Probabilistic Methods in Topology
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Integral homology of random simplicial complexes

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The random complex process

Definition

By $\mathcal{Y}(n) = \{Y(n, M) : 0 \leq M \leq \binom{n}{3}\}$ we mean the (2-dim) random complex process in which we add one by one, in a random order, 2-dim simplices to the full 1-dim skeleton on n points.

We say that $\mathcal{Y}(n)$ has some property whp if the probability that $\mathcal{Y}(n)$ has this property tends to 1 as $n \to \infty$.

Definition

The hitting time $\tau(A)$ for some complex property A is the random variable defined as

$$\tau(A) = \min\{M : Y(n, M) \text{ has } A\}.$$

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$$\tau(A) = \min\{M : Y(n,M) \text{ has } A\}.$$

Main Result

Theorem Łuczak, Peled'16+

Let $\mathcal{Y}(n) = \{Y(n, M) : 0 \leq M \leq \binom{n}{3}\}$ be the random complex process. Then whp

$$\tau(H_1(Y(n, M), \mathbb{Z}) = 0) = \tau(\delta_2 > 0)\,,$$

where $\delta_2 > 0$ is the property that each pair of vertices $\{i, j\} \in [n]^2$ is contained in at least one simplex from $Y(n, M)$.

Main Result: comments

Theorem Łuczak, Peled'16+

Whp $\tau(H_1(Y(n,M),\mathbb{Z})=0)=\tau(\delta_2>0)$.

Remark

Clearly, $\tau(H_1(Y(n,M),\mathbb{Z})=0)\geq\tau(\delta_2>0)$.

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$\tau(H_1(Y(n,M),\mathbb{Z})=0)=\tau(\delta_2>0)=\frac{2\log n+O(1)}{n}\binom{n}{3}$.

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We prove a slightly stronger result about the structure of $H_1(Y(n,M),\mathbb{Z})$ just slightly below the threshold for $\delta_2>0$.

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History

Theorem Linial Meshulam'06

The threshold for the property that $H_1(Y(n,p),\mathbb{F}_2) = 0$ is the same as for the property that $\delta_2 > 0$, i.e. if $pn = 2\log n + \omega(n)$, then

$$\Pr(H_1(Y(n,p),\mathbb{F}_2) = 0) \to \begin{cases} 0 & \text{if} \quad \omega(n) \to -\infty\,, \\ 1 & \text{if} \quad \omega(n) \to \infty\,. \end{cases}$$

Theorem Meshulam, Wallach'09

$\mathbb{F}_2$ can be replaced by any given finite field $\mathbb{F}_q$.
An analogous result holds in higher dimensions.

Theorem Kahle, Pittel'16

Whp $\tau(H_1(Y(n,p),\mathbb{F}_2) = 0) = \tau(\delta_2 > 0)$.

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In the above results one can always replace $\mathbb{F}_2$ by $\mathbb{Q}$.

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As for the homology over $\mathbb{Z}$ ...

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For a given q and $pn = 2\log n + \omega(n)$, we have

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A QUOTE

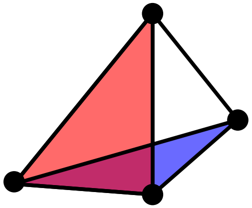
Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different .

Johann Wolfgang von Goethe

A QUOTE

Combinatorialists are like Frenchmen: whatever you say to them they translate into their own language (and forthwith it is something entirely different).

Topology for combinatorialists



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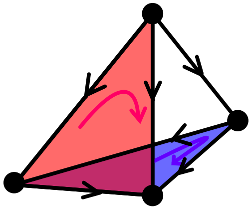

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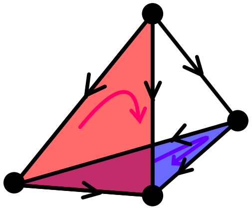
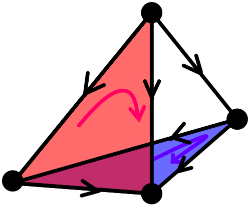


-1	-1
-1	0
$+1$	0
0	-1
0	$+1$
0	0

Topology for combinatorialists



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Definition of the shadow of Y (over Γ)

$$\mathrm{SH}(Y, \Gamma) = \{f : H_1(Y \cup f, \Gamma) = H_1(Y, \Gamma)\}$$

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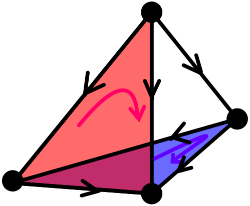

