Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture

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Outline

\{2\} \prec \{1, 3, 5\} \prec \{4, 6\}

‘Ordered set partitions contain interesting mathematics.’

1. The classical coinvariant algebra

2. A generalized coinvariant algebra

3. Connections with the Delta Conjecture

4. Variations
   (Jonathan Chan, Jia Huang, Travis Scrimshaw, Andy Wilson)
Symmetric polynomials

\[ \mathcal{S}_n \text{ acts on } \mathbb{Q}[x_n] := \mathbb{Q}[x_1, \ldots, x_n] \text{ by permuting variables.} \]

\[ \mathbb{Q}[x_n]^{\mathcal{S}_n} = \{\text{symmetric polynomials in } x_1, \ldots, x_n\}. \]

\[ e_d(x_n) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}. \]

**Thm:** [Newton] \( \{e_1(x_n), e_2(x_n), \ldots, e_n(x_n)\} \) is an algebraically independent generating set of \( \mathbb{Q}[x_n]^{\mathcal{S}_n} \).
Coinvariant algebra

\( \mathfrak{S}_n \) acts on \( \mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n] \) by permuting variables.

The \textit{invariant ideal} \( l_n \subseteq \mathbb{Q}[\mathbf{x}_n] \) is

\[
l_n := \langle \mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n} \rangle = \langle e_1(\mathbf{x}_n), e_2(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n) \rangle
\]

The \textit{coinvariant algebra} is \( R_n := \frac{\mathbb{Q}[\mathbf{x}_n]}{l_n} \).

**Thm:** (Chevalley) We have

\[ R_n \cong \mathbb{Q}[\mathfrak{S}_n]. \]
Artin Basis

**Thm:** [E. Artin] The set of *sub-staircase* monomials

\[ \{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} : 0 \leq i_j < j \} \]

is a basis for \( R_n \).

e.g. If \( n = 5 \),

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

so \( x_1^0 x_2^0 x_3^2 x_4^1 x_5^2 \) is an Artin basis element.
Hilbert series

Thm: [E. Artin] The set of sub-staircase monomials

$$\{x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n} : 0 \leq i_j < j\}$$

is a basis for $R_n$.

Cor: The Hilbert series of $R_n$ is

$$\text{Hilb}(R_n; q) = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) =: [n]!_q.$$
MacMahon’s Theorem

\[
\text{inv}(25143) = 0 + 0 + 2 + 1 + 2 = 5
\]
\[
\text{maj}(25143) = 2 + 4 = 6
\]

**Thm:** [MacMahon] \(\text{inv}\) and \(\text{maj}\) are equidistributed on \(\mathfrak{S}_n\):

\[
\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)} = [n]!_q.
\]

**Q:** Is there a basis of \(R_n\) which reflects the statistic \(\text{maj}\)?
Garsia-Stanton Basis

Let \( \pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n \). The GS monomial is

\[
gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}.
\]

\[\pi = 25143 \Rightarrow gs_\pi = (x_2x_5)(x_2x_5x_1x_4)\]

\[
\deg(gs_\pi) = \text{maj}(\pi).
\]

**Thm:** [Garsia] The set \( \{gs_\pi : \pi \in \mathfrak{S}_n\} \) descends to a basis for \( R_n \).
Q: Recall $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$. What about the graded isomorphism type?

\[
T = \begin{bmatrix}
1 & 2 & 3 & 5 \\
4 & 6 & 8 & \\
7 &
\end{bmatrix}
\]

\[
\text{maj}(T) = 3 + 5 + 6 = 14.
\]
Q: Recall $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$. What about the graded isomorphism type?

Thm: (Lusztig-Stanley) We have

$$\text{grFrob}(R_n; q) = \sum_{T \in SYT(n)} q^{\text{maj}(T)} \cdot s_{\text{sh}(T)}(x).$$

\[
\begin{array}{cccc}
1 & 2 & 3 \\
2 & & & 1 \\
3 & & & 2 \\
\end{array}
\]

$$\text{grFrob}(R_3; q) = q^0 \cdot s_{(3)}(x) + q^1 \cdot s_{(2,1)}(x) + q^2 \cdot s_{(2,1)}(x) + q^3 \cdot s_{(1,1,1)}(x).$$
Ordered Set Partitions

**Def:** An *ordered set partition* is set partition of \([n]\) with a total order on its blocks.

**Ex:**

\[
\{1, 3, 5\} \prec \{6\} \prec \{2, 4\} = (135 \mid 6 \mid 24)
\]

is an ordered set partition of \([6]\) with 3 blocks.

\[
\mathcal{OP}_{n,k} := \{\text{all ordered set partitions } \sigma \models [n] \text{ with } k \text{ blocks}\}.
\]

\[
|\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k).
\]

(No nice product formula.)

