Survey on the square and hexagonal model for random groups

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We want to consider questions:

- What does a typical group look like?
- What properties are typical for groups?
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- What properties are typical for groups?

Random groups:

- have some „exotic” properties,
- provide examples hard to construct without random groups
What are random groups?
Isoperimetric inequality
Property (T) and Haagerup Property
Walls and hypergraphs
Other models

Motivations
Definition
Square model - introduction

Definition (General definition of random group)

\[ G = \langle S \mid R \rangle \]

- \( S \) is a finite set of generators
- \( R \) is a random set of relations

We need some distribution to draw \(|R|\) at random.
Definition (General definition of random group)

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- \( S \) is a finite set of generators
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We need some distribution to draw \(|R|\) at random.

Question:
What properties hold with high probability when \(|R| \to \infty\)?
Definition (The square model)

\[ G(n, d) = \langle S | R \rangle \]

- \( S \) is a finite set of \( n \) generators
- \( R \) is a set of \((2n - 1)^{4d}\) relators chosen uniformly at random among about \((2n - 1)^4\) words of length 4.
- \( d \in (0, 1) \) is called the density

A property \( \mathcal{P} \) occurs with overwhelming probability (w.o.p.) if

\[ \mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \to 1, \]

as \( n \to \infty \).
The square model - basic results

- For $d < \frac{1}{2}$ hyperbolic, torsion-free, of dimension 2 [O. ’13]
- For $d > \frac{1}{2}$ w.o.p. trivial [O. ’13]
- For $d < \frac{1}{4}$ w.o.p. free [O. ’13]
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We will show the idea of proving the hyperbolicity.
van Kampen diagram

**Definition**

Let $G = \langle S | R \rangle$. For every relation $r \in R$ we have a polygon with as many edges as letters in $r$. On every edge there is a letter such that the boundary word is $r$. A van Kampen diagram is a planar diagram obtained by gluing these polygons along corresponding edges.

**Example**

$$G = \langle a, b | aba^{-1}b^{-1} \rangle$$
Theorem (O. ’13, based on Ollivier ’07)

For any $\varepsilon > 0$ in the square model at density $d < \frac{1}{2}$ w.o.p. every reduced van Kampen diagram $\mathcal{D}$ w.r.t. the group presentation satisfies

$$|\partial \mathcal{D}| > 4(1 - 2d - \varepsilon)|\mathcal{D}| \quad (1)$$

- $|\partial \mathcal{D}|$ - number of edges in the boundary of $\mathcal{D}$
- $|\mathcal{D}|$ - number of 2-cells of $\mathcal{D}$
Theorem (O. ’13, based on Ollivier ’07)

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$$|\partial D| > 4(1 - 2d - \varepsilon)|D|$$

- $|\partial D|$ - number of edges in the boundary of $D$
- $|D|$ - number of 2-cells of $D$

Corollary

Random group in the square model at density $d < \frac{1}{2}$ is w.o.p. hyperbolic.
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A group $G$ has Property (T) if for every real Hilbert space $\mathcal{H}$ every action of $G$ on $\mathcal{H}$ via affine isometries has a fixed point.
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- Haagerup property is a strong negation of Property (T)
- If a group has Haagerup property then it satisfies Baum-Connes conjecture and Novikov conjecture.
- Property (T) was used to find an explicit family of expanders
The square model - further results

- For $d < \frac{3}{8}$ w.o.p. no Property (T) [O. '16]
- For $d < \frac{3}{10}$ w.o.p. Haagerup property [O. '16]
- For $d > \frac{5}{12}$ w.o.p. has (T) [Przytycki, Orlef, O. '16]
Space with walls

**Definition**

*Space with walls* is a set $Y$ with a nonempty family $\mathcal{H}$ of nonempty subsets satisfying: for every $h \in \mathcal{H}$, $h'$ (completion in $Y$) belongs to $\mathcal{H}$. 

Example

$A_i = \{ (x, y) : x < i \}$

$B_i = \{ (x, y) : y < i \}$

For $i \in \mathbb{Z}$ pairs $\{A_i, A'_i\}$ and $\{B_i, B'_i\}$ form walls on $\mathbb{R}^2$. 

