Homology of Random Simplicial Complexes Based on Steiner Systems

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Outline

Random Graphs - The $G(n, p)$ Model

$G(n, p)$ is the probability space of all graphs on the vertex set $[n]$ with indepndent edge probability p.

Theorem [Erdős-Rényi '59]:

Let $G \in G(n, p)$. For any function $\omega(n) \to \infty$, the following holds:

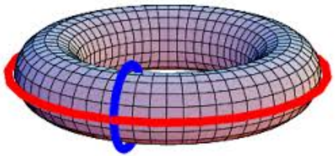
$$\lim_{n \to \infty} \Pr\left[G \text{ is connected}\right] = \begin{cases} 0 & p = \frac{\ln n - \omega(n)}{n}, \\ 1 & p = \frac{\ln n + \omega(n)}{n}. \end{cases}$$

Simplicial Homology - I

Let X be a simplicial complex and let $\mathbb{K}$ be a field.

$\dim \tilde{\mathrm{H}}_0(X;\mathbb{K}) + 1$ is the number of path-components of X.

$\dim \tilde{\mathrm{H}}_1(X;\mathbb{K})$ is the number of 1-dimensional "holes over $\mathbb{K}$" in X.



The *k-th reduced Betti number* $\tilde{\beta}_k(X) = \tilde{\beta}_k(X;\mathbb{K}) = \dim \tilde{\mathrm{H}}_k(X;\mathbb{K})$ is the number of k-dimensional "holes over $\mathbb{K}$" in X.

Simplicial Homology - II

Let X be a simplicial complex and let $\mathbb{K}$ be a field.
$C_k(X;\mathbb{K})$ is the vector space over $\mathbb{K}$ generated by the oriented k-simplices of X.
Indentify

$$[v_0 v_1 \dots v_k] = \mathrm{sgn}(\pi)[v_{\pi(0)} v_{\pi(1)} \dots v_{\pi(k)}].$$

The boundary map $\partial_k : C_k(X;\mathbb{K}) \to C_{k-1}(X;\mathbb{K})$ is the linear extension of the formula

$$\partial_k([v_0 v_1 \dots v_k]) = \sum_{i=0}^{k} (-1)^i [v_0 v_1 \dots \hat{v}_i \dots v_k].$$

Simplicial Homology - III

$B_k(X;\mathbb{K}) := \mathrm{Im}\ \partial_{k+1}$, the space of *k-boundaries*.
$Z_k(X;\mathbb{K}) := \ker \partial_k$, the space of *k-cycles*.

The boundary of a boundary is 0, i.e.

$$\partial_k \partial_{k+1} \equiv 0.$$

$$\implies B_k(X;\mathbb{K}) \subset Z_k(X;\mathbb{K}).$$

The *k-th reduced homology group* of X over $\mathbb{K}$ is

$$\tilde{\mathrm{H}}_k(X;\mathbb{K}) = \frac{Z_k(X;\mathbb{K})}{B_k(X;\mathbb{K})}.$$

Simplicial Cohomology

$C^k(X;\mathbb{K})$ is the dual space of $C_k(X;\mathbb{K})$, e.g. the space of all skew-symmetric functions on $X(k)$.
The coboundary map $d_k : C^k(X;\mathbb{K}) \to C^{k+1}(X;\mathbb{K})$ is given by

$$d_k(\phi)([v_0 v_1 \ldots v_i \ldots v_{k+1}]) = \sum_{i=0}^{k+1} (-1)^i \phi([v_0 v_1 \ldots \hat{v}_i \ldots v_{k+1}]).$$

$B^k(X;\mathbb{K}) := \mathrm{Im}\ d_{k-1}$, the space of k-coboundaries.
$Z^k(X;\mathbb{K}) := \ker d_k$, the space of k-cocycles.

The k-th reduced cohomology group of X over $\mathbb{K}$ is

$$\tilde{\mathrm{H}}^k(X;\mathbb{K}) = \frac{Z^k(X;\mathbb{K})}{B^k(X;\mathbb{K})}.$$

Fact:

$$\tilde{\mathrm{H}}_k(X;\mathbb{K}) \cong \tilde{\mathrm{H}}^k(X;\mathbb{K}).$$

Random Simplicial Complexes

Linial-Meshulam Model

$Y_k(n,p)$ is the probability space of all simplicial complexes $\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$ with independent k-simplex probability p.