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$$\mathrm{SH}(Y,\Gamma) = \{f : H_1(Y \cup f, \Gamma) = H_1(Y,\Gamma)\}$$
$$= \{f : f \in \mathrm{Span}(Y,\Gamma)\}$$

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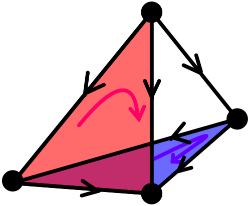

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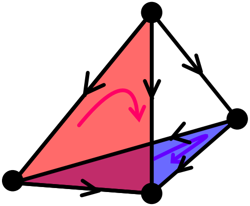
$$\begin{aligned}\mathrm{SH}(Y,\Gamma) &= \{f : H_1(Y \cup f, \Gamma) = H_1(Y,\Gamma)\}\\ &= \{f : f \in \mathrm{Span}(Y,\Gamma)\}\end{aligned}$$

Observation

$$H_1(Y,\Gamma) = 0 \quad \Longleftrightarrow \quad \mathrm{SH}(Y,\Gamma) = \binom{[n]}{3}.$$

A torsion factor



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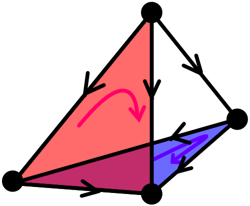

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Definition of the shadow of Y (over Γ)

$\mathrm{SH}(Y, \Gamma) = \{f : f \in \mathrm{Span}(Y, \Gamma)\}$

Definition of the weak shadow of Y (over Γ)

$\mathrm{SH}_w(Y, \Gamma) = \{f : \exists_{k \in \Gamma} \quad k \cdot f \in \mathrm{Span}(Y, \Gamma)\}$

Integral homology

Theorem Kahle, Pittel'16

Whp $\tau(\mathrm{SH}(Y(n,M),\mathbb{Q}) = \binom{[n]}{3})) = \tau(\delta_2 > 0)$.

Corollary Kahle, Pittel'16

Whp $\tau(\mathrm{SH}_w(Y(n,M),\mathbb{Z}) = \binom{[n]}{3})) = \tau(\delta_2 > 0)$.

Theorem Hoffman, Kahle, Paquette'16+

If $np \geq 80 \log n$, then whp $\mathrm{SH}((Y(n,p),\mathbb{Z})) = \binom{[n]}{3})$.

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INTEGRAL HOMOLOGY

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Integral homology

Theorem UCT

Let Y be a 2-dim simplicial complex on n vertices. Then, if for all primes q

$$\mathrm{SH}(Y, \mathbb{F}_q) = \binom{[n]}{3},$$

then also

$$\mathrm{SH}(Y, \mathbb{Z}) = \binom{[n]}{3}.$$

INTEGRAL HOMOLOGY

THEOREM KALAI'83

Let Y be a 2-dim simplicial complex on n vertices. Then, if for all primes q such that $q \leq \sqrt{3}^{\binom{n-1}{2}} \leq \exp(n^2)$

$$\mathrm{SH}(Y, \mathbb{F}_q) = \binom{[n]}{3},$$

then also

$$\mathrm{SH}(Y, \mathbb{Z}) = \binom{[n]}{3}.$$

An idea!

Idea

Let $A(\mathbb{F})$ denote the event that at the threshold $\tau_2 = \tau(\delta_2 > 0)$ we have $\mathrm{SH}(Y(n, \tau_2), \mathbb{F})) = \binom{[n]}{3}$.
If we prove that for every $\mathbb{F} = \mathbb{F}_q$ we have

$$\mathbb{P}(\overline{A(\mathbb{F}_q)}) \leq \exp(-2n^2)$$

we are done!

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Indeed, if this is the case

$$\mathbb{P}\left(\mathrm{SH}(Y(n,\tau_2),\mathbb{Z}) < \binom{[n]}{3}\right) \leq \mathbb{P}\left(\overline{\bigcap_{q \leq \exp(n^2)} A(\mathbb{F}_q)}\right)$$

$$\leq \mathbb{P}\left(\bigcup_{q \leq \exp(n^2)} \overline{A(\mathbb{F}_q)}\right) \leq \sum_{q \leq \exp(n^2)} \mathbb{P}(\overline{A(\mathbb{F}_q)}) = o(1).$$

