Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture

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Outline

$$\{2\} \prec \{1,3,5\} \prec \{4,6\}$$

'Ordered set partitions contain interesting mathematics.'

1. The classical coinvariant algebra

2. A generalized coinvariant algebra

3. Connections with the Delta Conjecture

4. Variations
(Jonathan Chan, Jia Huang, Travis Scrimshaw, Andy Wilson)

Symmetric polynomials

$\mathfrak{S}_n$ acts on $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ by *permuting variables*.

$\mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n} = \{\text{symmetric polynomials in } x_1, \ldots, x_n\}$.

$$e_d(\mathbf{x}_n) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}.$$

Thm: [Newton] $\{e_1(\mathbf{x}_n), e_2(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)\}$ is an algebraically independent generating set of $\mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n}$.

Coinvariant algebra

$\mathfrak{S}_n$ acts on $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ by *permuting variables*.

The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is

$$\begin{aligned} I_n &:= \langle \mathbb{Q}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle \\ &= \langle e_1(\mathbf{x}_n), e_2(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n) \rangle \end{aligned}$$

The *coinvariant algebra* is $R_n := \frac{\mathbb{Q}[\mathbf{x}_n]}{I_n}$.

Thm: (Chevalley) We have

$$R_n \cong \mathbb{Q}[\mathfrak{S}_n].$$

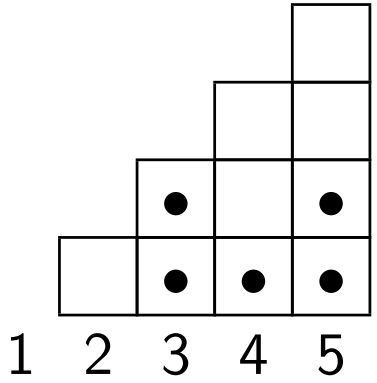
Artin Basis

Thm: [E. Artin] The set of *sub-staircase* monomials

$$\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \ : \ 0 \leq i_j < j\}$$

is a basis for R_n.

e.g. If $n = 5$,



so $x_1^0 x_2^0 x_3^2 x_4^1 x_5^2$ is an Artin basis element.

Hilbert series

Thm: [E. Artin] The set of *sub-staircase* monomials

$$\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \ : \ 0 \leq i_j < j\}$$

is a basis for R_n.

Cor: The *Hilbert series* of R_n is

$$\mathrm{Hilb}(R_n; q) = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) =: [n]!_q.$$

$$\mathrm{inv}(25143) = 0 + 0 + 2 + 1 + 2 = 5$$

$$\mathrm{Hilb}(R_n; q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)}.$$

MacMahon's Theorem

$$\mathrm{inv}(25143) = 0 + 0 + 2 + 1 + 2 = 5$$

$$\mathrm{maj}(25143) = 2 + 4 = 6$$

Thm: [MacMahon] inv and maj are equidistributed on $\mathfrak{S}_n$:

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} = [n]!_q.$$

Q: Is there a basis of R_n which reflects the statistic maj?

Garsia-Stanton Basis

Let $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$. The *GS monomial* is

$$gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}.$$

$$\pi = 25143 \Rightarrow gs_\pi = (x_2 x_5)(x_2 x_5 x_1 x_4)$$

$$\deg(gs_\pi) = \mathrm{maj}(\pi).$$

Thm: [Garsia] The set $\{gs_\pi : \pi \in \mathfrak{S}_n\}$ descends to a basis for R_n.

Graded Frobenius Image

Q: Recall $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$. What about the *graded* isomorphism type?

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 8 & \\ \cline{1-3} 7 & & & \\ \cline{1-1} \end{array}$$

$$\mathrm{maj}(T) = 3 + 5 + 6 = 14.$$

Graded Frobenius Image

Q: Recall $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$. What about the *graded* isomorphism type?

