VaR bounds for joint portfolios with \textit{(in)}dependence constraints

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https://sites.google.com/site/giovannipuccetti/
1. No dependence information


2. Independent subgroups


1. No dependence information

DNB risk portfolio used for ICAAP; see Aas & Puccetti (2014).

\[ S_d = X_1 + \cdots + X_d \]  

(total loss exposure)

Basel II(I) requirement: compute and reserve based on

\[ \text{VaR}_\alpha(S_d) \quad \text{or} \quad \text{TVaR}_\alpha(S_d) \]
Risk Measures

The *Value-at-Risk* (VaR) of a random loss $Y$ with distribution $F_Y$, computed at a probability level $\alpha \in (0, 1)$, is defined as

$$ VaR_\alpha(Y) = F_Y^{-1}(\alpha) = \inf \{ x \in \mathbb{R} : F_Y(x) > \alpha \}. $$

The *Tail-Value-at-Risk* (TVaR) or *Expected Shortfall* of a random loss $Y$ with finite mean $\mathbb{E}(Y)$ is defined by

$$ TVaR_\alpha(Y) = \frac{1}{1 - \alpha} \int_\alpha^1 VaR_q(Y) \, dq, $$

while the *Left Tail-Value-at-Risk* (LTVaR) of $Y$ is defined by

$$ LTVaR_\alpha(Y) = \frac{1}{\alpha} \int_0^\alpha VaR_q(Y) \, dq. $$
One period risks with statistically estimated marginals
and unknown dependence structure

\[ \mathbf{X} = (X_1, \ldots, X_d) \in \mathcal{F}, \]
\[ \mathcal{F} = \{(X_1, \ldots, X_d) : X_i \sim F_i, 1 \leq i \leq d\} \]
$X = (X_1, \ldots, X_d) \in \mathcal{F},$

$\mathcal{F} = \{(X_1, \ldots, X_d) : X_i \sim F_i, 1 \leq i \leq d\}$

one period risks with statistically estimated marginals

and unknown dependence structure

DU-SPREAD for VaR

$\overline{\text{VaR}}_\alpha(S_d) = \sup \{ \text{VaR}_\alpha(X_1 + \cdots + X_d) : X \in \mathcal{F} \}$

$\underline{\text{VaR}}_\alpha(S_d) = \inf \{ \text{VaR}_\alpha(X_1 + \cdots + X_d) : X \in \mathcal{F} \}$
\[ X = (X_1, \ldots, X_d) \in \mathcal{F}, \]

\[ \mathcal{F} = \{(X_1, \ldots, X_d) : X_i \sim F_i, 1 \leq i \leq d\} \]

one period risks with statistically estimated marginals

and unknown dependence structure

<table>
<thead>
<tr>
<th>DU-SPREAD for VaR</th>
<th>superadditive models</th>
<th>DU-SPREAD for TVAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{VaR}_\alpha(S_d) )</td>
<td>( \sum_{i=1}^{d} \text{VaR}_\alpha(X_i) )</td>
<td>( \text{TVaR}_\alpha(S_d) )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{d} \text{VaR}_\alpha(X_i) )</td>
<td>( \overline{\text{VaR}}_\alpha(S_d) )</td>
<td>( \sum_{i=1}^{d} \text{TVaR}<em>\alpha(X_i) = \overline{\text{TVaR}}</em>\alpha(S_d) )</td>
</tr>
</tbody>
</table>
DU-SPREAD for the 0.9997-VaR with marginals information only

Credit Risk
2.5e06 simulations

Market Risk
2.5e06 simulations

Ownership Risk
2.5e06 simulations

Operational Risk
LogNormal distribution

Business Risk
LogNormal distribution

Insurance Risk
LogNormal distribution

DU-SPREAD for the 0.9997-TVAR with marginals information only

X₁
Credit Risk
2.5e06 simulations

X₂
Market Risk
2.5e06 simulations

X₃
Ownership Risk
2.5e06 simulations

X₄
Operational Risk
LogNormal distribution

X₅
Business Risk
LogNormal distribution

X₆
Insurance Risk
LogNormal distribution

62,156.4

93,755.0

105,878.2

74,354.7

110,588.8
DU-SPREAD for the 0.9997-VaR with marginals information only

\[
\begin{align*}
X_1 & \quad \text{Credit Risk} \\
X_2 & \quad \text{Market Risk} \\
X_3 & \quad \text{Ownership Risk} \\
X_4 & \quad \text{Operational Risk} \\
X_5 & \quad \text{Business Risk} \\
X_6 & \quad \text{Insurance Risk}
\end{align*}
\]