**Q:** Is there a nice quotient of \(\mathbb{Q}[x_n]\) reflecting the combinatorics of \(\mathcal{OP}_{n,k}\)?
New Generalized Coinvariant Algebra

Defn: [HRS] For \( k \leq n \), \( I_{n,k} \subseteq \mathbb{Q}[x_n] \) is the ideal

\[
I_{n,k} := \langle e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-k+1}(x_n), x_1^k, x_2^k, \ldots, x_n^k \rangle.
\]

The ring \( R_{n,k} \) is the corresponding quotient.

\[
R_{n,k} = \frac{\mathbb{Q}[x_n]}{I_{n,k}}.
\]

- \( R_{n,k} \) is a graded \( \mathfrak{S}_n \)-module.
- \( R_{n,1} = \frac{\mathbb{Q}[x_n]}{\langle x_1, x_2, \ldots, x_n \rangle} \cong \mathbb{Q} \).
- \( I_{n,n} = I_n \) and \( R_{n,n} = R_n \).

Thm: [HRS] We have \( \dim(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k) \).
Inversions in OSPs

\[ \text{inv}(135 \mid 6 \mid 24) = \#\{(2, 3), (2, 5), (2, 6)\} = 3. \]

We have

\[ \sum_{\sigma \in \mathcal{O}_n} q^{\text{inv}(\sigma)} = [k]_q! \cdot \text{Stir}_q(n, k). \]

Here \( \text{Stir}_q(0, k) = \delta_{0,k} \) and

\[ \text{Stir}_q(n, k) = \text{Stir}_q(n - 1, k - 1) + [k]_q \cdot \text{Stir}_q(n - 1, k). \]

**Thm:** [HRS] We have \( \text{Hilb}(R_{n,k}; q) = \text{rev}_q([k]_q! \cdot \text{Stir}_q(n, k)). \)
Artin Basis

**Def:** A \((n, k)\)-staircase is a shuffle of \((0, 1, \ldots, k - 1)\) and \(((k - 1)^{n-k})\).

**Ex:** \((n, k) = (5, 3)\) – shuffles of \((0, 1, 2)\) and \((2, 2)\):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

**Thm:** [HRS] The *generalized Artin monomials*

\[
A_{n,k} := \{x_1^{i_1} \cdots x_n^{i_n} : (i_1, \ldots, i_n) \text{ fits below some } (n, k)\)-staircase\}
\]

descends to a basis of \(R_{n,k}\).
Major index on OSPs

Represent OSPs as ‘descent starred permutations’.

\[
\{1, 3, 5\} \prec \{6\} \prec \{2, 4\} \leadsto 5 \ast 3 \ast 1 \ 6 \ 4 \ast 2 \\
\leadsto 5^0 \ast 3^0 \ast 1^1 \ 6^2 \ 4^2 \ast 2^3
\]

\[
\text{maj}(5 \ast 3 \ast 1 \ 6 \ 4 \ast 2) = 0 + 0 + 2 + 2 = 4.
\]

**Thm:** [Remmel-Wilson] \( \text{inv} \) and \( \text{maj} \) are equidistributed on \( \mathcal{OP}_{n,k} \).

\[
\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{maj}(\sigma)} = [k]!_q \cdot \text{Stir}_q(n, k).
\]
Garsia-Stanton basis

Recall $g_s\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}$.

**Def:** The $(n, k)$-GS monomials are

$G_S n, k = \{g_s\pi \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-k}}^{i_{n-k}} : \pi \in S_n, \ k - \text{des}(\pi) > i_1 \geq \cdots \geq i_{n-k} \geq 0\}$.

**Ex:** $(n, k) = (7, 5), \pi = 2561734$.

$$g_s\pi = (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7)$$

$(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \quad (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2 \quad (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2$

$(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2 x_5 \quad (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2 x_5 \quad (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2 x_5^2$
Garsia-Stanton basis

Recall $gs_{\pi} = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}$.

**Def:** The $(n, k)$-GS monomials are

$G_{S_{n,k}} = \{ gs_{\pi} \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-k}}^{i_{n-k}} : \pi \in S_n, \; k - \text{des}(\pi) > i_1 \geq \cdots \geq i_{n-k} \geq 0 \}$.