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For $i \in \mathbb{Z}$ pairs $\{A_i, A'_i\}$ and $\{B_i, B'_i\}$ form *walls* on $\mathbb{R}^2$. 
We say that a wall \( \{ h, h' \} \) separates points \( x, y \in Y \) if \( x \in h, y \in h' \) or \( x \in h', y \in h \).
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The distance between \(x\) and \(y\) in the wall metric \(d_{\text{wall}}\) is the number of walls separating \(x\) and \(y\).
**Metric on a space with walls**

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**Example**

Wall distance between \( x \) and \( y \) is 5.
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A group $G$ acts on a space with walls if the action preserves walls, i.e. for every $g \in G$ and wall $\{h, h'\}$ the pair $\{gh, gh'\}$ is a wall.
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Action of $G$ on a space with walls is proper if it is metrically proper, i.e. for every $p \in Y$ and a sequence $g_n$ of elements $G$ s.t. $d_G(e, g_n) \to \infty$ holds $d_{\text{wall}}(p, g_n(p)) \to \infty$. 
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Theorem (Chatterji, Niblo ’04)

If a discrete group $G$ acts properly on a space with walls then it acts properly on \text{CAT}(0) cube complex, so has Haagerup Property.
Theorem (Niblo, Roller '98)

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If a group has a subgroup with at least two relative ends then it acts nontrivially on CAT(0) cube complex.
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Finding subgroup with $\geq 2$ relative ends - using walls.
Definition

The Cayley complex $\tilde{X}$ of a group $G = \langle S | R \rangle$ is the universal cover of the presentation complex of $G$. 
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Definition (Ollivier, Wise, ’11)

We define graph $\Gamma$:

- $V(\Gamma)$ - set of midpoints of edges of $\tilde{\mathcal{X}}$.
- Vertices $x, y$ are jointed if are antipodal points of a 2-cell of $\tilde{\mathcal{X}}$. 

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Properties of hypergraphs

- Action of group on $\tilde{X}$ preserves the system of hypergraphs
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- Action of group on \( \tilde{X} \) preserves the system of hypergraphs

**Theorem (O. ’13)**

*In the square model at density \( d < \frac{1}{3} \) w.o.p. hypergraphs are embedded trees in \( \tilde{X} \).*
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**Theorem (O. ’16)**

*In the square model at density $d < \frac{3}{8}$ w.o.p. hypergraphs can be corrected to be embedded trees in $\tilde{X}$.***
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- Embedded trees $\rightarrow$ split $X$ into two connected components
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In the square model at density $d < \frac{3}{8}$ w.o.p. hypergraphs can be corrected to be embedded trees in $\tilde{X}$.

- Embedded trees $\rightarrow$ split $X$ into two connected components
- Two connected components $\rightarrow$ structure of a space with walls on $X$
Collared diagrams

Definition

A **collared diagram** is a disc diagram such that the corresponding hypergraph segment passes through all exterior 2-cells but no internal 2-cells.
Collared diagrams

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A collared diagram is a disc diagram such that the corresponding hypergraph segment passes through all exterior 2-cells but no internal 2-cells.
Theorem (Ollivier, Wise ’11, generalization O. ’13)

A hypergraph is not an embedded tree iff there exists a reduced diagram collared by this hypergraph.
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A hypergraph is not an embedded tree iff there exists a reduced diagram collared by this hypergraph.

Remark

For density $d < \frac{1}{3}$ diagrams a), b) and c) violate Isoperimetric Inequality
Theorem (Ollivier, Wise ’11, generalization O. ’13)

A hypergraph is not an embedded tree iff there exists a reduced diagram collared by this hypergraph.