Unfortunately, it does not quite work

Problem

It is not true that for $\mathbb{F} = \mathbb{F}_q$ we have

$$\mathbb{P}(\overline{A(\mathbb{F}_q)}) \leq \exp(-2n^2).$$

In fact,

$$\mathbb{P}(\overline{A(\mathbb{F}_q)}) \geq n^{-4}.$$

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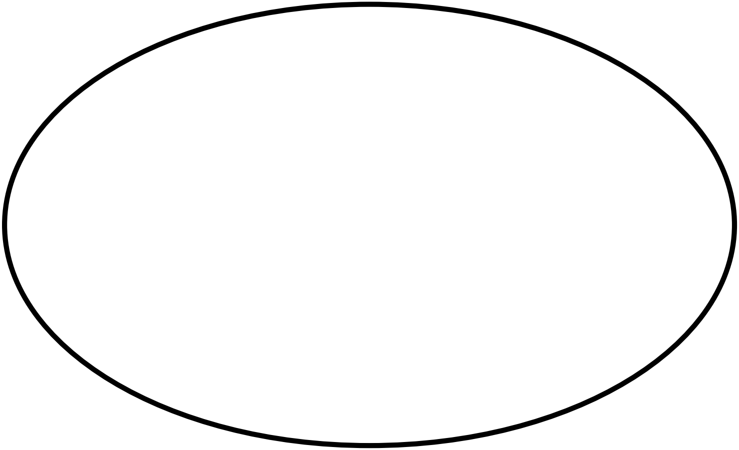

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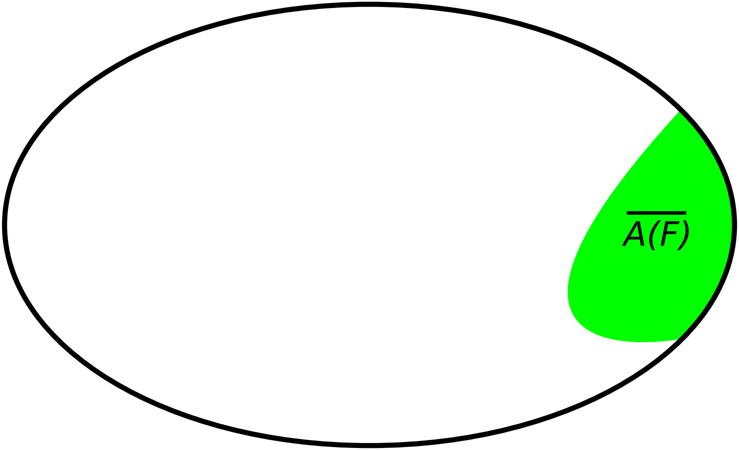

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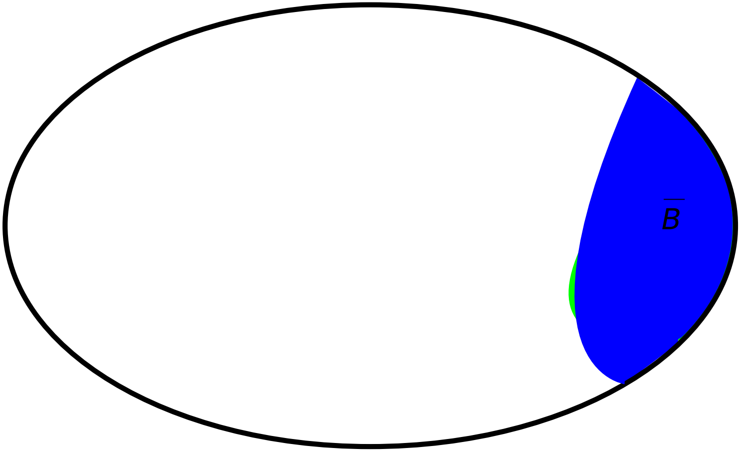

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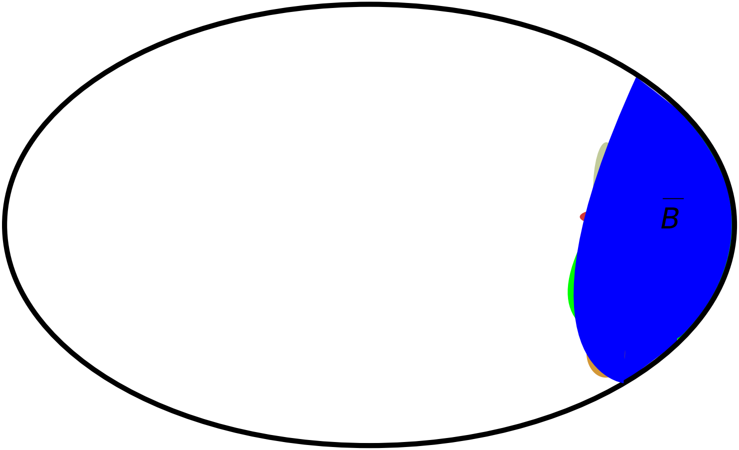

BUT, THERE MIGHT BE A CHANCE...

