Survey on the square and hexagonal model for random groups

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- What does a typical group look like?
- What properties are typical for groups?

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- What properties are typical for groups?

Random groups:

- have some „exotic” properties,
- provide examples hard to construct without random groups

Definition (General definition of random group)

$$G = \langle S | R \rangle$$

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- *R is a random set of relations*

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Question:
What properties hold with high probability when $|R| \to \infty$?

Definition (The square model)

$$G(n, d) = \langle S | R \rangle$$

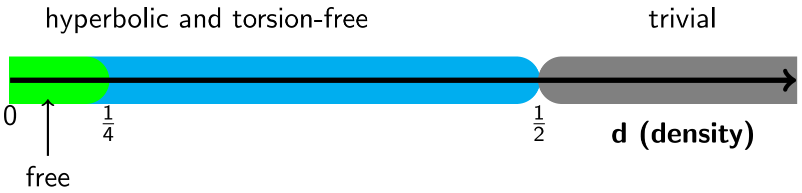
- *S is a finite set of n generators*
- *R is a set of $(2n-1)^{4d}$ relators chosen uniformly at random among about $(2n-1)^4$ words of length 4.*
- $d \in (0, 1)$ *is called the density*

A property $\mathcal{P}$ occurs with overwhelming probability (w.o.p.) if

$$\mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \longrightarrow 1,$$

as $n \longrightarrow \infty$.

The square model - basic results



- For $d < \frac{1}{2}$ hyperbolic, torsion-free, of dimension 2 [O. '13]
- For $d > \frac{1}{2}$ w.o.p. trivial [O. '13]
- For $d < \frac{1}{4}$ w.o.p. free [O. '13]

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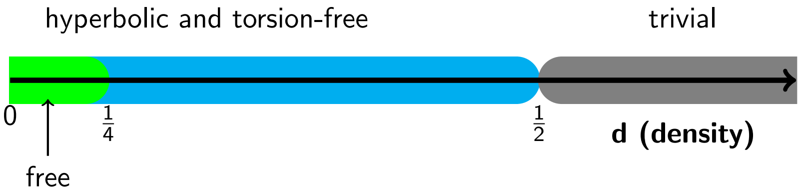

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We will show the idea of proving the hyperbolicity.

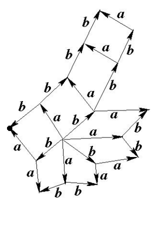
van Kampen diagram

Definition

Let $G = \langle S|R \rangle$. For every relation $r \in R$ we have a polygon with as many edges as letters in r. On every edge there is letter s.t. the the boundary word is r. A van Kampen diagram is a planar diagram obtained by gluing this polygons along corresponding edges.

Example

$$G = \langle a, b | aba^{-1}b^{-1} \rangle$$



Theorem (O. '13, based on Ollivier '07)

For any $\varepsilon > 0$ in the square model at density $d < \frac{1}{2}$ w.o.p. every reduced van Kampen diagram $\mathcal{D}$ w.r.t. the group presentation satisfies

$$|\partial\mathcal{D}| > 4(1 - 2d - \varepsilon)|\mathcal{D}| \qquad (1)$$

- $|\partial\mathcal{D}|$ - *number of edges in the boundary of* $\mathcal{D}$
- $|\mathcal{D}|$ - *number of 2-cells of* $\mathcal{D}$

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Corollary

Random group in the square model at density $d < \frac{1}{2}$ is w.o.p. hyperbolic.

Definition

A group G has Property (T) if for every real Hilbert space $\mathcal{H}$ every action of G on $\mathcal{H}$ via affine isometries has a fixed point.

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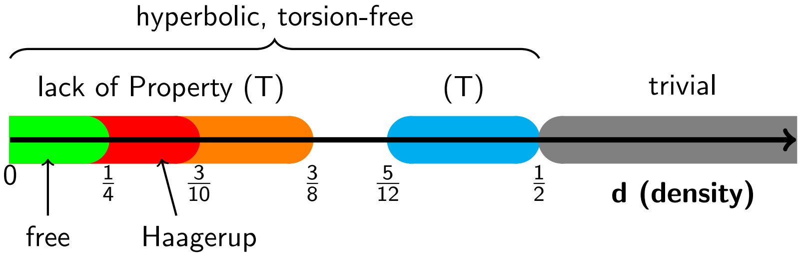
- Haagerup property is a strong negation of Property (T)
- If a group has Haagerup property then it satisfies Baum-Connes conjecture and Novikov conjecture.
- Property (T) was used to find an explicit family of expanders

The square model - further results



- For $d < \frac{3}{8}$ w.o.p. no Property (T) [O. '16]
- For $d < \frac{3}{10}$ w.o.p. Haagerup property [O. '16]
- For $d > \frac{5}{12}$ w.o.p. has (T) [Przytycki, Orlef, O. '16]

Space with walls

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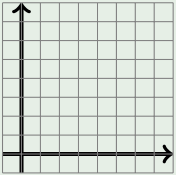
Example



$$A_i = \{(x, y) : x < i\}$$

$$B_i = \{(x, y) : y < i\}$$

For $i \in \mathbb{Z}$ pairs $\{A_i, A_i'\}$ and $\{B_i, B_i'\}$ form walls on $\mathbb{R}^2$.