Theorem [Linial-Meshulam-Wallach '09]:

Let $Y \in Y_k(n,p)$ and let $\mathbb{F}_q$ be a fixed finite field. For any function $\omega(n) \to \infty$, the following holds.

$$\lim_{n\to\infty} \Pr\left[\tilde{\mathrm{H}}_{k-1}(Y;\mathbb{F}_q) = 0\right] = \begin{cases} 0 & p = \frac{k \ln n - \omega(n)}{n}, \\ 1 & p = \frac{k \ln n + \omega(n)}{n}. \end{cases}$$

Random Graphs with Bounded Degree

Let M be a perfect matching on $[n]$, where n is even.
Take d random permutations $\pi_1, ..., \pi_d \in S_n$.
Define

$$G = \bigcup_{i=1}^{d} \pi_i(M).$$

Denote by $\mathcal{I}(n, d)$ the probability space of all graphs formed by this process.

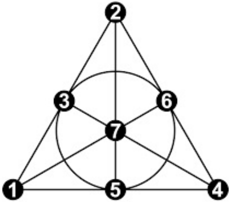
Theorem:
Fix $d \geq 3$ and let $G \in \mathcal{I}(n, d)$. Then

$$\lim_{n \to \infty} \Pr[G \text{ is connected}] = 1.$$

Steiner Systems

A Steiner system of type $S(k, k+1, n)$ is a family $S \subset \binom{[n]}{k+1}$ such that every $\tau \in \binom{[n]}{k}$ is contained in exactly one set in S.

S(2,3,7) – The Fano Plane.



Random Complexes with Bounded Degree

Let S be a Steiner system of type $S(k, k+1, n)$.
Take d random permutations $\pi_1, ..., \pi_d \in S_n$.
Define

$$X = \Delta_{n-1}^{(k-1)} \cup \bigcup_{i=1}^{d} \pi_i(S).$$

Denote by $\mathcal{S}_k(n, d)$ the probability space of all complexes formed by this process.

Theorem [A-Meshulam]:

Fix $k \geq 1$ and let $d \geq c(k) = 6k^2$.
Let $X \in \mathcal{S}_k(n, d)$. Then

$$\lim_{n \to \infty} \Pr\left[\tilde{\mathrm{H}}_{k-1}(X; \mathbb{R}) = 0\right] = 1.$$

Graph Laplacians

$G = (V, E)$ a graph

The Laplacian of G is the $V \times V$ matrix L_G:

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of L_G: $\quad 0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$

The Spectral Gap

$\lambda_2(G)$ controls the expansion of G and the convergence rate of a random walk on G. In particular, $\lambda_2(G) > 0 \Leftrightarrow G$ connected.

Higher Laplacians

A positive weight function $c(\sigma)$ on the simplices of X induces an Inner product on $C^k(X) = C^k(X;\mathbb{R})$:

$$(\phi, \psi) = \sum_{\sigma \in X(k)} c(\sigma)\phi(\sigma)\psi(\sigma) \quad .$$

Adjoint $\quad d_k^* : C^{k+1}(X) \to C^k(X)$

$$(d_k\phi, \psi) = (\phi, d_k^*\psi) \ .$$

$$C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X)$$

The reduced k-Laplacian of X is the positive semidefinite operator

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X) \quad .$$

Harmonic Cochains

The space of Harmonic k-cochains

$$\ker \Delta_k = \{\phi \in C^k(X) : d_k\phi = 0 \ , \ d_{k-1}^*\phi = 0\}.$$

Simplicial Hodge Theorem:

$$C^k(X) = \mathrm{Im}\, d_{k-1} \oplus \ker \Delta_k \oplus \mathrm{Im}\, d_k^* \ .$$

$$\ker \Delta_k \cong \tilde{\mathrm{H}}^k(X;\mathbb{R}).$$

$\mu_k(X) =$ minimal eigenvalue of Δ_k.