\[
\begin{align*}
2.5\times10^6 & \text{ simulations} \\
2.5\times10^6 & \text{ simulations} \\
2.5\times10^6 & \text{ simulations} \\
\text{LogNormal distribution} \\
\text{LogNormal distribution} \\
\text{LogNormal distribution}
\end{align*}
\]

\[
\begin{align*}
62,156.4 & \quad 77,896.33 \\
& \text{(t-copula)} \\
& 93,755.0 \\
& 105,878.2
\end{align*}
\]

DU-SPREAD for the 0.9997-TVAR with marginals information only

\[
\begin{align*}
74,354.7 & \quad 110,588.8
\end{align*}
\]
Moscadelli(2004)’s operational risk portfolio

<table>
<thead>
<tr>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>X5</th>
<th>X6</th>
<th>X7</th>
<th>X8</th>
</tr>
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<tr>
<td>Corporate Finance</td>
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<td>Retail Brokerage</td>
</tr>
<tr>
<td>GPD(1.19)</td>
<td>GPD(1.17)</td>
<td>GPD(1.01)</td>
<td>GPD(1.39)</td>
<td>GPD(1.23)</td>
<td>GPD(1.22)</td>
<td>GPD(0.85)</td>
<td>GPD(0.98)</td>
</tr>
</tbody>
</table>
Moscadelli (2004)’s operational risk portfolio

DU-SPREAD for the 0.999-VaR with marginals information only

$4.38 \times 10^6$ $9.33 \times 10^6$ $43.4 \times 10^6$
Moscadelli(2004)’s operational risk portfolio

DU-SPREAD for the 0.999-VaR with marginals information only

4.38x10^6  9.33x10^6  43.4*10^6

VaR/TVaR DU-spread can be easily computed via the

The Rearrangement Algorithm project

https://sites.google.com/site/rearrangementalgorithm/
Additional info can reduce DU-spreads:

\( \tilde{\mathcal{F}}' \subset \tilde{\mathcal{F}} \)
Positive dependence assumptions do not help

\[ \leq_{\text{co}} (X_1, \ldots, X_8) \]

DU-S for the 0.999-VaR with marginals information only

\[ 30.62 \leq 150.12 \leq 248.24 \]
Positive dependence assumptions do not help

\[
\begin{align*}
X_1, X_2, X_3, X_4 & \subseteq_{\text{comonotonicity within}} X_5, X_6, X_7, X_8 \\
\text{Pareto(0.5) marginals} & \subseteq_{\text{independence between}} \text{Exp(1) marginals} \\
\end{align*}
\]

DU-S for the 0.999-VaR with marginals information only

\begin{align*}
30.62 & \quad 150.12 & \quad 248.24
\end{align*}

DU-S for the 0.999-VaR with marginals information and \( \leq_{\text{co assumption}} \) assumption

\begin{align*}
122.49 & \quad 150.12 & \quad 205.27
\end{align*}

Why positive dependence does not help?

- Positive dependence assumption:

\[
\text{If } (X_1^+, X_2^+) \leq_{\text{co}} (X_1, X_2) \\
\text{then } \text{TVaR}_\alpha(X_1^+ + X_2^+) \leq \text{TVaR}_\alpha(X_1 + X_2)
\]

- Negative dependence assumption:

\[
\text{If } (X_1, X_2) \leq_{\text{co}} (X_1^+, X_2^+) \\
\text{then } \text{TVaR}_\alpha(X_1 + X_2) \leq \text{TVaR}_\alpha(X_1^+ + X_2^+)
\]

It is easy to see that these ordering results can be generalized to arbitrary dimensions and law invariant, convex risk measure using the supermodular order.
2. Independent subgroups

In actuarial practice, very often some subgroups of the marginal risks can be assumed to be independent:

- inhomogeneous sources of risks are gathered in a top-level portfolio where e.g. the marginal risks represent loss types aggregated over different subsidiary companies.