Let us recall the problem: the following estimate did not work since $\mathbb{P}(\overline{A(\mathbb{F}_q)})$ is too large.

$$\mathbb{P}\Big(\bigcup_{q \leq \exp(n^2)} \overline{A(\mathbb{F}_q)}\Big) \leq \sum_{q \leq \exp(n^2)} \mathbb{P}(\overline{A(\mathbb{F}_q)}).$$

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But we can estimate it more carefully

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{q\le\exp(n^2)}\overline{A(\mathbb{F}_q)}\Big)&=\mathbb{P}\Big(\overline{B}\cup\bigcup_{q\le\exp(n^2)}(\overline{A(\mathbb{F}_q)}\setminus\overline{B})\Big)\\&\le\mathbb{P}(\overline{B})+\sum_{q\le\exp(n^2)}\mathbb{P}(\overline{A(\mathbb{F}_q)}\setminus\overline{B}).\end{aligned}$$

The above expression tends to 0 provided:

- $\mathbb{P}(\overline{B})\to 0$,
- for each $\mathbb{F}$ we have $\mathbb{P}(\overline{A(\mathbb{F}_q)}\setminus\overline{B})\le\exp(-2n^2)$.

THE IDEA OF THE PROOF

Instead of $A(\mathbb{F})$, which means

$$\mathrm{SH}(Y(n,\tau_2),\mathbb{F}) = \binom{[n]}{3}, \tag{1}$$

we consider events $C(\mathbb{F})$, which say that

$$\mathrm{SH}(Y(n,\tau_2),\mathbb{F}) \geq \binom{[n]}{3} - \frac{n^3}{\log\log n}, \tag{2}$$

and B which states that **for each field** $\mathbb{F}$**,** (2) implies (1).

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Then the result would follow from the following two lemmata:

ALGEBRAIC LEMMA

For any field $\mathbb{F}$, we have $\mathbb{P}(\overline{C(\mathbb{F})}) \leq \exp(-2n^2)$.

COMBINATORIAL LEMMA

$\mathbb{P}(\overline{B}) = o(1)$.

Algebraic part of the proof

Algebraic Lemma

For any field $\mathbb{F}$, we have

$$\mathbb{P}\left(\mathrm{SH}(Y(n,\tau_2),\mathbb{F}) \leq \binom{[n]}{3} - \frac{n^3}{\log\log n}\right) \leq \exp(-2n^2)\,.$$

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Proof Let X be the random variable which count the number of steps in the process when we have chosen a simplex outside the shadow.
If at τ_2 the shadow is not too large, then $X \ge B(\tau_2, 6/\log\log n)$, where the binomial r.v. $B(\tau_2, 6/\log\log n)$ has the expectation $6\tau_2/\log\log n \gg n^2$.

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However, each time we add a simplex which is not in the shadow we increase it, so, clearly, we must have $X \le \binom{n}{2}$!

Algebraic part of the proof

Algebraic Lemma

For any field $\mathbb{F}$, we have

$$\mathbb{P}\left(\mathrm{SH}(Y(n,\tau_2),\mathbb{F}) \geq \binom{[n]}{3} - \frac{n^3}{\log\log n}\right) \leq \exp(-2n^2)\,.$$

Proof (cont.) Thus

$$\mathbb{P}\left(\mathrm{SH}(Y(n,\tau_2),\mathbb{F}) \leq \binom{[n]}{3} - \frac{n^3}{\log\log n}\right)$$
$$\leq \mathbb{P}\left(B\left(\tau_2, \frac{6}{\log\log n}\right) \leq \binom{n}{2}\right) \leq \exp(-2n^2). \quad \square$$

Some combinatorics (at last)!

Definition

Let Y be a 2-dim simplicial complex on n points. We say that a family of triples D is rich, if it has the following three properties:

- each 2-dim simplex from Y is in D,
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- if T is a triangulation of a sphere and $|T| - 1$ elements of T belong to D, then the remaining element of T also lies in D.

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An (unpleasant) observation

A crude bound for the number of rich families is

$$\binom{n^3}{n^3/\log\log n} = \exp(\bar{O}(n^3))$$

while no natural interesting property of $Y(n, \tau_2)$ has probability ρ such that

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A compact description of a rich family

Definition

Let D be a rich family in Y.
A triple is blue if it is in D and red otherwise.
An edge is red if it belongs to at least $n/\log_{(5)} n$ red triangles, and blue otherwise.
A point is red if it belongs to at least $n/\log_{(4)} n$ red edges, and blue otherwise.

