The Krein–von Neumann Extension: Weyl Asymptotics and Eigenvalue Counting Function Bounds for 2nd Order Uniformly Elliptic PDEs

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Topics involved:

- **Weyl asymptotics:** a bit of history.
- **Self-adjoint extensions of semibounded operators** in a Hilbert space, particularly, **Friedrichs** and **Krein–von Neumann** extensions.
- **Examples.**
- **Spectral connections** between the **Perturbed Krein Laplacian** in **bounded Lipschitz** domains in $\mathbb{R}^n$, $n \in \mathbb{N}$, and the **buckling** of a **clamped plate** ($\rightarrow$ elasticity applications).
- **Weyl asymptotics** of **perturbed Krein Laplacians** in **bounded Lipschitz** domains in $\mathbb{R}^n$.
- **Eigenvalue Counting Function Bounds** for **perturbed Krein Laplacians** in **arbitrary finite volume** domains in $\mathbb{R}^n$.

Based on various collaborations since 2010:

[AGMST10] M. S. Ashbaugh, F.G., M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein—von Neumann extension and its connection to an abstract buckling problem*, Math. Nachr. **283**, 165–179 (2010).

[AGMT10] M. S. Ashbaugh, F.G., M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223**, 1372–1467 (2010).

[GLMS15] F.G., A. Laptev, M. Mitrea, and S. Sukhtaiev, *A bound for the eigenvalue counting function for higher-order Krein Laplacians on open sets*, in *Mathematical Results in Quantum Mechanics, Proceedings of the QMath12 Conference*, P. Exner, W. König, and H. Neidhardt (eds.), World Scientific, Singapore, 2015, pp. 3–29.

[BGMM16] J. Behrndt, F.G., T. Micheler, and M. Mitrea, *The Krein-von Neumann realization of perturbed Laplacians on bounded Lipschitz domains*, IWOTA 2014 proceedings, Birkhäuser, OTAA **255**, 49–66 (2016).

[AGLMS16] M. Ashbaugh, F.G., A. Laptev, M. Mitrea, and S. Sukhtaiev, *A bound for the eigenvalue counting function for Krein–von Neumann and Friedrichs extensions*, arXiv:1605.01170, Adv. Math. (to appear).

Weyl asymptotics for the Laplacian $-\Delta$:

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be a **bounded, smooth** domain, and $-\Delta_{BC,\Omega}$ be the **Laplacian** on Ω in the Hilbert space $L^2(\Omega)$ with appropriate classical boundary conditions (**BC**), e.g., **Dirichlet**, **Neumann**, **Robin-type**, on the boundary $\partial\Omega$.

Let $\lambda_{BC,\Omega,j}$, $j \in \mathbb{N}$, be the eigenvalues of $-\Delta_{BC,\Omega}$ enumerated according to their multiplicity.

Introduce the **eigenvalue counting function** for $-\Delta_{BC,\Omega}$ by

$$N(\lambda; -\Delta_{BC,\Omega}) = \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{BC,\Omega,j} \leq \lambda\}, \quad \lambda \in \mathbb{R}.$$

In 1911, Weyl proved his celebrated leading order asymptotics

$$N(\lambda; -\Delta_{BC,\Omega}) = (2\pi)^{-n} v_n |\Omega| \, \lambda^{n/2} + o\big(\lambda^{n/2}\big) \text{ as } \lambda \to \infty,$$

where, $v_n = \pi^{n/2}/\Gamma((n/2)+1)$ is the volume of the unit ball in $\mathbb{R}^n$.

Note. The leading order is **shape and BC-independent** for a large class of **BC**s and **only** depends on the **volume** $|\Omega|$ of Ω.

Weyl also initiated a study of the **remainder term** in his asymptotics.

Weyl asymptotics for the Laplacian $-\Delta$ (contd.):

A bit of history: it all started in **Theoretical Physics**

The problem to determine the eigenvalue asymptotics was apparently independently posed by

Arnold Sommerfeld (September 1910)

and

Hendrik Antoon Lorentz (October 1910), Nobel Prize in Physics in 1902.

Hermann Weyl (1885–1955) solved it in less than 4 months by **February 1911** (in the Dirichlet case for $n = 2$) and wrote 6 papers on this subject from 1911–15 extending his result to other boundary conditions and to $n = 3$.

Rumor has it that **David Hilbert** (1862–1943) did not think he would live to see the problem solved ......

Weyl asymptotics for the Laplacian $-\Delta$ (contd.):

The problem originated in **Physics** in connection with (forced) **vibrations** and **black body radiation** (standing electromagnetic waves in a cavity with reflecting surface walls):

Gustav Kirchhoff (1859), Josef Stefan (1879), Ludwig Boltzmann (1884), Wilhelm Wien (1893, 1896), Lord Rayleigh (1900, 1905), Sir James Jeans (1905), Albert Einstein (1905), especially,

Max Planck (October 1900), who was led to the **Discovery of Quantum Theory** ($E = h\nu$, h - Planck's constant, **energy quantization,** one of the fathers of quantum physics, Nobel Prize in Physics in 1918).