**Thm:** [HRS] The set $G_{S_{n,k}}$ descends to a basis of $R_{n,k}$.

**Rmk:** Daniël Kroes has a Stanley-Reisner style version of this result.
Module structure

\( \mathfrak{S}_n \) acts on \( \mathcal{OP}_{n,k} \) by letter permutation.

\[
(125 \mid 4 \mid 36) \xrightarrow{(1,3,4)} (235 \mid 1 \mid 46)
\]

Obs: \( k = n \leadsto \) recover regular representation of \( \mathfrak{S}_n \).

Thm: [HRS] We have \( R_{n,k} \cong \mathcal{OP}_{n,k} \) as ungraded \( \mathfrak{S}_n \)-modules.

Q: What about the graded \( \mathfrak{S}_n \)-module structure?
Ordered multiset partitions

**Def:** An *ordered multiset partition* is a list of nonempty sets of positive integers.

$$\mu = (145 \mid 13 \mid 246)$$

**Def-Thm:** [Wilson] Let $C_{n,k}(x; q)$ be the symmetric function

$$C_{n,k}(x; q) = \sum_{\mu} q^{\text{inv}(\mu)} x^\mu = \sum_{\mu} q^{\text{maj}(\mu)} x^\mu,$$

where $\sum$ is over all OMPs of size $n$ with $k$ blocks.

**Rmk:** Also have two other Mahonian-type statistics $\text{dinv}$ and $\text{minimaj}$. (Haglund-Remmel-Wilson, Wilson, R.)
Def-Thm: [Wilson] Let $C_{n,k}(x; q)$ be the symmetric function

$$C_{n,k}(x; q) = \sum_{\mu} q^{\text{inv}(\mu)} x^\mu = \sum_{\mu} q^{\text{maj}(\mu)} x^\mu,$$

where $\sum$ is over all OMPs of size $n$ with $k$ blocks.

Thm: [HRS] We have

$$\text{grFrob}(R_{n,k}; q) = (\text{rev}_q \circ \omega) C_{n,k}(x; q).$$
Schur expansion

**Thm:** [HRS] We have

$$\text{grFrob}(R_n,k; q) = (\text{rev}_q \circ \omega) C_{n,k}(x; q).$$

(Apply RSK . . .)

**Thm:** [HRS] The Schur expansion of $\text{grFrob}(R_n,k; q)$ is

$$\text{grFrob}(R_n,k; q) = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} \left[\frac{n - \text{des}(T) - 1}{n - k}\right]_q s_{\text{sh}(T)}(x).$$

**Rmk:** Kyle Meyer has an Adin-Brenti-Roichman style refinement of this result.
Hall-Littlewood expansion

For $\lambda \vdash n$, let $Q'_\lambda(x; q)$ be the dual Hall-Littlewood polynomial.

**Thm:** [Lusztig-Stanley] The graded Frobenius image of $R_n$ is

$$\text{grFrob}(R_n; q) = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} s_{\text{sh}(T)}(x) = \text{rev}_q Q'_{(1^n)}(x; q).$$

**Thm:** [HRS] The graded Frobenius image of $R_{n,k}$ is

$$\text{grFrob}(R_{n,k}; q) = \text{rev}_q \sum_{\substack{\lambda \vdash n \\ell(\lambda) = k \\ell(\lambda) = k}} q^{\sum(i-1)(\lambda_i-1)}\left[ m_1(\lambda), \ldots, m_n(\lambda) \right]_q Q'_\lambda(x; q).$$
Skip compositions

For $S \subseteq [n]$, let $\gamma(S) = (\gamma_1, \ldots, \gamma_n)$ be the associated skip composition.

**Ex:** $n = 8$, $S = \{2, 4, 5, 8\}$:

$\gamma(S) = (0, 2, 0, 3, 3, 0, 0, 5)$.

If $\gamma$ is a length $n$ composition, let $\kappa_\gamma(x_n) \in \mathbb{Q}[x_n]$ be the Demazure character.

- $\kappa_\gamma(x_n)$ is character of indecomposable rep’n of $B \subseteq GL_n(\mathbb{C})$.
- $\kappa_\gamma(x_n)$ defined recursively with *Demazure character formula*.
- $\kappa_\gamma(x_n)$ defined combinatorially with ‘skyline fillings’ [Mason].
Gröbner basis

Give monomials in $\mathbb{Q}[x_n]$ lexicographic order.

**Thm:** [HRS] A Gröbner basis for $I_{n,k}$ is given by the variable powers

$$x_1^k, x_2^k, \ldots, x_n^k$$

together with the variable reversed Demazure characters

$$\kappa_{\gamma}(S)^*(x_n^*)$$

for $S \subseteq [n-1]$ with $|S| = n - k + 1$. If $k < n$, this Gröbner basis is reduced.