Thm: (Lusztig-Stanley) We have

$$\mathrm{grFrob}(R_n; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot s_{\mathrm{sh}(T)}(\mathbf{x}).$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\mathrm{grFrob}(R_3; q) = q^0 \cdot s_{(3)}(\mathbf{x}) + q^1 \cdot s_{(2,1)}(\mathbf{x}) + q^2 \cdot s_{(2,1)}(\mathbf{x}) + q^3 \cdot s_{(1,1,1)}(\mathbf{x}).$$

Ordered Set Partitions

Def: An *ordered set partition* is set partition of $[n]$ with a total order on its blocks.

Ex:

$$\{1,3,5\} \prec \{6\} \prec \{2,4\} = (135 \mid 6 \mid 24)$$

is an ordered set partition of $[6]$ with 3 blocks.

$$\mathcal{OP}_{n,k} := \{\text{all ordered set partitions } \sigma \models [n] \text{ with } k \text{ blocks}\}.$$

$$|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k).$$

(No nice product formula.)

Q: Is there a nice quotient of $\mathbb{Q}[\mathbf{x}_n]$ reflecting the combinatorics of $\mathcal{OP}_{n,k}$?

New Generalized Coinvariant Algebra

Defn: [HRS] For $k \leq n$, $I_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal

$$I_{n,k} := \langle e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n), x_1^k, x_2^k, \ldots, x_n^k \rangle.$$

The ring $R_{n,k}$ is the corresponding quotient.

$$R_{n,k} = \frac{\mathbb{Q}[\mathbf{x}_n]}{I_{n,k}}.$$

- $R_{n,k}$ is a graded $\mathfrak{S}_n$-module.
- $R_{n,1} = \frac{\mathbb{Q}[\mathbf{x}_n]}{\langle x_1, x_2, \ldots, x_n \rangle} \cong \mathbb{Q}$.
- $I_{n,n} = I_n$ and $R_{n,n} = R_n$.

Thm: [HRS] We have $\dim(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$.

Inversions in OSPs

$$\mathrm{inv}(135 \mid 6 \mid 24) = \#\{(2,3),(2,5),(2,6)\} = 3.$$

We have

$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(\sigma)} = [k]!_q \cdot \mathrm{Stir}_q(n,k).$$

Here $\mathrm{Stir}_q(0,k) = \delta_{0,k}$ and

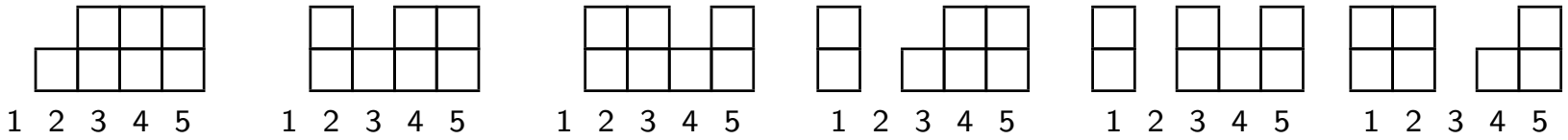
$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k).$$

Thm: [HRS] We have $\mathrm{Hilb}(R_{n,k};q) = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k))$.

Artin Basis

Def: A (n,k)-*staircase* is a shuffle of $(0,1,\ldots,k-1)$ and $((k-1)^{n-k})$.

Ex: $(n,k)=(5,3)$ – shuffles of $(0,1,2)$ and $(2,2)$:



Thm: [HRS] The *generalized Artin monomials*

$$\mathcal{A}_{n,k} := \{x_1^{i_1}\cdots x_n^{i_n} \,:\, (i_1,\ldots,i_n) \text{ fits below } \textit{some}\ (n,k)\text{-staircase}\}$$

descends to a basis of $R_{n,k}$.