This reads like a list of Who's Who in Theoretical Physics just before Quantum Mechanics entered the scene.

The subject is still so popular as there are numerous **applications/ramifications** in areas such as: Acoustics (Music: Lord Raleigh, high overtones for violins), Radiation, (Inverse) Spectral Geometry (Mark Kac's "Can one hear the shape of a drum?" in 1966, perhaps, one of the most cited spectral theory papers ?), Analytic Number Theory (automorphic forms), etc.

Notation for the abstract part:

Let $\mathcal{H}$ be a separable complex Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and identity operator $I_{\mathcal{H}}$ in $\mathcal{H}$.

$\dot{+}$ denotes the direct sum (not necessarily orthogonal) in $\mathcal{H}$.

S denotes a **symmetric, closed, densely defined** operator in $\mathcal{H}$ **bounded from below** (typically, $S \geq 0$ or $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$).

$\widetilde{S}$ a self-adjoint extension of S.

S_F denotes the **Friedrichs extension** of S.

S_K denotes the **Krein-von Neumann extension** of S.

Basic notions: symmetric and nonnegative operators

A linear operator $S : \mathrm{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called **non-negative**, $S \geq 0$, if

$$(u, Su)_{\mathcal{H}} \geq 0, \quad u \in \mathrm{dom}(S).$$

S is called **strictly positive**, if for some $\varepsilon > 0$, $(u, Su)_{\mathcal{H}} \geq \varepsilon \|u\|_{\mathcal{H}}^2$, $u \in \mathrm{dom}(S)$. One then writes $S \geq \varepsilon I_{\mathcal{H}}$. Similarly, one defines $S \geq T$ (but there are technical details concerning (quadratic form) domains........)

Notation: $A \subseteq B$ denotes $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and $Af = Bf$ for all $f \in \mathrm{dom}(A)$. I.e., B is an **extension** of A (equivalently, A is a **restriction** of B).

S **symmetric** in $\mathcal{H}$: $\quad S \subseteq S^*$.

T **self-adjoint** in $\mathcal{H}$: $\quad T = T^*$.

Basic principle of self-adjoint extensions:

Let S_j, $j = 1, 2$, be **symmetric**, $\quad S_1 \subseteq S_2 \Longrightarrow S_2^* \subseteq S_1^*$

Thus,

$$S_1 \subseteq \boxed{S_2 \subseteq S_2^*} \subseteq S_1^*$$

Existence of self-adjoint extensions was completely settled in

J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102**, 49–131 (1929–30)

in terms of equal deficiency indices.

This marked the **birth** of **Mathematical Quantum Theory**.

In particular, he introduced the notion of **closed operators** and hence enabled the notion of **spectral theory** for unbounded linear operators in Hilbert spaces.

Mark Krein's 1947 result:

Theorem (M. Krein, Mat. Sb. 20 (1947)).

$S \geq 0$ densely defined, closed in $\mathcal{H}$. Then, among all **non-negative self-adjoint extensions** of S, there exist **two distinguished** (**extremal**) ones, S_K and S_F, which are the smallest and largest (in the sense of order between the quadratic forms associated with self-adjoint operators) such extensions. Any non-negative self-adjoint extension $\widetilde{S} \geq 0$ of S necessarily satisfies

$$0 \leq S_K \leq \widetilde{S} \leq S_F, \quad \text{i.e., } \forall a > 0, \ (S_F + aI_{\mathcal{H}})^{-1} \leq (\widetilde{S} + aI_{\mathcal{H}})^{-1} \leq (S_K + aI_{\mathcal{H}})^{-1}.$$

In particular, this determines S_K and S_F uniquely.
In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one has

$$\begin{aligned}
&\operatorname{dom}(S_F) = \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*), \\
&\operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \\
&\operatorname{dom}(S^*) = \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*) \dotplus \ker(S^*), \\
&\ker(S_K) = \ker\big((S_K)^{1/2}\big) = \ker(S^*) = \operatorname{ran}(S)^{\perp} = \operatorname{def}(S),
\end{aligned}$$

this null space might well be **infinite-dimensional** (it **is** for PDEs) !!!