- some risk categories such as catastrophic risk, operational risk and reinsurance risk can be assumed to be independent from market-driven losses deriving from market, credit, ownership risk.

Once decomposed a **risk portfolio into independent subgroups**, a popular assumption would be to put some **positive (maximal) dependence structure within** the subgroups.
Our model:

Let $\mathcal{I} = \{ I_1, \ldots, I_k \}$ be a partition of $\{1, \ldots, d\}$.

Assumptions:

1. The risk vector $(X_1, \ldots, X_d)$ have fixed distributions $F_1, \ldots, F_d$;
2. Let $X_{I_i} := (X_j, j \in I_i)$ denote the risk subvector of the $i$-th subgroup. The risk subvectors $X_{I_1}, \ldots, X_{I_k}$ are independent;
3. The risks $(X_j, j \in I_i)$ inside each subgroup are comonotonic.
Our model:

Let \( \mathcal{I} = \{I_1, \ldots, I_k\} \) be a partition of \( \{1, \ldots, d\} \).

Assumptions:

1. The risk vector \((X_1, \ldots, X_d)\) have fixed distributions \(F_1, \ldots, F_d\);

2. Let \( X_{I_i} := (X_j, j \in I_i) \) denote the risk subvector of the \( i \)-th subgroup. The risk subvectors \( X_{I_1}, \ldots, X_{I_k} \) are independent;

3. The risks \((X_j, j \in I_i)\) inside each subgroup are comonotonic.

For a given partition \( \mathcal{I} \), denote by

\[
\text{VaR}_{\alpha}^{\mathcal{I}}(S_d) \quad \text{and} \quad \text{TVaR}_{\alpha}^{\mathcal{I}}(S_d)
\]

the VaR and the TVaR of the aggregate position \( S_d \) under the above assumptions.
Let $X_1, \ldots, X_5$ be Pareto$(1/3)$ risks with $\overline{\text{VaR}}_{0.975}(S_5) = 19.7897$.

<table>
<thead>
<tr>
<th>partition</th>
<th>$\text{VaR}^\tau_{\alpha}(S_5)$</th>
<th>$\text{TVaR}^\tau_{\alpha}(S_5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 2, 3, 4, 5}$</td>
<td>12.1003</td>
<td>20.6481</td>
</tr>
<tr>
<td>${1, 2, 3, 4}, {5}$</td>
<td>10.4506</td>
<td>17.3080</td>
</tr>
<tr>
<td>${1, 2, 3}, {4, 5}$</td>
<td>9.6377</td>
<td>15.3021</td>
</tr>
<tr>
<td>${1, 2, 3}, {4}, {5}$</td>
<td>8.9734</td>
<td>14.2156</td>
</tr>
<tr>
<td>${1, 2}, {3, 4}, {5}$</td>
<td>8.4992</td>
<td>12.9841</td>
</tr>
<tr>
<td>${1, 2}, {3}, {4}, {5}$</td>
<td>7.7798</td>
<td>11.6279</td>
</tr>
<tr>
<td>${1}, {2}, {3}, {4}, {5}$</td>
<td>7.0264</td>
<td>10.0471</td>
</tr>
</tbody>
</table>
Pareto with infinite mean

Let $X_1, \ldots, X_5$ be Pareto(1.02) risks with $\overline{\text{VaR}}_{0.975}(S_5) = 786.60$.

<table>
<thead>
<tr>
<th>partition</th>
<th>$\text{VaR}_{\alpha}^T(S_5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 2, 3, 4, 5}$</td>
<td>210.6203</td>
</tr>
<tr>
<td>${1, 2, 3, 4}, {5}$</td>
<td>220.3223</td>
</tr>
<tr>
<td>${1, 2, 3}, {4, 5}$</td>
<td>224.1656</td>
</tr>
<tr>
<td>${1, 2, 3}, {4}, {5}$</td>
<td>228.5470</td>
</tr>
<tr>
<td>${1, 2}, {3, 4}, {5}$</td>
<td>230.7306</td>
</tr>
<tr>
<td>${1, 2}, {3}, {4}, {5}$</td>
<td>235.0663</td>
</tr>
<tr>
<td>${1}, {2}, {3}, {4}, {5}$</td>
<td>239.4450</td>
</tr>
</tbody>
</table>
$X_i \sim \text{Pareto}(\theta_i); \quad \theta_1 = 1.05, \theta_2 = 1.02, \theta_3 = 0.625, \theta_4 = 0.25, \theta_5 = 0.2.$