Fact 0

Every rich family contains

- at most $n/\log\log n$ red triangles,
- at most $n/\log_{(3)} n$ red edges,
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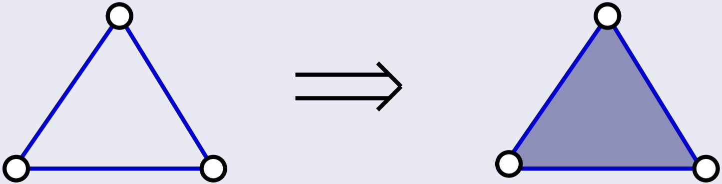
Reconstructing the family

Fact 1



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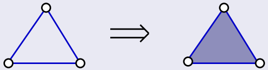

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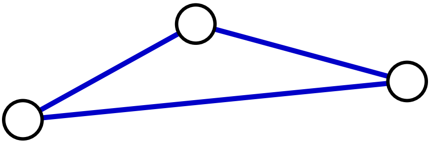

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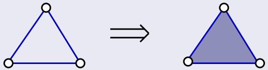

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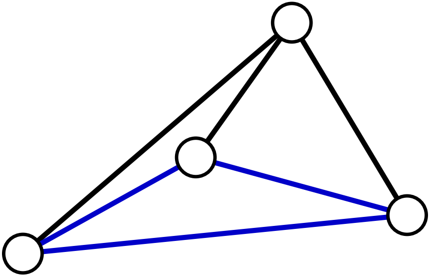

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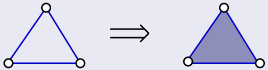

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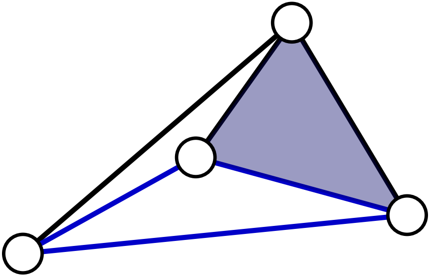

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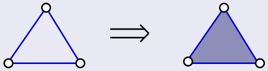

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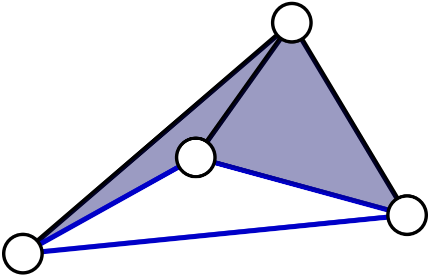

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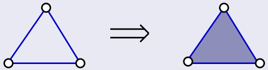

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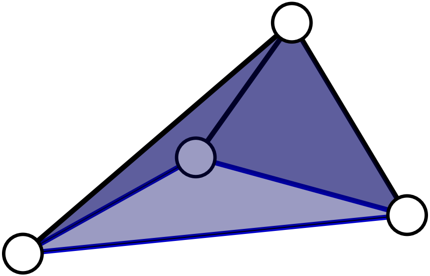

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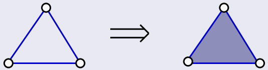

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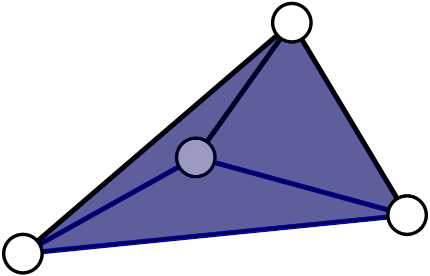

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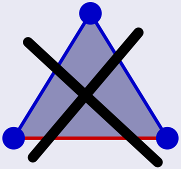

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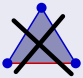

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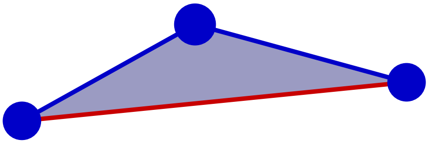

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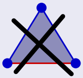

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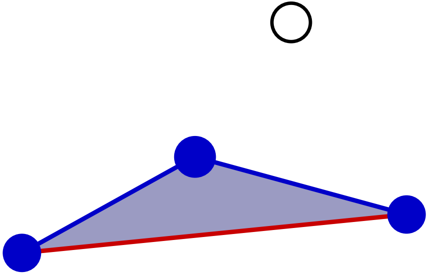

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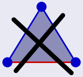

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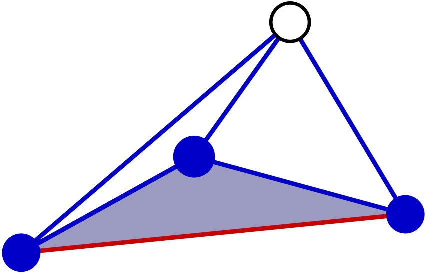