1d Example: $\Omega = (a, b)$, $-\infty < a < b < \infty$, $V = 0$:

$$-\Delta_{min,(a,b)} = \overline{-\frac{d^2}{dx^2}\Big|_{C_0^\infty((a,b))}}, \quad \mathrm{dom}(-\Delta_{min,(a,b)}) = W_0^{2,2}((a,b)),$$

$$-\Delta_{F,(a,b)} u = -\Delta_{D,(a,b)} u = -u'',$$

$$u \in \mathrm{dom}(-\Delta_{F,(a,b)}) = \big\{ v \in L^2((a,b)) \,\big|\, v, v' \in AC([a,b]);$$

Dirichlet b.c.s $\boxed{\boldsymbol{v(a) = v(b) = 0};}\; v'' \in L^2((a,b)) \big\},$

$$-\Delta_{N,(a,b)} u = -u'',$$

$$u \in \mathrm{dom}(-\Delta_{N,(a,b)}) = \big\{ v \in L^2((a,b)) \,\big|\, v, v' \in AC([a,b]);$$

Neumann b.c.s $\boxed{\boldsymbol{v'(a) = v'(b) = 0};}\; v'' \in L^2((a,b)) \big\},$

$$-\Delta_{K,(a,b)} u = -u'',$$

$$u \in \mathrm{dom}(-\Delta_{K,(a,b)}) = \big\{ v \in L^2((a,b)) \,\big|\, v, v' \in AC([a,b]);$$

Krein b.c.s $\boxed{\boldsymbol{v'(a) = v'(b) = [v(b) - v(a)]/(b-a)};}\; v'' \in L^2((a,b)) \big\}.$

This settles the one-dimensional case $n = 1$. (Hence, $\boxed{n \geq 2}$ from now on.)

Perturbed Krein Laplacians on bounded Lipschitz (resp., minimally smooth) domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

We assume for the remainder of the Weyl asymptotics part:

Hypothesis.

Let $n \in \mathbb{N}$, $n \geq 2$.

(i) $\Omega \subset \mathbb{R}^n$ is a nonempty, open, bounded, Lipschitz domain.

(ii) $V \in L^\infty(\Omega)$ is nonnegative, $V \geq 0$.

The case $n = 1$, $\Omega = (a, b)$, $-\infty < a < b < \infty$, was settled in the previous example.

This permits one to make the step from the **Laplacian,** $-\Delta$, to a **perturbed Laplacian,** $-\Delta + V$, i.e., a **Schrödinger**-type operator.

Bounded Lipschitz domains:

Definition.

Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ open and $R > 0$ fixed. Ω is called a **bounded Lipschitz domain**, if there exists $r \in (0, R)$ such that for every $x_0 \in \partial\Omega$ one can find a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$T\big(\Omega \cap B(x_0, r)\big) = T\big(B(x_0, r)\big) \cap \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\big|\, x_n > \varphi(x')\big\}.$$

Examples of **bounded Lipschitz** domains include:

(i) All bounded (geometrically) **convex** domains.

(ii) All open sets which are the image of a domain as in (i) above under a $C^{1,1}$-diffeomorphism.

(iii) All bounded domains of class $C^{1,r}$ for some $r > 1/2$.

Considering only **smooth** Ω could not even handle a **rectangle** in $\mathbb{R}^2$!!!

So studying **Lipschitz domains** is not just a technicality.

The domain of the perturbed Krein Laplacian $H_{K,\Omega}$:

The **minimal** and **maximal** perturbed Laplacian in $L^2(\Omega)$ are defined by

$$\begin{aligned} &H_{min,\Omega} u := (-\Delta + V)u, \quad u \in \operatorname{dom}(H_{min,\Omega}) := W_0^{2,2}(\Omega), \\ &H_{max,\Omega} u := (-\Delta + V)u, \\ &u \in \operatorname{dom}(H_{max,\Omega}) = \big\{ v \in L^2(\Omega) \,\big|\, \Delta v \in L^2(\Omega) \big\}. \end{aligned}$$

Then

$$H_{min,\Omega} = \overline{(-\Delta + V)|_{C_0^\infty(\Omega)}} = H_{max,\Omega}{}^* \ \text{ and } \ H_{max,\Omega} = H_{min,\Omega}{}^*.$$

The **Krein–von Neumann** extension $H_{K,\Omega}$ of $H_{min,\Omega}$, i.e., the **perturbed Krein Laplacian** in $L^2(\Omega)$ is then given by

$$\begin{aligned} &H_{K,\Omega} u := (-\Delta + V)u, \\ &u \in \operatorname{dom}(H_{K,\Omega}) = \operatorname{dom}(H_{min,\Omega}) \dot{+} \ker(H_{max,\Omega}) \\ &\qquad\qquad\qquad\quad = W_0^{2,2}(\Omega) \dot{+} \big\{ v \in L^2(\Omega) \,\big|\, (-\Delta + V)v = 0 \text{ in } \Omega \big\}. \end{aligned}$$

(One recalls, $\dot{+}$ denotes the direct sum (not necessarily orthogonal) in $L^2(\Omega)$.)

The domain of $H_{K,\Omega}$ (contd.): nonlocal b.c.'s

NONLOCAL and ***V*-DEPENDENT boundary condition** for $\mathbf{H_{K,\Omega}}$:

$$\mathrm{dom}(H_{K,\Omega}) := \big\{ v \in \mathrm{dom}(H_{max,\Omega}) \,\big|\, \gamma_N v + M^{WT}_{D,N,\Omega,V}(0)(\gamma_D v) = 0 \big\},$$

i.e., these b.c.'s are **not** of **local Robin-type**.