Remark

For density \( d < \frac{1}{3} \) diagrams a), b) and c) violate Isoperimetric Inequality \( \Rightarrow \) hypergraphs are embedded trees.
If the density \( d < \frac{3}{8} \) only a) can occur. We can correct it to omit self-intersection.
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To sum up

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- Group acts on a space with walls
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- Subgroup having $\leq 2$ relative ends is a stabilizer of some hypergraph
To sum up

- Hypergraphs are embedded trees →
- They split $\tilde{X}$ into two connected components →
- Group acts on a space with walls →
- We check properties of this action (is it proper action?) →
- Subgroup having $\leq 2$ relative ends is a stabilizer of some hypergraph →
- We conclude lack of (T) or Haagerup property.
Definition (The hexagonal model)

\[ G(n, d) = \langle S | R \rangle \]

- \( S \) is a finite set of \( n \) generators
- \( R \) is a set of \((2n - 1)^{6d}\) relators chosen uniformly at random among about \((2n - 1)^6\) words of length 6.
- \( d \in (0, 1) \) is called the density

A property \( \mathcal{P} \) occurs with overwhelming probability (w.o.p.) if

\[ \mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \to 1, \]

as \( n \to \infty \).
The hexagonal model - results

- For $d < \frac{1}{6}$ free, and for $d > \frac{1}{2}$ trivial [O. ’16]
- For $d < \frac{1}{3}$ w.o.p. no (T) [O. ’16]
- For $d > \frac{1}{3}$ w.o.p. Property (T) [easy observation]

Sharp threshold for Property (T) is $\frac{1}{3}$. 
Definition (The Gromov density model)

\[ G(n, l, d) = \langle S|R \rangle \]

- \( S \) is a finite set of \( n \) generators
- \( R \) is a set of \((2n - 1)^d l\) relators chosen uniformly at random among about \((2n - 1)^l\) words of length \( l \).
- \( d \in (0, 1) \) is called the density
- \( n \) is fixed but \( l \) goes to infinity.

A property \( \mathcal{P} \) occurs with overwhelming probability (w.o.p.) if

\[ \mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \to 1, \]

as \( l \to \infty \).
Results in the Gromov model

- for $d < \frac{1}{6}$ w.o.p. it has Haagerup property [Ollivier, Wise '08]
- for $d < \frac{5}{24}$ w.o.p. it does not have (T) [Przytycki, Mackay '14]
- For $d > \frac{1}{3}$ w.o.p. it has (T) [Żuk '03, Kotowski and Kotowski '13]
Definition (The $k$-angular model model)

$$G(n, d) = \langle S \mid R \rangle$$

- $S$ is a finite set of $n$ generators
- $R$ is a set of $(2n - 1)^{dk}$ relators chosen uniformly at random among about $(2n - 1)^k$ words of length $k$.
- $d \in (0, 1)$ is called the density

A property $\mathcal{P}$ occurs with overwhelming probability (w.o.p.) if

$$\mathbb{P}(\mathcal{P} \text{ holds for } G(n, d)) \to 1,$$

as $n \to \infty$. 

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Let $k = mk'$. 
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- If at density $d$ Property (T) holds w.o.p. in the $k'$-angular model it holds w.o.p. in the $k$-angular model.
Let $k = mk'$. 

- If at density $d$ Property (T) holds w.o.p. in the $k'$-angular model it holds w.o.p. in the $k$-angular model.
- If at density $d$ Property (T) does not hold w.o.p. in the $k$-angular model it w.o.p. does not hold in the $k'$-angular model.
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- If at density $d$ Property (T) holds w.o.p. in the $k'$-angular model it holds w.o.p. in the $k$-angular model.
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**Definition**

We say that $d_T \in (0,1)$ is a **sharp threshold** in a random group model for a property $\mathcal{P}$ if

- for densities $d < d_T$ w.o.p. property $\mathcal{P}$ does not hold
- for densities $d > d_T$ w.o.p. property $\mathcal{P}$ holds.
Sharp threshold for Property (T)

The graph shows the sharp threshold for Property (T) in the Gromov model. The x-axis represents the parameter k, and the y-axis represents the value of $d(T)$. The graph indicates critical values for different values of k, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{24}$, and ... for various values of k.
The triangular model \((k=3)\)

- For \(d > \frac{1}{3}\) w.o.p. Property (T) [Żuk ’03, Kotowski and Kotowski ’13]
- More detailed picture - Antoniuk, Łuczak, Świątkowski, Friedgut (series of papers).