**Q:** Why do Demazure characters show up in the Gröbner basis?
The Diagonal Coinvariants

Let $\mathfrak{S}_n$ act on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ diagonally:

\[
\pi.x_i := x_{\pi(i)} \quad \quad \pi.y_i := y_{\pi(i)}.
\]

**Def:** The *diagonal coinvariant module* is the bigraded $\mathfrak{S}_n$-representation

\[
DR_n := \mathbb{Q}[x_n, y_n]/\langle \mathbb{Q}[x_n, y_n]_{\mathfrak{S}_n} \rangle.
\]

**Thm:** [Haiman] We have $\dim(DR_n) = (n + 1)^{n-1}$. In fact (up to sign twist), $DR_n$ is isomorphic to the permutation action of $\mathfrak{S}_n$ on size $n$ parking functions.
Bigraded Characters

**Q:** What is the bigraded $\mathfrak{S}_n$-module structure of $DR_n$?

**Thm:** [Haiman] The bigraded Frobenius series of $DR_n$ is $\nabla(e_n)$, where $\nabla$ is the Bergeron-Garsia nabla operator on symmetric functions (a Macdonald eigenoperator).

**Problem:** Expand $\nabla(e_n)$ in the Schur basis $\{s_\lambda : \lambda \vdash n\}$:

$$\nabla(e_n) = \sum_{\lambda \vdash n} c_\lambda(q, t)s_\lambda.$$ 

When $\lambda$ is a hook, Haglund’s $q, t$-Schröder Theorem gives an answer. No conjecture in general.
The Shuffle Theorem

‘If symmetric functions are too hard, work with quasisymmetric functions.’

**Thm:** [Carlsson-Mellit] (‘Shuffle Theorem’) We have that

$$\nabla(e_n) = \sum_{P \in \text{Park}_n} q^{\text{area}(P)} t^{\text{dinv}(P)} F_{\text{iDes}}(P).$$

The *Delta Conjecture* is a (conjectural) generalization of the Shuffle Theorem.
Delta Operators

- $\Lambda_n$ = symmetric functions in $(x_1, x_2, \ldots)$ of degree $n$.
- $\{\tilde{H}_\mu : \mu \vdash n\}$ = modified Macdonald basis.
- $f = f(x_1, x_2, \ldots)$ a symmetric function.

**Def:** $\Delta'_f : \Lambda_n \rightarrow \Lambda_n$ is the Macdonald eigenoperator defined by

$$\Delta'_f : \tilde{H}_\mu \mapsto f(\ldots, q^i t^j, \ldots)\tilde{H}_\mu,$$

where $(i, j)$ range over all cells $\neq (0, 0)$ of the Ferrers diagram of $\mu$.

**Ex:** $\mu = (4, 2) \vdash 6$.

<table>
<thead>
<tr>
<th></th>
<th>q</th>
<th>q^2</th>
<th>q^3</th>
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<tr>
<td>t</td>
<td>q</td>
<td>t</td>
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<td></td>
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<td>qt</td>
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</tr>
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</table>

$$\Delta'_f(\tilde{H}_\mu) = f(q, q^2, q^3, t, qt)\tilde{H}_\mu$$

**Fact:** $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$. 
Delta Conjecture

**Conj:** [Haglund-Remmel-Wilson] For any $k \leq n$ we have

$$\Delta'_{e_{k-1}}(e_n) = \{z^{n-k}\} \left[ \sum_{P \in LD_n} q^{d_{\text{inv}}(P)} t^{\text{area}(P)} \prod_{i: a_i(P) > a_{i-1}(P)} \left( 1 + \frac{z}{t^{a_i(P)}} \right) x^P \right]$$

$$= \{z^{n-k}\} \left[ \sum_{P \in LD_n} q^{d_{\text{inv}}(P)} t^{\text{area}(P)} \prod_{i \in \text{Val}(P)} \left( 1 + \frac{z}{q^{d_i(P)+1}} \right) x^P \right],$$

where $\{z^{n-k}\}$ extracts the coefficient of $z^{n-k}$.