• γ_D and γ_N are "appropriate" Dirichlet and Neumann trace operators in $L^2(\partial\Omega)$, i.e., suitable generalizations of

$$\gamma_D u = u|_{\partial\Omega} \;\text{ and }\; \gamma_N u = \nu \cdot \nabla u|_{\partial\Omega},$$

with ν the outward pointing unit vector on $\partial\Omega$ (which exists a.e. on $\partial\Omega$).

• $M^{WT}_{D,N,\Omega,V}(z)$ is an "appropriate" z-dependent **Dirichlet-to-Neumann operator** (i.e., an operator-valued **Weyl–Titchmarsh operator**) in $L^2(\partial\Omega)$,

$$M^{WT}_{D,N,\Omega,V}(z) \text{ “}=\text{” } \gamma_N \big[\gamma_N (H_{D,\Omega} - \overline{z} I_{L^2(\Omega)})^{-1}\big]^*, \quad z \in \rho(H_{D,\Omega}).$$

Formally, if Ω is **smooth**, and in the special case $V = 0$, this **nonlocal** b.c. is of the form

$$\frac{\partial v}{\partial \nu}(x) - \frac{\partial (Hv)}{\partial \nu}(x) = 0, \quad x \in \partial\Omega,$$

where (Hv) denotes the **harmonic extension** of $v|_{\partial\Omega}$ into Ω.

General spectral properties of $H_{K,\Omega}$:

Theorem ([AGMT10], [BGMM16]).

The perturbed Krein Laplacian $H_{K,\Omega}$ is a self-adjoint operator on $L^2(\Omega)$ which satisfies

$$H_{K,\Omega} \geq 0 \text{ and } H_{min,\Omega} \subseteq H_{K,\Omega} \subseteq H_{max,\Omega},$$

$$H_{K,\Omega} \text{ has a purely discrete spectrum in } (0,\infty),$$

$$\sigma_{ess}(H_{K,\Omega}) = \{0\} \text{ if } n \geq 2.$$

For **any non-negative** self-adjoint extension $\widetilde{S}$ of $H_{min,\Omega} = \overline{(-\Delta + V)|_{C_0^\infty(\Omega)}}$ on $\mathrm{dom}(H_{min,\Omega}) = W_0^{2,2}(\Omega)$ one has

$$H_{K,\Omega} \leq \widetilde{S} \leq H_{D,\Omega}.$$

Connection: Krein Laplacian $\longleftrightarrow$ Buckling Problem:

Given $\lambda \in \mathbb{C}$, consider the **eigenvalue problem** for the **generalized buckling** of a **clamped plate** in $\Omega \subset \mathbb{R}^n$,

$$(-\Delta + V)^2 u = \lambda (-\Delta + V) u \text{ in } \Omega, \ u \in \operatorname{dom}(-\Delta_{max,\Omega}), \ \gamma_D u = 0, \ \gamma_N u = 0,$$

where $(-\Delta + V)^2 u := (-\Delta + V)(-\Delta u + Vu)$ in the sense of distributions in Ω.

Equivalently,

$$(-\Delta + V)^2 u = \lambda (-\Delta + V) u \ \text{ in } \ \Omega, \quad u \in W_0^{2,2}(\Omega). \qquad (*)$$

This is a **generalized** eigenvalue problem of the type $\boxed{Au = \lambda Bu}$ in the sense that $B \neq I_{L^2(\Omega)}$ in general. Equivalently, it's a **linear operator pencil** problem ....

One can show that **necessarily**, $\lambda > 0$.

Theorem ([AGMT10], [BGMM16]).

$\lambda > 0$ is an eigenvalue (necessarily discrete) of $H_{K,\Omega}$ **if and only if** $\lambda > 0$ is an eigenvalue of the **(generalized) buckling problem** $(*)$. Moreover, the multiplicities coincide.

In particular, $H_{K,\Omega}$ and the **(generalized) buckling problem** $(*)$ are what **Percy Deift** called **"essentially isospectral"** in 1978 ......

Notes:

(i) The equivalence:

Krein Laplacian eigenvalue problem $\longleftrightarrow$ **buckling problem**

shows that the **nonlocal** and **V-dependent boundary condition** associated with the **second-order** perturbed **Krein Laplacian eigenvalue problem** can be turned into a **fourth-order buckling problem** with **(local) Dirichlet** and **Neumann** boundary conditions.

(ii) The **(generalized) buckling problem** comes from a **linear operator pencil** problem $\boxed{Au = \lambda Bu,}$ where $A = (-\Delta + V)^2$, $B = (-\Delta + V)$.

(iii) **G. Grubb,** J. Oper. Th. **10** (1983) developed the intimate connection between S_K and the **(generalized) buckling problem** for even-order elliptic partial differential operators on **smooth** bounded domains $\Omega \subset \mathbb{R}^n$.