<table>
<thead>
<tr>
<th>$d = 5$</th>
<th>$\text{VaR}_{\alpha}(S_d)$</th>
<th>$\text{VaR}_{\alpha}^{\text{P}}(S_d)$</th>
<th>best partition</th>
<th>$\text{VaR}_{\alpha}^{\text{P}}(S_d)$</th>
<th>worst partition</th>
<th>$\text{VaR}_{\alpha}^{\text{+}}(S_d)$</th>
<th>$\text{VaR}_{\alpha}(S_d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.950$</td>
<td>22.41</td>
<td>47.10</td>
<td>${1, 2}, {3}, {4}, {5}$</td>
<td>54.21</td>
<td>${1}, {2, 3, 4, 5}$</td>
<td>50.11</td>
<td>121.83</td>
</tr>
<tr>
<td>$\alpha = 0.975$</td>
<td>47.57</td>
<td>94.83</td>
<td>${1, 2}, {3}, {4}, {5}$</td>
<td>106.56</td>
<td>${1, 3, 4, 5}, {2}$</td>
<td>101.33</td>
<td>238.56</td>
</tr>
<tr>
<td>$\alpha = 0.990$</td>
<td>126.42</td>
<td>241.58</td>
<td>${1, 2}, {3}, {4}, {5}$</td>
<td>263.84</td>
<td>${1, 3, 4, 5}, {2}$</td>
<td>255.74</td>
<td>584.84</td>
</tr>
</tbody>
</table>
Pareto, mixed

\[ X_i \sim \text{Pareto}(\theta_i); \quad \theta_1 = 1.05, \theta_2 = 1.02, \theta_3 = 0.625, \theta_4 = 0.25, \theta_5 = 0.2. \]

<table>
<thead>
<tr>
<th>( d = 5 )</th>
<th>( \text{VaR}_\alpha(S_d) )</th>
<th>( \text{VaR}^P_\alpha(S_d) )</th>
<th>best partition</th>
<th>( \text{VaR}^P_\alpha(S_d) )</th>
<th>worst partition</th>
<th>( \text{VaR}^+_{\alpha}(S_d) )</th>
<th>( \text{VaR}_\alpha(S_d) )</th>
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<td>584.84</td>
</tr>
</tbody>
</table>

- **Asymptotic rule of thumb**: the least VaR-based capital estimate (at the high regulatory confidence levels typically used) is produced by assuming that the **infinite-mean risks are comonotonic** and the **finite-mean risks are independent**. The greatest by obtaining the maximum number of independent infinite mean sums.

- **In general**: how to select a partition?
An OpRisk case study

We consider the database of OpRisk losses collected by Willis Professional Risks from public media (957 inflation-adjusted gross losses in Million GBP from 1974–2013)

<table>
<thead>
<tr>
<th>Business line (BL)</th>
<th>data size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Available (NA)</td>
<td>61</td>
</tr>
<tr>
<td>Corporate Finance</td>
<td>42</td>
</tr>
<tr>
<td>Trading &amp; Sales</td>
<td>201</td>
</tr>
<tr>
<td>Retail Banking</td>
<td>233</td>
</tr>
<tr>
<td>Commercial Banking</td>
<td>210</td>
</tr>
<tr>
<td>Payment &amp; Settlement</td>
<td>12</td>
</tr>
<tr>
<td>Agency Services</td>
<td>18</td>
</tr>
<tr>
<td>Asset Management</td>
<td>79</td>
</tr>
<tr>
<td>Retail Brokerage</td>
<td>49</td>
</tr>
<tr>
<td>Insurance (life)</td>
<td>17</td>
</tr>
<tr>
<td>Insurance (non-life)</td>
<td>35</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>957</strong></td>
</tr>
</tbody>
</table>

Table 1 Data size for each BL in the Willis dataset.

