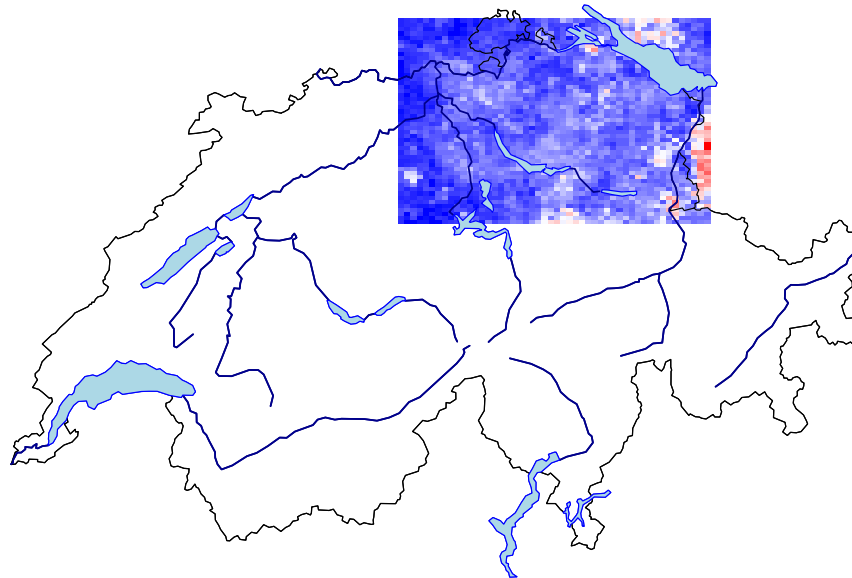


Figure 10: One simulation from our fitted extremal- t model with trend surfaces.

0. Max-stable process

1. Data and
descriptive analysis

2. Simple max-stable
processes

3. Trends surfaces

4. General max-stable
processes

▷ 5. Conclusion

References

5. Conclusion

What we haven't seen

- ❑ Using weighted pairwise likelihood;
- ❑ Many (many!) utility functions. Highly recommended to have a look at the documentation;
- ❑ The package has a vignette: `vignette("SpatialExtremesGuide")`;
- ❑ Copula models—although I do not recommend their use for spatial extremes;
- ❑ Bayesian hierarchical models;
- ❑ Unconditional simulations: several implementations (including exact simulations)
- ❑ Conditional simulations—really CPU demanding.

What we haven't seen

- ❑ Using weighted pairwise likelihood;
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 - ❑ Copula models—although I do not recommend their use for spatial extremes;
 - ❑ Bayesian hierarchical models;
 - ❑ Unconditional simulations: several implementations (including exact simulations)
 - ❑ Conditional simulations—really CPU demanding.
- 👉 A rather recent review on max-stable processes with R code is given by Ribatet [2013]

THANK YOU!

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