We found the extension to **abstract** Hilbert space operators and hence, an **abstract buckling problem**, in 2010, so this is now **a general phenomenon**.

Moreover, Grubb's results for **smooth** bounded domains were extended to bounded **Lipschitz domains** in [BGMM16], see the next result:

Weyl asymptotics for positive e.v.'s of $H_{K,\Omega}$:

Theorem ([AGMT10], [BGMM16]).

Assume that Ω is a bounded **Lipschitz domain** in $\mathbb{R}^n$, $n \geq 2$, and let $N(\lambda; H_{K,\Omega}) = \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} \leq \lambda\}$, $\lambda \in \mathbb{R}$, with $\lambda_{K,\Omega,j}$ enumerated according to multiplicity. Then

$$N(\lambda; H_{K,\Omega}) = (2\pi)^{-n} v_n |\Omega| \, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big) \text{ as } \lambda \to \infty,$$

where, $v_n = \pi^{n/2}/\Gamma((n/2)+1)$ is the volume of the unit ball in $\mathbb{R}^n$.

This is based on the one-to-one connection with the **(generalized) buckling problem** combined with results of **V. A. Kozlov** and new results on Dirichlet and Neumann traces for Lipschitz domains in [BGMM16].

Note. The **remainder estimate** $O\big(\lambda^{(n-(1/2))/2}\big)$ is possible because of the **Lipschitz** hypothesis on Ω. For **arbitrary** bounded, open Ω one obtains only $o\big(\lambda^{n/2}\big)$ for the **remainder estimate**.

Epilogue: All this was inspired by

A. Alonso and B. Simon, *The Birman–Krein–Vishik theory of selfadjoint extensions of semibounded operators*, J. Oper. Th. **4**, 251–270 (1980).

"It seems to us that the Krein extension of $-\Delta$, i.e., $-\Delta$ with the boundary condition $\partial_\nu u = \partial_\nu H(u)$ on $\partial\Omega$, is a natural object and therefore worthy of further study. For example: Are the asymptotics of its nonzero eigenvalues given by Weyl's formula?"

First solved 3 years later by **G. Grubb**, *Spectral asymptotics for the "soft" selfadjoint extension of a symmetric elliptic differential operator*, J. Oper. Th. **10**, 9–20 (1983), for the case of even-order elliptic partial differential operators on **smooth, bounded** domains $\Omega \subset \mathbb{R}^n$.

V. A. Kozlov, *Estimates of the remainder in formulas for the asymptotic behavior of the spectrum for lin. operator bundles*, Funct. Anal. Appl. **17**, 147-149 (1983).

Our Weyl asymptotics results for $H_{K,\Omega}$ in 2010 ([AGMT10], Adv. Math.) were the first for certain **nonsmooth** bounded domains Ω (close to Lipschitz); the corresponding bounded **Lipschitz domain** results are from 2015 (see [BGMM16]).

Back to Eigenvalue Counting Functions:

Hypothesis.

Let $n \in \mathbb{N}$, $n \geq 2$, $\emptyset \neq \Omega \subset \mathbb{R}^n$ open and of finite (Euclidean) volume.

Note. (i) **No** assumptions are made on $\partial\Omega$.
(ii) Since $W_0^{2m,2}(\Omega)$ embeds compactly into $L^2(\Omega)$, $-\Delta_{F,\Omega}$ has purely discrete spectrum and hence so does $-\Delta_{K,\Omega}$ in $(0,\infty)$ (i.e., away from 0).

For **simplicity**, we start with the **Laplacian** (i.e., $V=0$): Recall the **eigenvalue counting function** for $-\Delta_{K,\Omega}$,

$$N(\lambda; -\Delta_{K,\Omega}) = \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} \leq \lambda\}, \quad \lambda \in \mathbb{R},$$

$\lambda_{K,\Omega,j}$, $j \in \mathbb{N}$, the eigenvalues of $-\Delta_{K,\Omega}$ enumerated according to multiplicity.

Theorem (Krein Laplacian, [GLMS15]).

$$N(\lambda; -\Delta_{K,\Omega}) \leq (2\pi)^{-n} v_n |\Omega| \{1 + [2/(2+n)]\}^{n/2} \lambda^{n/2}, \quad \lambda > 0,$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ is the (Euclidean) volume of the unit ball in $\mathbb{R}^n$.

This seemed to be a new result, no matter what assumptions on $\partial\Omega$.

Eigenvalue Counting Functions (contd.):

Sketch of proof.