Willis = 958 obs.; Moscadelli = 47,000 obs.
<table>
<thead>
<tr>
<th>BL (initials)</th>
<th></th>
<th>$\xi_i$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF + P&amp;S + AS + INS</td>
<td>1</td>
<td>1.41</td>
<td>22.56</td>
</tr>
<tr>
<td>T&amp;S</td>
<td>2</td>
<td>0.88</td>
<td>128.47</td>
</tr>
<tr>
<td>RB</td>
<td>3</td>
<td>1.66</td>
<td>28.76</td>
</tr>
<tr>
<td>CB</td>
<td>4</td>
<td>1.20</td>
<td>83.49</td>
</tr>
<tr>
<td>AM + RB</td>
<td>5</td>
<td>1.06</td>
<td>20.18</td>
</tr>
</tbody>
</table>

**Table 2** Estimated parameter values assuming BLs to be GPD-distributed risks.

We select partitions (out of 52) applying a **hierarchical clustering**

<table>
<thead>
<tr>
<th>OpRisk, $d = 5$</th>
<th>$\alpha = 0.99$</th>
<th>Cl</th>
<th>$\alpha = 0.999$</th>
<th>Cl</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{VaR}_\alpha(S_d)$</td>
<td>$3.440 \cdot 10^5$</td>
<td>$8.677 \cdot 10^6$</td>
<td>$2.991 \cdot 10^6$</td>
<td>$[2.904, 3.084] \cdot 10^6$</td>
</tr>
<tr>
<td>${1, 4}, {2}$, ${3}$, ${5}$</td>
<td>$1.010 \cdot 10^5$</td>
<td>$[1.011, 1.028] \cdot 10^5$</td>
<td>$2.970 \cdot 10^6$</td>
<td>$[2.883, 3.061] \cdot 10^6$</td>
</tr>
<tr>
<td>${1, 4, 5}$, ${2}$, ${3}$</td>
<td>$1.004 \cdot 10^5$</td>
<td>$[0.995, 1.012] \cdot 10^5$</td>
<td>$2.976 \cdot 10^6$</td>
<td>$[2.889, 3.069] \cdot 10^6$</td>
</tr>
<tr>
<td>${1, 2, 4, 5}$, ${3}$</td>
<td>$1.001 \cdot 10^5$</td>
<td>$[0.992, 1.009] \cdot 10^5$</td>
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<tr>
<td>$\text{VaR}_\alpha(S_d)$</td>
<td>$3.618 \cdot 10^4$</td>
<td>$1.654 \cdot 10^6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
We select partitions (out of 52) applying a hierarchical clustering.

<table>
<thead>
<tr>
<th>BL (initials)</th>
<th>i</th>
<th>$\xi_i$</th>
<th>$\beta_i$</th>
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<tbody>
<tr>
<td>CF + P&amp;S + AS + INS</td>
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<td>4 {1, 4}, {2}, {3}, {5}</td>
<td>$1.010 \cdot 10^5$</td>
<td>[1.011, 1.028] · $10^5$</td>
<td>[1.022, 1.062] · $10^6$</td>
<td></td>
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<tr>
<td>3 {1, 4, 5}, {2}, {3}</td>
<td>$1.004 \cdot 10^5$</td>
<td>[0.995, 1.012] · $10^5$</td>
<td>[2.883, 3.061] · $10^6$</td>
<td></td>
</tr>
<tr>
<td>2 {1, 2, 4, 5}, {3}</td>
<td>$1.001 \cdot 10^5$</td>
<td>[0.992, 1.009] · $10^5$</td>
<td>[2.976, 3.062] · $10^6$</td>
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<td>1.41</td>
<td>22.56</td>
</tr>
<tr>
<td>T&amp;S</td>
<td>2</td>
<td>0.88</td>
<td>128.47</td>
</tr>
<tr>
<td>RB</td>
<td>3</td>
<td>1.66</td>
<td>28.76</td>
</tr>
<tr>
<td>CB</td>
<td>4</td>
<td>1.20</td>
<td>83.49</td>
</tr>
<tr>
<td>AM + RB</td>
<td>5</td>
<td>1.06</td>
<td>20.18</td>
</tr>
</tbody>
</table>

Table 2 Estimated parameter values assuming BLs to be GPD-distributed risks.