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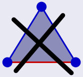

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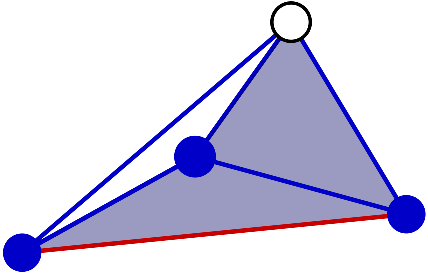

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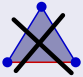

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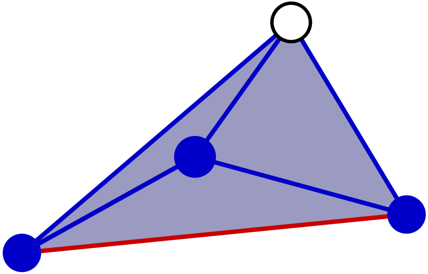

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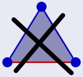

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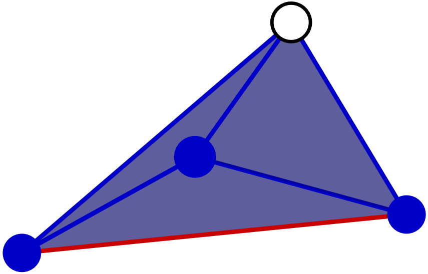

Red edges with blue ends

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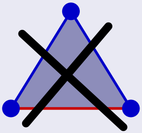

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Whp $Y(n, \tau_2)$ contains at most $n/\log\log n$ red edges with two blue ends.

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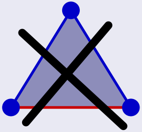

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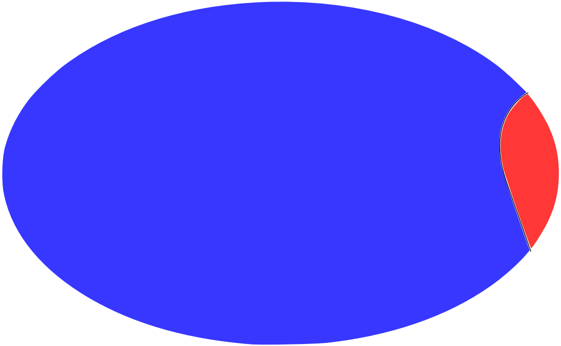

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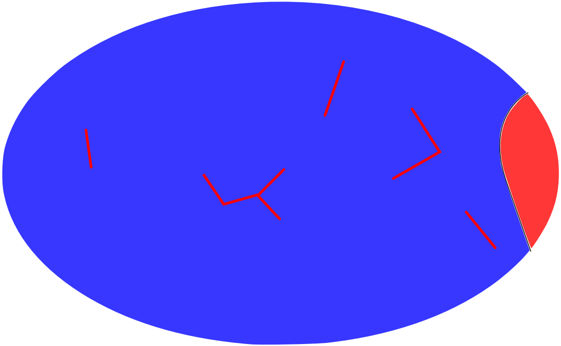

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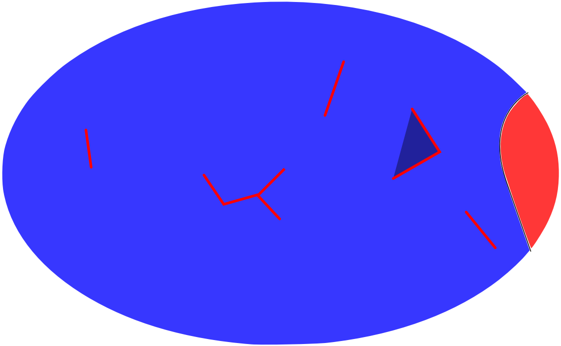

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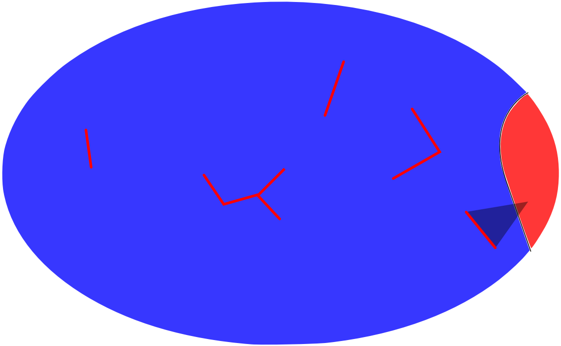

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Whp $Y(n, \tau_2)$ contains a subset W of at least $n - 2n/\log\log n$ vertices which is "totally blue", i.e. all vertices, edges, and triangles contained in it are blue.