Major index on OSPs

Represent OSPs as 'descent starred permutations'.

$$\{1,3,5\} \prec \{6\} \prec \{2,4\} \rightsquigarrow 5_*3_*1\ 6\ 4_*2$$
$$\rightsquigarrow 5^0{}_*3^0{}_*1^1\ 6^2\ 4^2{}_*2^3$$

$$\mathrm{maj}(5_*3_*1\ 6\ 4_*2) = 0 + 0 + 2 + 2 = 4.$$

Thm: [Remmel-Wilson] inv and maj are equidistributed on $\mathcal{OP}_{n,k}$.

$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = [k]!_q \cdot \mathrm{Stir}_q(n,k).$$

Garsia-Stanton basis

Recall $gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}$.

Def: The (n,k)-*GS monomials* are

$$\mathcal{GS}_{n,k} = \{gs_\pi \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-k}}^{i_{n-k}} \ : \ \pi \in \mathfrak{S}_n, \ k - \mathrm{des}(\pi) > i_1 \geq \cdots \geq i_{n-k} \geq 0\}.$$

Ex: $(n,k) = (7,5)$, $\pi = 2561734$.

$$gs_\pi = (x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7)$$

$(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7)$ $\quad$ $(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2$ $\quad$ $(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2$

$(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2 x_5$ $\quad$ $(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2 x_5$ $\quad$ $(x_2 x_5 x_6) \cdot (x_2 x_5 x_6 x_1 x_7) \cdot x_2^2 x_5^2$

Garsia-Stanton basis

Recall $gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}$.

Def: The (n,k)-*GS monomials* are

$$\mathcal{GS}_{n,k} = \{gs_\pi \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-k}}^{i_{n-k}} \ : \ \pi \in \mathfrak{S}_n, \ k - \mathrm{des}(\pi) > i_1 \geq \cdots \geq i_{n-k} \geq 0\}.$$

Thm: [HRS] The set $\mathcal{GS}_{n,k}$ descends to a basis of $R_{n,k}$.

Rmk: Daniël Kroes has a Stanley-Reisner style version of this result.

Module structure

$\mathfrak{S}_n$ acts on $\mathcal{OP}_{n,k}$ by *letter permutation*.

$$(125 \mid 4 \mid 36) \xrightarrow{(1,3,4)\cdot} (235 \mid 1 \mid 46)$$

Obs: $k = n \rightsquigarrow$ recover regular representation of $\mathfrak{S}_n$.

Thm: [HRS] We have $R_{n,k} \cong \mathcal{OP}_{n,k}$ as *ungraded* $\mathfrak{S}_n$-modules.

Q: What about the *graded* $\mathfrak{S}_n$-module structure?

Ordered multiset partitions

Def: An *ordered multiset partition* is a list of nonempty sets of positive integers.

$$\mu = (145 \mid 13 \mid 246)$$

Def-Thm: [Wilson] Let $C_{n,k}(\mathbf{x}; q)$ be the symmetric function

$$C_{n,k}(\mathbf{x}; q) = \sum_{\mu} q^{\mathrm{inv}(\mu)} \mathbf{x}^{\mu} = \sum_{\mu} q^{\mathrm{maj}(\mu)} \mathbf{x}^{\mu},$$

where $\sum$ is over all OMPs of size n with k blocks.

Rmk: Also have two other Mahonian-type statistics dinv and $\mathrm{minimaj}$. (Haglund-Remmel-Wilson, Wilson, R.)

Graded Frobenius image

Def-Thm: [Wilson] Let $C_{n,k}(\mathbf{x};q)$ be the symmetric function

$$C_{n,k}(\mathbf{x};q) = \sum_{\mu} q^{\mathrm{inv}(\mu)}\mathbf{x}^{\mu} = \sum_{\mu} q^{\mathrm{maj}(\mu)}\mathbf{x}^{\mu},$$

where $\sum$ is over all OMPs of size n with k blocks.

Thm: [HRS] We have

$$\mathrm{grFrob}(R_{n,k};q) = (\mathrm{rev}_q \circ \omega)C_{n,k}(\mathbf{x};q).$$

Schur expansion

Thm: [HRS] We have

$$\mathrm{grFrob}(R_{n,k};q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x};q).$$

(Apply RSK ...)