A Vanishing Criterion:

$$\mu_k(X) > 0 \Leftrightarrow \tilde{\mathrm{H}}_k(X;\mathbb{R}) = 0.$$

Hodge Decomposition

$$C^k(X) = \mathrm{Im}\, d_{k-1} \oplus \ker \Delta_k \oplus \mathrm{Im}\, d_k^* \ .$$

Proof:

Let $\alpha \in C^{k-1}(X), \beta \in \ker \Delta_k, \gamma \in C^{k+1}(X)$. Then:

$$\begin{aligned}(d_{k-1}\alpha, \beta) = (\alpha, d_{k-1}^*\beta) = 0 &\Rightarrow \mathrm{Im}\, d_{k-1} \perp \ker \Delta_k \\ (\beta, d_k^*\gamma) = (d_k\beta, \gamma) = 0 &\Rightarrow \ker \Delta_k \perp \mathrm{Im}\, d_k^* \\ (d_{k-1}\alpha, d_k^*\gamma) = (d_k d_{k-1}\alpha, \gamma) = 0 &\Rightarrow \mathrm{Im}\, d_{k-1} \perp \mathrm{Im}\, d_k^*.\end{aligned}$$

If $\phi \perp \mathrm{Im}\, d_{k-1} \oplus \mathrm{Im}\, d_k^*$ then

$$\begin{aligned}0 = (\phi, d_{k-1}d_{k-1}^*\phi) = (d_{k-1}^*\phi, d_{k-1}^*\phi) &\Rightarrow d_{k-1}^*\phi = 0 \\ 0 = (\phi, d_k^* d_k\phi) = (d_k\phi, d_k\phi) &\Rightarrow d_k\phi = 0.\end{aligned}$$

The Garland Method

Let X be a pure n-dimensional complex with weight function:

$$c(\sigma) = (n - \dim \sigma)!|\{\tau \in X(n) : \tau \supset \sigma\}|.$$

For $\tau \in X$ consider the link $X_\tau = \mathrm{lk}(X, \tau)$ with a weight function given by $c_\tau(\alpha) = c(\tau\alpha)$.

Theorem [Garland '72]:

Let $0 \leq \ell < k < n$. Then:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell+1}{k+1} \quad \Rightarrow \quad \tilde{\mathrm{H}}^k(X;\mathbb{R}) = 0.$$

In particular:

$$\min_{\tau \in X(n-2)} \mu_0(X_\tau) > \frac{n-1}{n} \quad \Rightarrow \quad \tilde{\mathrm{H}}^{n-1}(X;\mathbb{R}) = 0.$$

Links of Random Steiner Complexes

Fix $0 \le \ell < k$.
Let $X \in \mathcal{S}_k(n, d)$ and let $\tau \in X(\ell)$. Then

$$\mathrm{lk}(X, \tau) \in \mathcal{S}_{k-\ell-1}(n - \ell - 1, d).$$

In particular when $\ell = k - 2$,

$$\mathrm{lk}(X, \tau) \in \mathcal{I}(n - k + 1, d).$$

By Garland it is enough to study the spectral gap of $\mathcal{I}(n, d)$.

The Spectral Gap of $\mathcal{I}(n, d)$

The following theorem uses Friedman's celebrated proof of Alon's conjecture on the expansion of random d-regular graphs.

Theorem:
Fix $k > 0$ and let $d \geq c(k) = 6k^2$. Let n be sufficiently large.
For $G \in \mathcal{I}(n, d)$,

$$\Pr\left[\mu_0(G) > \frac{k-1}{k}\right] \geq 1 - O(n^{-k}).$$

Outline of Proof

Fix integers k and $d \geq c(k)$. Let n be sufficiently large.
Let $X \in \mathcal{S}_k(n,d)$.

Using the Corollary to Friedman's Theorem:

For $\tau \in X(k-2)$,

$$\Pr\left[\mu_0(\mathrm{lk}(X,\tau)) > \frac{k-1}{k}\right] \geq 1 - O(n^{-k}).$$

$$\Rightarrow \Pr\left[\min_{\tau \in X(k-2)} \mu_0(\mathrm{lk}(X,\tau)) > \frac{k-1}{k}\right] \geq 1 - O(n^{-1}).$$

Using Garland's Theorem:

$$\Rightarrow \lim_{n\to\infty} \Pr\left[\tilde{\mathrm{H}}_{k-1}(X;\mathbb{R}) = 0\right] = 1.$$

Betti Numbers and Local Connectivity

Garland's Theorem:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-2}(\mathrm{lk}(X,\tau)) > \frac{\ell+1}{k} \quad \Rightarrow \quad \tilde{\beta}_{k-1}(X;\mathbb{R}) = 0.$$

Question:

Let $\mathbb{K}$ be any field.
Suppose that we only assume $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau);\mathbb{K}) = 0$ for all $\tau \in X(\ell)$.