Introduce the minimal operator $-\Delta_{min,\Omega} = -\Delta$, $\operatorname{dom}(-\Delta_{min,\Omega}) = W_0^{2,2}(\Omega)$, then $-\Delta_{min,\Omega} \geq \varepsilon I_\Omega$, and introduce the symmetric forms in $L^2(\Omega)$,

$$\mathfrak{a}_\Omega(f,g) = (-\Delta_{min,\Omega} f, -\Delta_{min,\Omega} g)_{L^2(\Omega)},\ f,g \in \operatorname{dom}(\mathfrak{a}_\Omega) = \operatorname{dom}(-\Delta_{min,\Omega}),$$

$$\mathfrak{b}_\Omega(f,g) = (f, -\Delta_{min,\Omega} g)_{L^2(\Omega)},\ f,g \in \operatorname{dom}(\mathfrak{b}_\Omega) = \operatorname{dom}(-\Delta_{min,\Omega}).$$

On can show (and this is key, but requires some efforts!) that

$$N(\lambda; -\Delta_{K,\Omega}) \leq \max\big(\dim\,\big\{f \in \operatorname{dom}(-\Delta_{min,\Omega}) \,\big|\, \mathfrak{a}_\Omega(f,f) - \lambda\,\mathfrak{b}_\Omega(f,f) < 0\big\}\big).$$

To further analyze this we now fix $\lambda \in (0,\infty)$ and introduce the auxiliary operator

$$L_{\Omega,\lambda} := (-\Delta_{min,\Omega})^*(-\Delta_{min,\Omega}) - \lambda(-\Delta_{min,\Omega}).$$

Then $L_{\Omega,\lambda}$ is self-adjoint, bounded from below, with purely discrete spectrum as its form domain, $W_0^{2,2}(\Omega)$, embeds compactly into $L^2(\Omega)$. We will study the auxiliary eigenvalue problem,

$$L_{\Omega,\lambda}\varphi_j = \mu_j\varphi_j, \quad \varphi_j \in \operatorname{dom}(L_{\Omega,\lambda}),$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and we repeat the eigenvalues μ_j of $L_{\Omega,\lambda}$ according to their multiplicity.

Eigenvalue Counting Functions (contd.):

Sketch of proof (contd.).

Since $\varphi_j \in W_0^{2,2}(\Omega)$, we denote by

$$\widetilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases}$$ their zero-extension of φ_j to all of $\mathbb{R}^n$.

Next, given $\mu > 0$, one estimates

$$\begin{aligned} \mu^{-1} \textstyle\sum_{j\in\mathbb{N},\, \mu_j<\mu} (\mu - \mu_j) &\geq \mu^{-1} \textstyle\sum_{j\in\mathbb{N},\, \mu_j<0,\, \mu_j<\mu} (\mu - \mu_j) \\ &\geq \mu^{-1} \textstyle\sum_{j\in\mathbb{N},\, \mu_j<0,\, \mu_j<\mu} \mu = n_-(L_{\Omega,\lambda}), \end{aligned}$$

where $n_-(L_{\Omega,\lambda})$ denotes the number of strictly negative eigenvalues of $L_{\Omega,\lambda}$. This implies

$$\begin{aligned} N(\lambda; -\Delta_{K,\Omega}) &\leq \max\big(\dim\big\{f \in \operatorname{dom}(-\Delta_{min,\Omega}) \,\big|\, \mathfrak{a}_\Omega(f,f) - \lambda\, \mathfrak{b}_\Omega(f,f) < 0\big\}\big) \\ &= n_-(L_{\Omega,\lambda}) \leq \mu^{-1} \textstyle\sum_{j\in\mathbb{N},\, \mu_j<\mu} (\mu - \mu_j) = \mu^{-1} \textstyle\sum_{j\in\mathbb{N}} [\mu - \mu_j]_+, \quad \mu > 0. \quad (*) \end{aligned}$$

Here, $x_+ := \max(0, x)$, $x \in \mathbb{R}$.

Next, we focus on estimating the right-hand side of $(*)$.

Eigenvalue Counting Functions (contd.):

Sketch of proof (contd.).

$$
\begin{aligned}
N(\lambda; -\Delta_{K,\Omega}) &\leq \mu^{-1} \sum_{j\in\mathbb{N}} (\mu - \mu_j)_+ = \mu^{-1} \sum_{j\in\mathbb{N}} \big[(\varphi_j, (\mu - \mu_j)\varphi_j)_{L^2(\Omega)}\big]_+ \\
&= \mu^{-1} \sum_{j\in\mathbb{N}} \Big[\mu\|\varphi_j\|^2_{L^2(\Omega)} - \big\|(-\Delta)\varphi_j\big\|^2_{L^2(\Omega)} + \lambda\big(\varphi_j, (-\Delta)\varphi_j\big)_{L^2(\Omega)}\Big]_+ \\
&= \mu^{-1} \sum_{j\in\mathbb{N}} \Big[\mu\|\widetilde{\varphi}_j\|^2_{L^2(\mathbb{R}^n)} - \big\|(-\Delta)\widetilde{\varphi}_j\big\|^2_{L^2(\mathbb{R}^n)} + \lambda\big(\widetilde{\varphi}_j, (-\Delta)\widetilde{\varphi}_j\big)_{L^2(\mathbb{R}^n)}\Big]_+ \\
&= \mu^{-1} \sum_{j\in\mathbb{N}} \bigg[\int_{\mathbb{R}^n} \big[\mu - \big(|\xi|^4 - \lambda|\xi|^2\big)\big] \big|\widehat{\widetilde{\varphi}}_j(\xi)\big|^2 \, d^n\xi\bigg]_+ \qquad \widehat{\ } = \text{Fourier Transf.} \\
&\leq \mu^{-1} \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^n} \big[\mu - \big(|\xi|^4 - \lambda|\xi|^2\big)\big]_+ \big|\widehat{\widetilde{\varphi}}_j(\xi)\big|^2 \, d^n\xi \\
&= \mu^{-1} \int_{\mathbb{R}^n} \big[\mu - |\xi|^4 + \lambda|\xi|^2\big]_+ \sum_{j\in\mathbb{N}} \big|\widehat{\widetilde{\varphi}}_j(\xi)\big|^2 \, d^n\xi.
\end{aligned}
$$