Thm: [HRS] The Schur expansion of $\mathrm{grFrob}(R_{n,k};q)$ is

$$\mathrm{grFrob}(R_{n,k};q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n-k \end{bmatrix}_q s_{\mathrm{sh}(T)}(\mathbf{x}).$$

Rmk: Kyle Meyer has an Adin-Brenti-Roichman style refinement of this result.

Hall-Littlewood expansion

For $\lambda \vdash n$, let $Q'_\lambda(\mathbf{x}; q)$ be the dual Hall-Littlewood polynomial.

Thm: [Lusztig-Stanley] The graded Frobenius image of R_n is

$$\begin{aligned}\mathrm{grFrob}(R_n; q) &= \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{sh}(T)}(\mathbf{x}) \\ &= \mathrm{rev}_q Q'_{(1^n)}(\mathbf{x}; q).\end{aligned}$$

Thm: [HRS] The graded Frobenius image of $R_{n,k}$ is

$$\mathrm{grFrob}(R_{n,k}; q) = \mathrm{rev}_q \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} q^{\sum (i-1)(\lambda_i - 1)} \begin{bmatrix} k \\ m_1(\lambda), \cdots, m_n(\lambda) \end{bmatrix}_q Q'_\lambda(\mathbf{x}; q).$$

Skip compositions

For $S \subseteq [n]$, let $\gamma(S) = (\gamma_1, \ldots, \gamma_n)$ be the associated *skip composition*.

Ex: $n = 8, S = \{2, 4, 5, 8\}$:

$$\gamma(S) = (0, 2, 0, 3, 3, 0, 0, 5).$$

If γ is a length n composition, let $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n]$ be the *Demazure character*.

- $\kappa_\gamma(\mathbf{x}_n)$ is character of indecomposable rep'n of $B \subseteq GL_n(\mathbb{C})$.
- $\kappa_\gamma(\mathbf{x}_n)$ defined recursively with *Demazure character formula*.
- $\kappa_\gamma(\mathbf{x}_n)$ defined combinatorially with 'skyline fillings' [Mason].

Gröbner basis

Give monomials in $\mathbb{Q}[\mathbf{x}_n]$ lexicographic order.

Thm: [HRS] A Gröbner basis for $I_{n,k}$ is given by the variable powers

$$x_1^k, x_2^k, \ldots, x_n^k$$

together with the variable reversed Demazure characters

$$\kappa_{\gamma(S)^*}(\mathbf{x}_n^*),$$

for $S \subseteq [n-1]$ with $|S| = n-k+1$. If $k < n$, this Gröbner basis is reduced.

Q: Why do Demazure characters show up in the Gröbner basis?

The Diagonal Coinvariants

Let $\mathfrak{S}_n$ act on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ *diagonally*:

$$\pi.x_i := x_{\pi(i)}$$
$$\pi.y_i := y_{\pi(i)}.$$

Def: The *diagonal coinvariant module* is the bigraded $\mathfrak{S}_n$-representation

$$DR_n := \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]_+^{\mathfrak{S}_n} \rangle.$$

Thm: [Haiman] We have $\dim(DR_n) = (n+1)^{n-1}$. In fact (up to sign twist), DR_n is isomorphic to the permutation action of $\mathfrak{S}_n$ on size n parking functions.

Bigraded Characters

Q: What is the bigraded $\mathfrak{S}_n$-module structure of DR_n?

Thm: [Haiman] The bigraded Frobenius series of DR_n is $\nabla(e_n)$, where ∇ is the Bergeron-Garsia nabla operator on symmetric functions (a *Macdonald eigenoperator*).

Problem: Expand $\nabla(e_n)$ in the *Schur basis* $\{s_\lambda : \lambda \vdash n\}$:

$$\nabla(e_n) = \sum_{\lambda \vdash n} c_\lambda(q,t) s_\lambda.$$

When λ is a *hook*, Haglund's q,t-Schröder Theorem gives an answer. No *conjecture* in general.