What can we say about $\tilde{\beta}_{k-1}(X;\mathbb{K})$?

The Upper Bound

Let $\mathbb{K}$ be a fixed field and let $\tilde{\beta}_j(X) = \tilde{\beta}_j(X;\mathbb{K})$.

Theorem [A-Meshulam]:

Fix $0 \leq \ell \leq k$.

If $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$ and $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau)) = 0$ for all $\tau \in X(\ell)$, then

$$\tilde{\beta}_{k-1}(X) \leq \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}}.$$

r-Hypertrees

Galai's Definition of Hypertrees:

$\Delta_{n-1}^{(r-1)} \subset X \subset \Delta_{n-1}^{(r)}$ is an r-hypertree over a field $\mathbb{K}$ if $\tilde{\mathrm{H}}_i(X;\mathbb{K}) = 0$ for all $0 \le i \le r$.

Fact:

For $\Delta_{n-1}^{(r-1)} \subset X \subset \Delta_{n-1}^{(r)}$ TFAE:

- X is an r-hypertree over $\mathbb{K}$.
- $\tilde{\mathrm{H}}_{r-1}(X;\mathbb{K}) = 0$ and $f_r(X) = \binom{n-1}{r}$.
- $\tilde{\mathrm{H}}_r(X;\mathbb{K}) = 0$ and $f_r(X) = \binom{n-1}{r}$.

Complexes with Hypertree Links

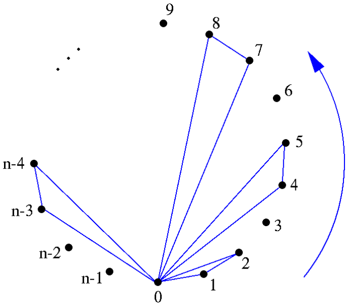
Theorem [A-Meshulam]:
Let $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$ and assume $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau)) = 0$ for all $\tau \in X(\ell)$. Then TFAE:

- $\tilde{\beta}_{k-1}(X) = \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}}$
- $\tilde{\beta}_k(X) = 0$
- $f_k(X) = \frac{\binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1}}{\binom{k+1}{\ell+1}}$
- $\mathrm{lk}(X,\tau)$ is a $(k-\ell-1)$-hypertree for all $\tau \in X(\ell)$.

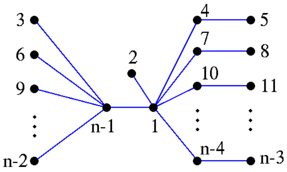
2-Complexes with 1-Tree Links

Fix an integer $t \geq 1$ and let $n = 3t + 2$.
$T_{n,2,0}$ is the 2-dimensional complex on the vertex set $\mathbb{Z}_n$ given by

$$T_{n,2,0} = \Delta_{n-1}^{(1)} \cup \{\{i, i+3j+1, i+3j+2\} : 0 \leq i \leq n-1 \, , \, 0 \leq j \leq t-1\}.$$



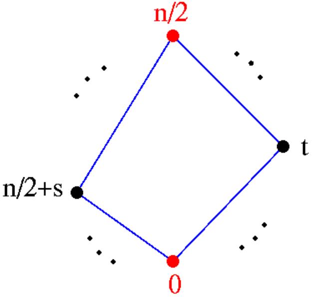
$\mathrm{lk}(T_{n,2,0}, 0)$



3-Complexes with 2-Hypertree Links

Fix an even integer $n \geq 4$.
$T_{n,3,0}$ is the 3-dimensional complex on the vertex set $\mathbb{Z}_n$ given by

$$T_{n,3,0} = \Delta_{n-1}^{(2)} \cup \{\{i, i+t, i+n/2, i+n/2+s\} : 0 \leq i \leq n-1 \text{ , } 0 < s, t < n/2\}.$$



Asymptotic Lower Bound

Theorem [A-Meshulam]:

Fix k, ℓ. Then for infinitely many n's there exist complexes $\Delta_{n-1}^{(k-1)} \subset X_{n,k,\ell} \subset \Delta_{n-1}^{(k)}$ such that $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X_{n,k,\ell},\tau)) = 0$ for all $\tau \in \Delta_{n-1}(\ell)$ and

$$\tilde{\beta}_{k-1}(X_{n,k,\ell}) \geq \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}} \left(1 - O\left(\frac{1}{n}\right)\right).$$

The proof uses the notion of sum complexes.

THANK YOU!