Eigenvalue Counting Functions (contd.):

Sketch of proof (contd.).

Here we used unitarity of the Fourier transform on $L^2(\mathbb{R}^n)$, the fact that $\big[\mu - |\xi|^4 + \lambda|\xi|^2\big]_+$ has compact support, and the monotone convergence theorem in the final step. Next,

$$\sum_{j\in\mathbb{N}} \Big|\widehat{\widetilde{\varphi}}_j(\xi)\Big|^2 = (2\pi)^{-n} \sum_{j\in\mathbb{N}} \Big|\big(e^{i\xi\cdot}, \widetilde{\varphi}_j\big)_{L^2(\mathbb{R}^n)}\Big|^2 = (2\pi)^{-n} \sum_{j\in\mathbb{N}} \Big|\big(e^{i\xi\cdot}, \varphi_j\big)_{L^2(\Omega)}\Big|^2$$
$$= (2\pi)^{-n} \big\|e^{i\xi\cdot}\big\|^2_{L^2(\Omega)} = (2\pi)^{-n}|\Omega|,$$

since $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$.

Introducing $\alpha = \lambda^{-2}\mu$, changing variables, $\xi = \lambda^{1/2}\eta$, and taking the minimum with respect to $\alpha > 0$, finally yields the bound,

$$N(\lambda; -\Delta_{K,\Omega}) \leq (2\pi)^{-n}|\Omega| \min_{\alpha>0} \bigg(\alpha^{-1} \int_{\mathbb{R}^n} \big[\alpha - |\xi|^4 + |\xi|^2\big]_+ d^n\xi\bigg) \lambda^{n/2}$$
$$= (2\pi)^{-n} v_n |\Omega| \{1 + [2/(2+n)]\}^{n/2} \lambda^{n/2}, \quad \lambda > 0.$$

Explicit minimization over α is a nice but nontrivial calculus exercise: we are indebted to **Mark Ashbaugh** for it! $\square$

Eigenvalue Counting Functions (contd.): Extensions

(i) Actually, in [GLMS15] we considered higher-order Krein Laplacians in $L^2(\Omega)$, denoted by $-\Delta_{K,\Omega,m}$, associated to the minimal operator

$$-\Delta_{min,\Omega,m} = (-\Delta)^m, \quad \operatorname{dom}((-\Delta)^m) = W_0^{2m,2}(\Omega),\ m \in \mathbb{N},$$

in [GLMS15] with the corresponding estimate,

$$N(\lambda; -\Delta_{K,\Omega,m}) \leq (2\pi)^{-n} v_n |\Omega| \{1 + [2m/(2m+n)]\}^{n/(2m)} \lambda^{n/(2m)}, \quad \lambda > 0,\ m \in \mathbb{N}.$$

(ii) More importantly, curently in [AGLMS16] we consider higher-order Krein extensions, denoted by $A_{K,\Omega,2m}(a,b,q)$, of the closed, strictly positive, higher-order differential operator in $L^2(\Omega)$ (the minimal operator)

$$A_{min,\Omega,2m}(a,b,q) = \tau_{2m}(a,b,q), \quad \operatorname{dom}(A_{min,\Omega,2m}(a,b,q)) = W_0^{2m,2}(\Omega),$$

corresponding to the higher-order differential expression

$$\tau_{2m}(a,b,q) = \Big(\sum_{j,k=1}^n (-i\partial_j - b_j(x)) a_{j,k}(x) (-i\partial_k - b_k(x)) + q(x) \Big)^m, \quad m \in \mathbb{N}.$$

Moreover, we need an "extension" of $A_{min,\Omega,2m}(a,b,q)$ to all of $\mathbb{R}^n$, denoted by the operator $\widetilde{A}_{2m}(a,b,q)$ in $L^2(\mathbb{R}^n)$,

$$\widetilde{A}_{2m}(a,b,q) = \tau_{2m}(a,b,q), \quad \operatorname{dom}\big(\widetilde{A}_{2m}(a,b,q)\big) = W^{2m,2}(\mathbb{R}^n).$$

Eigenvalue Counting Functions (contd.): Extensions

Hypothesis.

