The space of triangulations of a compact 4-dimensional manifold.

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Triangulations

Let T be a simplicial complex. We call T a triangulation of a PL-manifold M if there is a PL-homeomorphism $T \to M$.

Definition

For a PL-manifold M we define $T_n(M)$ to be the set of triangulations of M with the number of top-dimensional simplices $\leq n$ (up to simplicial isomorphisms). Let $T(M)$ be the union of $T_n(M)$.

Two simplicial complexes are triangulations of the same PL-manifold if and only if they have a common subdivision.

Pachner moves

Let $\partial\Delta^{d+1} = T_1 \sqcup T_2$, two disks. Let $T_1 \subset T$, where T is a triangulation of a PL-manifold of dimension d. Then we say T' is obtained by a bistellar transformation from T if $T' = T_2 \sqcup (T \setminus T_1)$.

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Every two triangulations of a closed PL manifold are related by a sequence of bistellar transformations.

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We put a metric on $T(M)$:

Definition

For $T_1, T_2 \in T(M)$ we set $d(T_1, T_2)$ to be the minimal number of bistellar transformations between them.

Results

Let $\exp_m(n)$ be defined by $\exp_m(n) = 2^{\exp_{m-1}(n)}$.

Theorem A (L-Nabutovsky)

There exists $C > 1$ such that for each closed PL 4-manifold M and for each m, there exists $S \subset T_n(M)$ with the property: $|S| > C^n$ and for any $T_1, T_2 \in S$, $d(T_1, T_2) > \exp_m(n)$.

This theorem is not true in dimension 3, at least for $M = S^3$ (Mijatovic).

For $d > 4$, $\exp_m$ can be replaced by any computable function (Nabutovsky).

Tietze transformations

Tietze transformations of group presentations:
1) A relator is conjugated by a generator.
2) A reletor is inverted.
3) A relator is replaced by a product with another relator.
4) A generator g and a relation $g = w$ are added, where w is a word not containing g, g^{-1}.
5) The addition of an empty relator.

Theorem (Tietze)

Any two finite presentations of the same group are connected by a sequence of Tietze transformations.

The moves 1)-4) applied to presentations of the trivial group are called (stable) Andrews-Curtis transformations. We say a balanced presentation of the trivial group is AC-reducible if it can be brought to the trivial presentation $\langle \mid \rangle$ by a sequence of transformations 1)-4).

Balanced presentations of the trivial group

For a finite presentation μ, denote by $l(\mu)$ the sum of the lengths of all relations plus the number of generators.
For presentations μ_1, μ_2, denote by $d(\mu_1, \mu_2)$ the minimal number of Tietze transformations between them.

Theorem B.1 (L-Nabutovsky)

There exists $C > 1$ such that for all m for all sufficiently large n, there exist more than C^n presentations of the trivial group of length $\leq n$ that are AC-reducible, have 4 generators and 4 relations and such that for any two of them, $d(\mu_1, \mu_2) > \exp_m(n)$.

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Theorem B.2 (L-Nabutovsky)

There exists $c > 0$ such that for all m for all sufficiently large n, there exist more than n^{cn} balanced presentations of the trivial group of length $\leq n$ that are AC-reducible and such that for any two of them, $d(\mu_1, \mu_2) > \exp_m(n)$.

Fast-growing Dehn function

If $w = 1$ over a presentation μ, we define the area of w in μ to be the minimal area of van Kampen diagrams for w.

Consider $G = \langle x, y, t | y^{-1}xy = x^2, t^{-1}x^{-1}t = y \rangle$ (the Baumslag-Gersten group). In this group, there is a word v_i of length $\sim 2^i$ representing $x^{\exp_i(2)}$.

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Consider $\mu_i = \langle G | t = [v_i, x] \rangle$ an AC-reducible presentation. Large area of t would imply $d(\mu_i, \langle \mid \rangle)$ is large, but it is not clear that $[v_i, x]$ has large area in μ_i.

Effective group theory

Approach 1 (L.): Consider instead $\mu_i = \langle G | t = [v_i, x^3][v_i, x^5][v_i, x^7] \rangle$. If we treat G as an "effective pseudogroup", i.e. treating words of large area as non-trivial, then we can define and prove that $t = [v_i, x^3][v_i, x^5][v_i, x^7]$ satisfies a small cancellation condition over the "effective" HNN extension G. Which implies x is non-trivial in the effective group $\mu_i = \langle G | t = [v_i, x^3][v_i, x^5][v_i, x^7] \rangle$, i.e. has large area.

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Approach 2 (Bridson): Consider instead $\mu_i = H_i \underset{\substack{s=\hat{x} \\ x=\hat{s}}}{*} \hat{H}_i$, where

$H_i = \langle G, s | s^{-1}[v_i, x]s = t \rangle$, and
$\hat{H}_i = \langle \hat{G}, \hat{s} | \hat{s}^{-1}[\hat{v}_i, \hat{x}]\hat{s} = \hat{t} \rangle$. (Note that $H_i = \mathbb{Z} = \hat{H}_i$)

Then use geometric proofs of the normal form theorems for HNN extensions and free products with amalgamation to show that x has nontrivial "effective" normal form, i.e. has large area.

From one to many

Mix the two approaches and add a dependence on the word v:

Define $H_{v,i} = \langle G, s | s^{-1} v w_i^{-1} v^{-1} w_i s = t \rangle$, and
$\hat{H}_{v,i} = \langle \hat{G}, \hat{s} | \hat{s}^{-1} \hat{v} \hat{w}_i^{-1} \hat{v}^{-1} \hat{w}_i \hat{s} = \hat{t} \rangle$, where $w_i = [v_i, x^3][v_i, x^5][v_i, x^7]$, and
v are different words on $\{y, yx\}$.
Again, $H_{v,i} = \mathbb{Z} = \hat{H}_{v,i}$.

Let $\mu_{v,i} = H_{v,i} \underset{\substack{s=\hat{x} \\ x=\hat{s}}}{*} \hat{H}_{v,i}$.

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Large areas of generators are not enough to show that $d(\mu_{v,i}, \mu_{v',i})$ is large for $v \neq v'$. We consider "effective" homomorphisms between different $\mu_{v,i}$. Whether there is one boils down to checking if areas of some images under these homomorphisms are large. We make such estimates using the approaches 1 and 2.

Note, we have very long words v made out of only 2 generators. We can abbreviate subwords of v by new generators:
Let $l(\mu_{v,i}) \sim n \log n$. Introduce $\frac{n}{\log n}$ new generators encoding words on x, y of length at most $\log(\frac{n}{\log n})$.
The total length of new relators is $< n$, while v can be rewritten with $O(n)$ new letters.

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Consider the graph Γ: the vertices are the generators. For each relation we add an edge or two connecting the generators used in the relation. We can make sure the diameter of Γ is $O(\log n)$.

From presentations to geometry

Given a finite presentation one can construct a closed 4-manifold embedded in $\mathbb{R}^5$, whose π_1 is naturally given by the presentation.

We first construct the presentation complex, then embed it into $\mathbb{R}^5$, take the boundary of its small neighbourhood and smooth it out.

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One bistellar transformation can not change the latter presentation by too many Tietze transformations.

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1) We want to list all closed 4-manifolds in some language.
2) Manifold presentations should be geometric: e.g. manifold presentation should produce triangulations in some natural way.
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All known to me presentations fail at 4) (Kirby diagrams, trisections, group trisections, triangulations). Is there some (maybe information-theoretic) connection between the fact that they all fail and complexity of the space of triangulations of a 4-manifold?