The Shuffle Theorem

'If symmetric functions are too hard, work with quasisymmetric functions.'

Thm: [Carlsson-Mellit] ('Shuffle Theorem') We have that

$$\nabla(e_n) = \sum_{P \in \mathrm{Park}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} F_{\mathrm{iDes}(P)}.$$

The *Delta Conjecture* is a (conjectural) generalization of the Shuffle Theorem.

Delta Operators

- Λ_n = symmetric functions in $(x_1, x_2, \dots)$ of degree n.
- $\{\widetilde{H}_\mu : \mu \vdash n\}$ = modified Macdonald basis.
- $f = f(x_1, x_2, \dots)$ a symmetric function.

Def: $\Delta'_f : \Lambda_n \to \Lambda_n$ is the Macdonald eigenoperator defined by

$$\Delta'_f : \widetilde{H}_\mu \mapsto f(\dots, q^i t^j, \dots)\widetilde{H}_\mu,$$

where (i, j) range over all cells $\neq (0, 0)$ of the Ferrers diagram of μ.
Ex: $\mu = (4, 2) \vdash 6$.

$\bullet$	q	q^2	q^3
t	qt		

$$\Delta'_f(\widetilde{H}_\mu) = f(q, q^2, q^3, t, qt)\widetilde{H}_\mu$$

Fact: $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$.

Delta Conjecture

Conj: [Haglund-Remmel-Wilson] For any $k \leq n$ we have

$$\Delta'_{e_{k-1}}(e_n) = \{z^{n-k}\}\left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i: a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P\right]$$
$$= \{z^{n-k}\}\left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i \in \mathrm{Val}(P)} \left(1 + z/q^{d_i(P)+1}\right) x^P\right],$$

where $\{z^{n-k}\}$ extracts the coefficient of z^{n-k}.

Rmk: When $k = n$, this is the Shuffle Theorem.

Def: Let $\mathrm{Rise}_{n,k}(\mathbf{x}; q, t), \mathrm{Val}_{n,k}(\mathbf{x}; q, t)$ denote the two right-hand sides.

$R_{n,k}$ and the Delta Conjecture

Conj: For all $k \leq n$,

$$\begin{aligned}\Delta'_{e_{k-1}}(e_n) &= \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) \\ &= \mathrm{Val}_{n,k}(\mathbf{x}; q, t).\end{aligned}$$

Thm: [Wilson, R.] We have

$$\mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q).$$

Thm: [HRS] Let $C_{n,k}(\mathbf{x}; q)$ be the common symmetric function above. We have

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q).$$

Variation 1: 0-Hecke algebras (w/ Jia Huang)

The *0-Hecke algebra* $H_n(0)$ over a field $\mathbb{F}$ has generators $\overline{\pi}_1, \overline{\pi}_2, \ldots, \overline{\pi}_{n-1}$ and relations

$$\begin{cases} \overline{\pi}_i^2 = -\overline{\pi}_i & 1 \leq i \leq n-1 \\ \overline{\pi}_i \overline{\pi}_j = \overline{\pi}_j \overline{\pi}_i & |i-j| > 1 \\ \overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1} & 1 \leq i \leq n-2 \end{cases}$$

$H_n(0)$ acts on $\mathbb{F}[\mathbf{x}_n]$ by

$$\overline{\pi}_i . f := \frac{x_{i+1} f - x_{i+1}(s_i . f)}{x_i - x_{i+1}}.$$

Variation 1: 0-Hecke algebras (w/ Jia Huang)

$H_n(0)$ acts on $\mathcal{OP}_{n,k}$:

$$\overline{\pi}_2.(13 \mid 2 \mid 45) = -(13 \mid 2 \mid 45)$$
$$\overline{\pi}_3.(13 \mid 2 \mid 45) = (14 \mid 2 \mid 35)$$
$$\overline{\pi}_4.(13 \mid 2 \mid 45) = 0$$

Q: Is there a 0-Hecke action on a quotient of $\mathbb{F}[\mathbf{x}_n]$ that reflects this?