(i) Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open and bounded, $n \in \mathbb{N}$.

(ii) Let $m \in \mathbb{N}$. Assume that

$$b = (b_1, b_2, \dots, b_n) \in \left[W^{(2m-1),\infty}(\mathbb{R}^n)\right]^n, \quad b_j \text{ real-valued}, \quad 1 \leq j \leq n,$$

$$0 \leq q \in W^{(2m-2),\infty}(\mathbb{R}^n), \quad a := \{a_{j,k}\}_{1 \leq j,k \leq n} \in C^{(2m-1)}\big(\mathbb{R}^n, \mathbb{R}^{n^2}\big),$$

and assume that there exists $\varepsilon_a > 0$, such that (uniform ellipticity ....)

$$\textstyle\sum_{j,k=1}^n a_{j,k}(x) y_j y_k \geq \varepsilon_a |y|^2, \quad x \in \mathbb{R}^n, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

In addition, assume that the $n \times n$ matrix-valued function a equals the identity I_n outside a ball $B_n(0; R)$ containing $\overline{\Omega}$, that is, there exists $R > 0$ such that

$$a(x) = I_n, \quad |x| \geq R, \quad \overline{\Omega} \subset B_n(0; R).$$

Given these assumptions, $0 \leq \widetilde{A}_{2m}(a, b, q)$ is self-adjoint in $L^2(\mathbb{R}^n)$.

We need one more spectral theory assumption on $\widetilde{A}_{2m}(a, b, q)$ so that the **Fourier transform** as the **eigenfunction transform** of $-\Delta$ on $\mathrm{dom}(-\Delta) = W^{2,2}(\mathbb{R}^n)$ can be replaced by the **eigenfunction transform**, i.e., a **distorted Fourier transfom**, associated with $\widetilde{A}_{2m}(a, b, q)$ on $\mathrm{dom}\big(\widetilde{A}_{2m}(a, b, q)\big) = W^{2m,2}(\mathbb{R}^n)$:

Eigenvalue Counting Functions (contd.): Extensions

Hypothesis.

In addition to the previous hypotheses on a, b, q and Ω, assume the following:
(i) Suppose there exists $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, such that the operator

$$(\mathbb{F}f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\overline{\phi(x,\xi)}\, d^n x, \quad \xi \in \mathbb{R}^n,$$

originally defined on functions $f \in L^2(\mathbb{R}^n)$ with compact support, can be extended to the unitary operator in $L^2(\mathbb{R}^n)$, such that

$$f \in W^{2,2}(\mathbb{R}^n; d^n x) \text{ if and only if } |\xi|^2(\mathbb{F}f)(\xi) \in L^2(\mathbb{R}^n; d^n\xi),$$

$$\text{and } \widetilde{A}_2(a,b,q) = \mathbb{F}^{-1} M_{|\xi|^2}\mathbb{F},$$

where $M_{|\xi|^2}$ represents the maximally defined operator of multiplication by $|\xi|^2$ in $L^2(\mathbb{R}^n; d^n\xi)$.

(ii) In addition, suppose that $\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)} < \infty$.

Note. Condition (ii) is tricky, but appropriate **limiting absorption theorems** can be shown to imply it in many cases of interest. Condition (i) implies the **absence** of **eigenvalues** and any **singular continuous spectrum** of $\widetilde{A}_2(a,b,q)$. It is simplest implemented by choosing $(a - I_n), b, q$ all of compact support.

Eigenvalue Counting Functions (contd.): Extensions

Theorem ([AGLMS16]).

Under these hypotheses, and with $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$ the Krein and Friedrichs extensions of $A_{\Omega,2m}(a,b,q)$, the following estimates holds:

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leq \frac{v_n}{(2\pi)^n}\left(1+\frac{2m}{2m+n}\right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \lambda^{n/(2m)},$$

$$N(\lambda, A_{F,\Omega,2m}(a,b,q)) \leq \frac{v_n}{(2\pi)^n}\left(1+\frac{2m}{n}\right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \lambda^{n/(2m)},$$

$$\lambda > 0,\ m \in \mathbb{N}.$$

Here $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$ and $\phi(\,\cdot\,,\,\cdot\,)$ represent the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a,b,q)$ satisfying

$$\widetilde{A}_2(a,b,q)\phi(\,\cdot\,,\xi) = |\xi|^2\phi(\,\cdot\,,\xi)$$

in the distributional sense. In particular, $\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} = |\Omega|$ if $a = I_n$, $b = q = 0$.

Some of these results are **new**, no matter what assumptions on $\partial\Omega$.

..... and, it really is all about:

HAPPY (belated) BIRTHDAY, Barry!!!!!