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Eliminating red vertices

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The link graph of the vertex is a random graph $G(n-1, p)$ with $p \geq \log n/n$. We shall use the following probabilistic fact.

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With probability at least $1 - o(n^{-1})$ if we remove from $G(n-1, p)$ any set of $2n/\log\log n$ vertices, the graph obtained in this way contains a component of size at least $n - 4n/\log\log n$. □

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Corollary

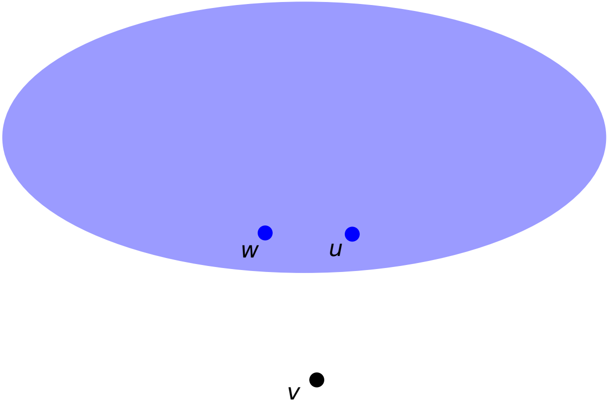
Whp the link of each vertex v of $Y(n, \tau_2)$ has a giant component $L(v)$ (with at least $n - 4n/\log\log n$ vertices) contained in the totally blue set W.

Giant component in the link

Observation

If the link of a vertex v has a component $L(v)$ contained in W then for all $w, u \in L(v)$ the triangle vwu is blue.

Proof:



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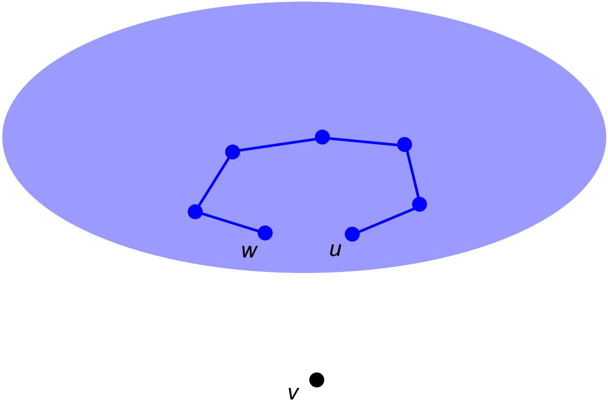

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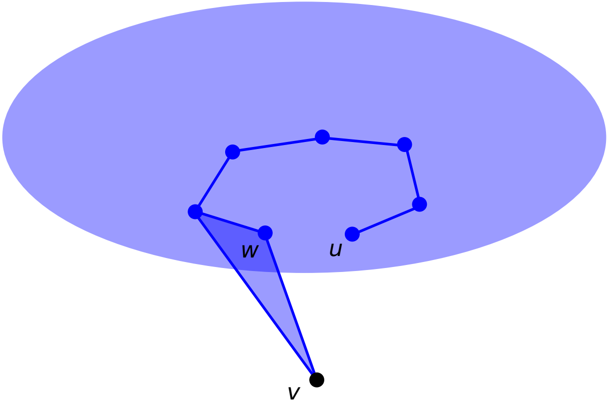

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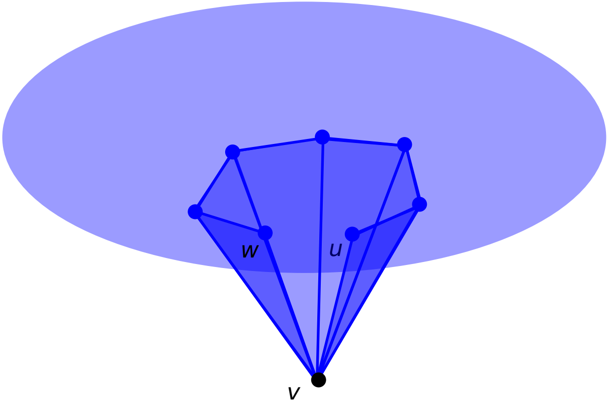