Metric on a space with walls

Definition

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The distance between x and y in the wall metric d_{wall} is the number of walls separating x and y.

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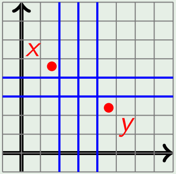

Wall distance between x and y is 5.

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Theorem (Chatterji, Niblo '04)

If a discrete group G acts properly on a space with walls then it acts properly on CAT(0) cube complex, so has Haagerup Property.

Theorem (Niblo, Roller '98)

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Finding subgroup with $\geqslant 2$ relative ends - using walls.

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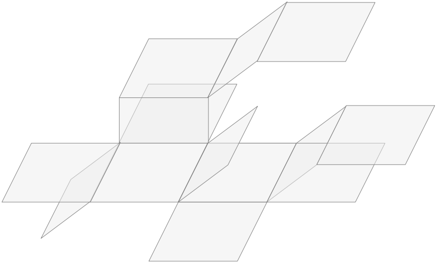

Definition (Ollivier, Wise, '11)

We define graph Γ*:*

- $V(\Gamma)$ *- set of midpoints of edges of* $\tilde{X}$.
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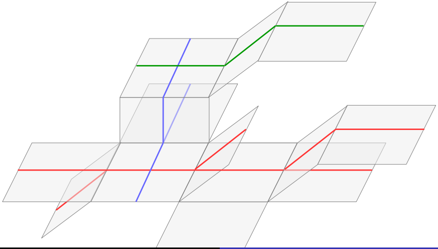

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- Embedded trees $\rightarrow$ split X into two connected components
- Two connected components $\rightarrow$ structure of a space with walls on X

Collared diagrams

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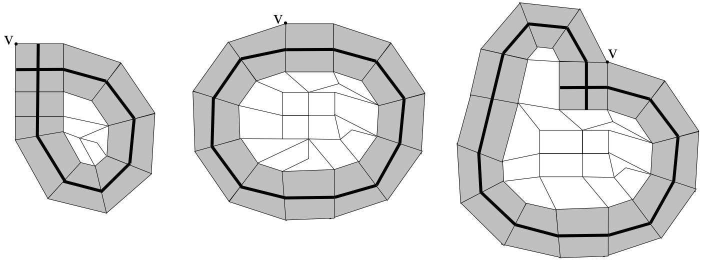

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Theorem (Ollivier, Wise '11, generalization O. '13)

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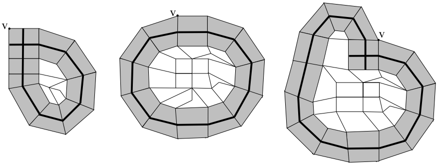

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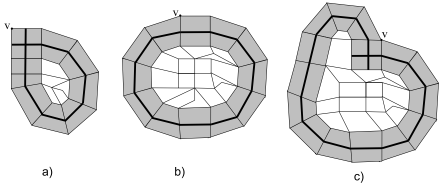
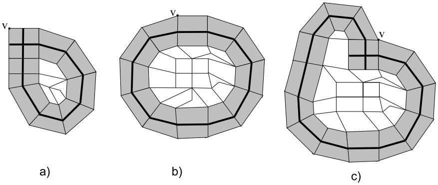
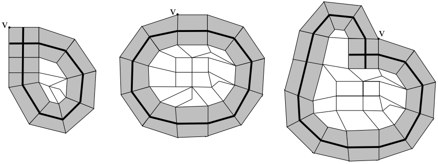


Remark

For density $d < \frac{1}{3}$ diagrams a), b) and c) violate Isoperimetric Inequality $\Rightarrow$ hypergarphs are embedded trees.



If the density $d < \frac{3}{8}$ only a) can occur. We can correct it to omit self-intersection.



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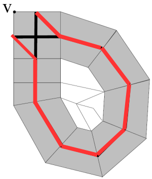

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- Group acts on a space with walls $\rightarrow$
- We check properties of this action (is it proper action?) $\rightarrow$
- Subgroup having $\leqslant 2$ relative ends is a stabilizer of some hypergraph $\rightarrow$
- We conclude lack of (T) or Haagerup property.