Variation 1: 0-Hecke algebras (w/ Jia Huang)

Def: Let $J_{n,k} \subseteq \mathbb{F}[\mathbf{x}_n]$ be the ideal

$$J_{n,k} := \left\langle \begin{array}{c} e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n), \\ h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n) \end{array} \right\rangle.$$

Let $S_{n,k} := \frac{\mathbb{F}[\mathbf{x}_n]}{J_{n,k}}$ be the corresponding quotient.

Thm: [Huang-R.] We have $S_{n,k} \cong \mathcal{OP}_{n,k}$ as $H_n(0)$-modules.

Have formulas for the characters $\mathbf{ch}_t(S_{n,k}), \mathrm{Ch}_t(S_{n,k})$, and $\mathrm{Ch}_{q,t}(S_{n,k})$. Moreover, have $\mathrm{Ch}_t(S_{n,k}) = \mathrm{grFrob}(R_{n,k}; t)$.

Also have a generalization to the full Hecke algebra (w/ J. Huang and T. Scrimshaw).

Variation 2: $\mathbb{Z}_r \wr \mathfrak{S}_n$ (w/ Jonathan Chan)

$\mathbb{Z}_r \wr \mathfrak{S}_n = n \times n$ monomial matrices over $\mathbb{C}$, nonzero entries satisfy $\zeta^r = 1$.

For $r \geq 2$, we have a notion of 'faces in the $\mathbb{Z}_r \wr \mathfrak{S}_n$ Coxeter complex'.

Ex: A 3-dimensional face attached to $\mathbb{Z}_3 \wr \mathfrak{S}_9$:

$$(47 \mid 1^2 \mid 2^0 3^0 8^1 \mid 5^1 6^2 9^1)$$

$$\mathcal{OP}^r_{n,k} := \{k\text{-dim'l faces in } \mathbb{Z}_r \wr \mathfrak{S}_n \text{ Coxeter complex}\}$$

carries an action of $\mathbb{Z}_r \wr \mathfrak{S}_n$.

Variation 2: $\mathbb{Z}_r \wr \mathfrak{S}_n$ (w/ Jonathan Chan)

Def: For $r \geq 2$, let $I^r_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal

$$I^r_{n,k} := \left\langle \begin{array}{c} x_1^{kr+1}, x_2^{kr+1}, \ldots, x_n^{kr+1} \\ e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \ldots, e_{n-k+1}(\mathbf{x}_n^r) \end{array} \right\rangle.$$

Let $R^r_{n,k} := \frac{\mathbb{C}[\mathbf{x}_n]}{I^r_{n,k}}$ be the corresponding quotient.

Thm: [Chan-R.] We have $R^r_{n,k} \cong \mathcal{OP}^r_{n,k}$ as $\mathbb{Z}_r \wr \mathfrak{S}_n$-modules. (Also formulas for graded character, bases, etc.)

Variation 3: Other Delta Operators (w/ Andy Wilson)

Def: Let n, k, r be nonnegative integers. Set

$$L_{n,k,r} := \left\langle \begin{array}{c} e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-r+1}(\mathbf{x}_n), \\ h_k(\mathbf{x}_n), h_{k+1}(\mathbf{x}_n), \ldots, h_{k+n}(\mathbf{x}_n) \end{array} \right\rangle,$$

and $T_{n,k,r} = \mathbb{Q}[\mathbf{x}_n]/L_{n,k,r}$.

Thm: [R.-Wilson] The algebra of $T_{n,k,r}$ is controlled by the combinatorics of 'tail positive words'.

Thm: [R.-Wilson] We have

$$\mathrm{grFrob}(T_{n,k,1}; q) = \Delta_{s_{k-1,1^{n-1}}} e_n(\mathbf{x}) \mid_{t=0} .$$

Thanks for listening!!