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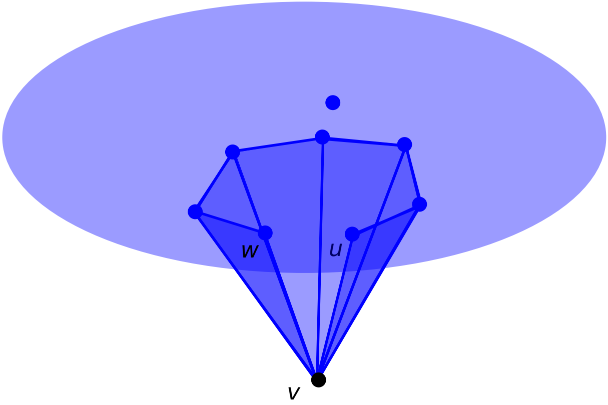

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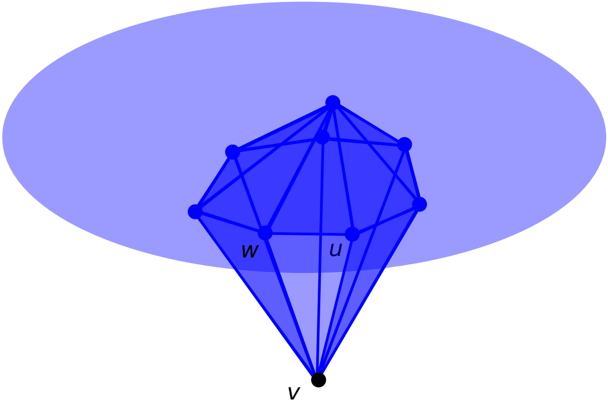

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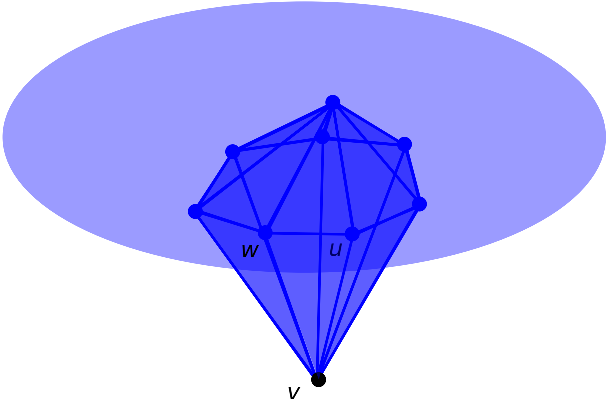

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Proof Whp the link of each vertex has a giant component in W, so it belongs to many blue triangles. But no red vertex can be contained in so many triangles, so the assertion follows. □

Completing the proof

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Whp $Y(n, \tau_2)$ contains no red edges (and so no red triangles).

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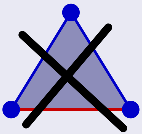

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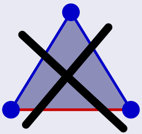

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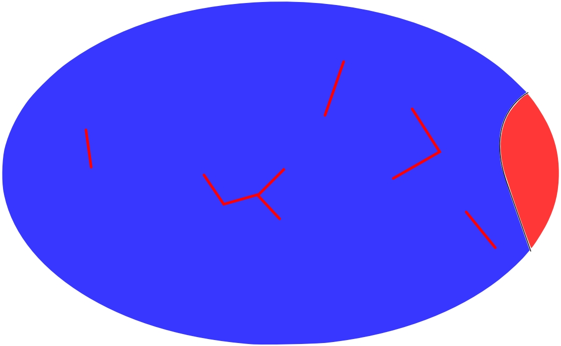

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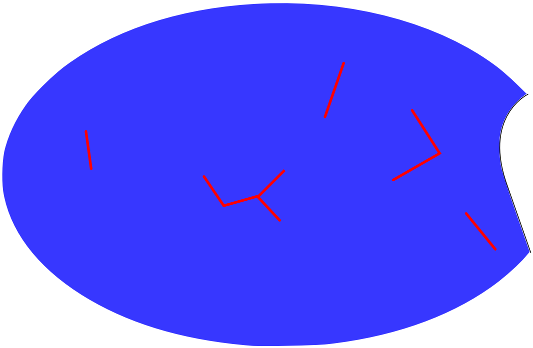

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Theorem Łuczak, Peled'16+

Whp $\tau(H_1(Y(n,M),\mathbb{Z})=0)=\tau(\delta_2>0)$.

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Theorem Łuczak, Peled'16+

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A few conjectures and open problems

Combinatorial Metaconjecture

(Most of) topological properties of sparse random structures do not depend on the choice of the field (or ring).

Conjecture

For every function $M = M(n)$ whp for each two fields $\mathbb{F}'$ and $\mathbb{F}''$

$$\mathrm{SH}(Y(n, M), \mathbb{F}') = \mathrm{SH}(Y(n, M), \mathbb{F}'') \, .$$

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A few conjectures and open problems

Problem

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THANK YOU!