Definition (The hexagonal model)

$$G(n,d) = \langle S | R \rangle$$

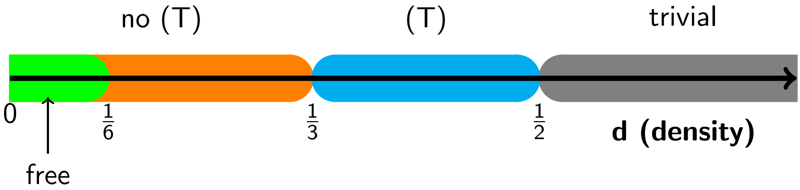
- *S is a finite set of n generators*
- *R is a set of $(2n-1)^{6d}$ relators chosen uniformly at random among about $(2n-1)^6$ words of length 6.*
- *$d \in (0,1)$ is called the density*

A property $\mathcal{P}$ occurs with overwhelming probability (w.o.p.) if

$$\mathbb{P}(\mathcal{P} \text{ holds for } G(n,d)) \longrightarrow 1,$$

as $n \longrightarrow \infty$.

The hexagonal model - results



- For $d < \frac{1}{6}$ free, and for $d > \frac{1}{2}$ trivial [O. '16]
- For $d < \frac{1}{3}$ w.o.p. no (T) [O. '16]
- For $d > \frac{1}{3}$ w.o.p. Property (T) [easy observation]

Sharp threshold for Property (T) is $\frac{1}{3}$.

Definition (The Gromov density model)

$$G(n, l, d) = \langle S | R \rangle$$

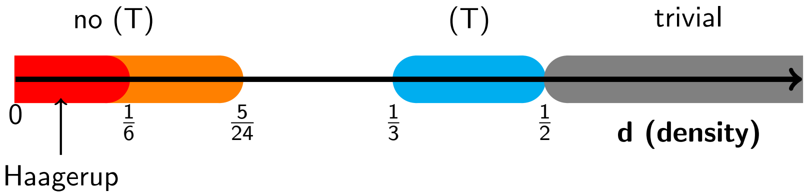
- *S is a finite set of n generators*
- *R is a set of $(2n-1)^{dl}$ relators chosen uniformly at random among about $(2n-1)^l$ words of length l.*
- *$d \in (0, 1)$ is called the density*
- *n is fixed but l goes to infinity.*

A property $\mathcal{P}$ occurs with overwhelming probability (w.o.p.) if

$$\mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \to 1,$$

as $l \to \infty$.

Results in the Gromov model



- for $d < \frac{1}{6}$ w.o.p. it has Haagerup property [Ollivier, Wise '08]
- for $d < \frac{5}{24}$ w.o.p. it does not have (T) [Przytycki, Mackay '14]
- For $d > \frac{1}{3}$ w.o.p. it has (T) [Żuk '03, Kotowski and Kotowski '13]

Definition (The k-angular model model)

$$G(n, d) = \langle S | R \rangle$$

- *S is a finite set of n generators*
- *R is a set of $(2n-1)^{dk}$ relators chosen uniformly at random among about $(2n-1)^k$ words of length k.*
- *$d \in (0, 1)$ is called the density*

A property $\mathcal{P}$ occurs with overwhelming probability (w.o.p.) if

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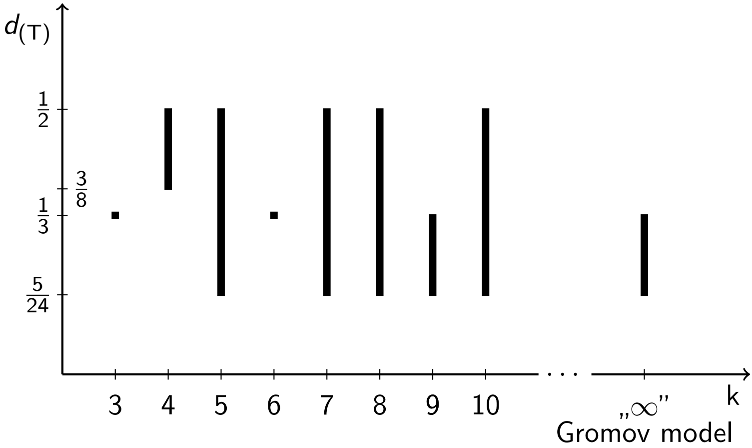
Definition

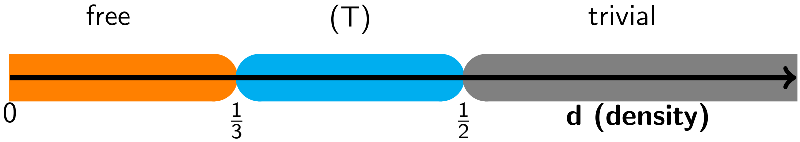
We say that $d_T \in (0,1)$ is a sharp threshold in a random group model for a property $\mathcal{P}$ if

- *for densities $d < d_T$ w.o.p. property $\mathcal{P}$ does not hold*
- *for densities $d > d_T$ w.o.p. property $\mathcal{P}$ holds.*

Sharp threshold for Property (T)



The triangular model (k=3)



- For $d > \frac{1}{3}$ w.o.p. Property (T) [Żuk '03, Kotowski and Kotowski '13]
- More detailed picture - Antoniuk, Łuczak, Świątkowski, Friedgut (series of papers).