We select partitions (out of 52) applying a **hierarchical clustering**

<table>
<thead>
<tr>
<th>OpRisk, $d = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>$\text{VaR}_\alpha(S_d)$</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

More details and examples coming soon in a preprint.
A more general model

Let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be a partition of $\{1, \ldots, d\}$.

**Assumptions:**

1. The risk vector $(X_1, \ldots, X_d)$ have fixed distributions $F_1, \ldots, F_d$;

2. Let $X_{I_i} := (X_j, j \in I_i)$ denote the risk subvector of the $i$-th subgroup. The risk subvectors $X_{I_1}, \ldots, X_{I_k}$ are independent;

3. The risks $(X_j, j \in I_i)$ inside each subgroup are *comonotonic*. 
Let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be a partition of $\{1, \ldots, d\}$.

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3. The risks $(X_j, j \in I_i)$ inside each subgroup are comonotonic.

In this case, the VaR of the aggregate position $\text{VaR}_\alpha(S_d)$ cannot be computed, but one still has, for a given partition $\mathcal{I}$, that

$$\text{VaR}_\alpha(S_d) \leq \text{TVaR}_\alpha(S_d) \leq \text{TVaR}_\alpha^\mathcal{I}(S_d);$$

see Th. 2.1 in Puccetti et al.(2017).
A more general model

Let \( \mathcal{I} = \{I_1, \ldots, I_k\} \) be a partition of \( \{1, \ldots, d\} \).

**Assumptions:**

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2. Let \( X_{I_i} := (X_j, j \in I_i) \) denote the risk subvector of the \( i \)-th subgroup. The risk subvectors \( X_{I_1}, \ldots, X_{I_k} \) are independent;

3. The risks \( (X_j, j \in I_i) \) inside each subgroup are comonotonic.

In this case, the VaR of the aggregate position \( \text{VaR}_\alpha(S_d) \) cannot be computed, but one still has, for a given partition \( \mathcal{I} \), that

\[
\text{LTVaR}_\mathcal{I}^\alpha(S_d) \leq \text{VaR}_\alpha(S_d) \leq \text{TVaR}_\alpha(S_d) \leq \text{TVaR}_\mathcal{I}^\alpha(S_d);
\]

see Th. 2.1 in Puccetti et al. (2017).
Example

DU-S for the 0.999-VaR with marginals info only

121.5  304.63  367.70
Example

DU-S for the 0.999-VaR with marginals info only

121.5 304.63 367.70

DU-S for the 0.999-VaR with marginals info AND independent subgroups

121.5 256.04
A QIS 5 Hierarchical risk model

Subset of the hierarchical aggregation structure used to determine the total solvency capital requirements of insurance companies using the standard model of Quantitative Impact Study (QIS) 5
Our assumptions:

Partial independence structure based on the correlation matrices given in QIS 5.

• = independent subgroups
○ = no dependence assumptions within each other or with respect to the other subgroups

DU-SPREAD for the 0.995-VaR with marginals information only

140.1  1178.9  1701.7

DU-SPREAD for the for the 0.995-VaR with extra **partial** independent substructures

142.0  1178.9  1317.4
Examples of more efficient additional information:


- Marginals information replaced by the knowledge of some moments; see Bernard, C., M. Denuit, and S. Vanduffel (2016)

- Risk model fixed on a trusted region and allowed to vary elsewhere; see Bernard, C. and S. Vanduffel (2015)

- **Risk bounds for factor models**: see Bernard, C., L. Rüschendorf, S. Vanduffel, and R. Wang (2017);

- **More examples**: Rüschendorf, L. & Witting, J. (2017);

- [ ... ]
- **Model risk numerical toolkit:** We have analytical and numerical techniques available for the computation of VaR/ES DU spreads.

- **Adding positive dependence info is not useful** to reduce the worst bounds on a risk measure: one should instead assume some independence/negative dependence structure in order to reduce the upper bound on a risk.

- **There’s more under the top of the iceberg:** The risk assessment of a multivariate bank portfolio cannot be reduced to a single VaR number. The VaR/ES DU spread might help to assess the implied model risk.