**Rmk:** When $k = n$, this is the Shuffle Theorem.

**Def:** Let $\text{Rise}_{n,k}(x; q, t)$, $\text{Val}_{n,k}(x; q, t)$ denote the two right-hand sides.
Conj: For all $k \leq n$,

$$
\Delta'_{e_{k-1}}(e_n) = \text{Rise}_{n,k}(x; q, t) \\
= \text{Val}_{n,k}(x; q, t).
$$

Thm: [Wilson, R.] We have

$$
\text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0) = \text{Val}_{n,k}(x; 0, q).
$$

Thm: [HRS] Let $C_{n,k}(x; q)$ be the common symmetric function above. We have

$$
\text{grFrob}(R_{n,k}; q) = (\text{rev}_q \circ \omega)C_{n,k}(x; q).
$$
The 0-Hecke algebra $H_n(0)$ over a field $\mathbb{F}$ has generators $\pi_1, \pi_2, \ldots, \pi_{n-1}$ and relations

\[
\begin{align*}
\pi_i^2 &= -\pi_i & 1 \leq i \leq n-1 \\
\pi_i \pi_j &= \pi_j \pi_i & |i - j| > 1 \\
\pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} & 1 \leq i \leq n-2
\end{align*}
\]

$H_n(0)$ acts on $\mathbb{F}[x_n]$ by

\[
\pi_i . f := \frac{x_{i+1} f - x_{i+1} (s_i . f)}{x_i - x_{i+1}}.
\]
$H_n(0)$ acts on $\mathcal{OP}_{n,k}$:

\[
\bar{\pi}_2.(13 \mid 2 \mid 45) = -(13 \mid 2 \mid 45), \\
\bar{\pi}_3.(13 \mid 2 \mid 45) = (14 \mid 2 \mid 35), \\
\bar{\pi}_4.(13 \mid 2 \mid 45) = 0
\]

**Q:** Is there a 0-Hecke action on a quotient of $\mathbb{F}[x_n]$ that reflects this?
Variation 1: 0-Hecke algebras (w/ Jia Huang)

**Def:** Let $J_{n,k} \subseteq \mathbb{F}[x_n]$ be the ideal

$$J_{n,k} := \left\langle e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-k+1}(x_n), h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n) \right\rangle.$$

Let $S_{n,k} := \frac{\mathbb{F}[x_n]}{J_{n,k}}$ be the corresponding quotient.

**Thm:** [Huang-R.] We have $S_{n,k} \cong \mathcal{OP}_{n,k}$ as $H_n(0)$-modules.

Have formulas for the characters $\text{ch}_t(S_{n,k}), \text{Ch}_t(S_{n,k}),$ and $\text{Ch}_{q,t}(S_{n,k})$. Moreover, have $\text{Ch}_t(S_{n,k}) = \text{grFrob}(R_{n,k}; t)$.

Also have a generalization to the full Hecke algebra (w/ J. Huang and T. Scrimshaw).
Variation 2: $\mathbb{Z}_r \wr S_n$ (w/ Jonathan Chan)

$\mathbb{Z}_r \wr S_n = n \times n$ monomial matrices over $\mathbb{C}$, nonzero entries satisfy $\zeta^r = 1$.

For $r \geq 2$, we have a notion of ‘faces in the $\mathbb{Z}_r \wr S_n$ Coxeter complex’.

**Ex:** A 3-dimensional face attached to $\mathbb{Z}_3 \wr S_9$:

$$(47 \mid 1^2 \mid 2^0 3^0 8^1 \mid 5^1 6^2 9^1)$$

$\mathcal{OP}^r_{n,k} := \{k\text{-dim}'l\ faces in $\mathbb{Z}_r \wr S_n$ Coxeter complex\}$

carries an action of $\mathbb{Z}_r \wr S_n$. 
Def: For \( r \geq 2 \), let \( I^r_{n,k} \subseteq \mathbb{C}[x_n] \) be the ideal

\[
I^r_{n,k} := \langle x_1^{kr+1}, x_2^{kr+1}, \ldots, x_n^{kr+1}, e_n(x_n^r), e_{n-1}(x_n^r), \ldots, e_{n-k+1}(x_n^r) \rangle.
\]

Let \( R^r_{n,k} := \frac{\mathbb{C}[x_n]}{I^r_{n,k}} \) be the corresponding quotient.

Thm: [Chan-R.] We have \( R^r_{n,k} \cong \mathcal{O} \mathcal{P}^r_{n,k} \) as \( \mathbb{Z}_r \wr \mathfrak{S}_n \)-modules. (Also formulas for graded character, bases, etc.)
Def: Let $n, k, r$ be nonnegative integers. Set 

$$L_{n,k,r} := \langle e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-r+1}(x_n), h_k(x_n), h_{k+1}(x_n), \ldots, h_{k+n}(x_n) \rangle,$$

and $T_{n,k,r} = \mathbb{Q}[x_n]/L_{n,k,r}$.

Thm: [R.-Wilson] The algebra of $T_{n,k,r}$ is controlled by the combinatorics of ‘tail positive words’.

Thm: [R.-Wilson] We have

$$\text{grFrob}(T_{n,k,1}; q) = \Delta_{s_{k-1,1n-1}} e_n(x) \mid_{t=0}.$$
Thanks for listening!!