Arm events in Invasion Percolation

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Invasion Percolation

A stochastic geometric model introduced by
Chandler-Koplick-Lerman-Willemsen, (1982) and Lenormand-Bories (1980).

The model exhibits "self-organized criticality". Its large scale behavior is similar to that of critical percolation, without any parameter being explicitly tuned.

Invasion percolation: Definition

Independent weights ω_e, uniform in $[0,1]$, attached to each edge $e \in \mathcal{E}(\mathbb{Z}^d)$. We will study the case $d = 2$ here.

For a graph $G = (V, E)$, define the *outer edge boundary*

$$\Delta G = \{e = (x, y) \notin \mathcal{E}(G), \text{ but } x \in G \text{ or } y \in G\}.$$

We define a sequence of graphs $G_i = (V_i, E_i)$ iteratively:

1. $G_0 = (\{\mathbf{0}\}, \emptyset)$.
2. The weight τ_i minimizes ω_e over $e \in \Delta G_i$.
3. $E_{i+1} = E_i \cup \{e_{i+1}\}$, and G_{i+1} the graph induced by E_{i+1}.

The invasion percolation cluster is

$$\mathcal{S} = \bigcup_{i=0}^{\infty} G_i.$$

Link with critical percolation

From the edge weights ω_e, we obtain a coupling of all independent percolation processes. The set of p-open edges $e \in \mathcal{E}(\mathbb{Z}^d)$, edges such that

$$\omega_e \le p,$$

has the same distribution as Bernoulli percolation with parameter p.

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Let $\mathcal{C}(\mathbf{0})$ be the cluster of $\mathbf{0}$. For every p greater than the critical probability

$$p_c = \inf\{0 \leq p \leq 1 : \mathbb{P}_p(\#\mathcal{C}(\mathbf{0}) = \infty) > 0\},$$

there is a unique p-open infinite cluster (Aizenman-Kesten-Newman, 1987).

Link with critical percolation

1. Once an edge e of the p-open infinite cluster is invaded, the invasion never takes an edge with weight $\omega_e > p$. Chayes-Chayes-Newman (1985) showed that the IPC intersect the p-open infinite cluster with probability 1 in general dimension.
   In dimension 2, this follows easily from Russo-Seymour-Welsh. So:

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   $$\limsup_{i\to\infty} \tau_i \leq p_c.$$
2. For $p \leq p_c$, any p-open cluster is finite. So
   $$\tau_i > p_c$$
   infinitely often. From this, it follows that any p_c-open cluster connected to the invasion graph is eventually invaded.

Critical percolation and IPC: similarities and differences

Let S_n be the intersection of the invaded region with $B(n) = [-n, n]^2$. Jarai (2003) proved

$$\mathbb{E}(\#S_n)^k \asymp n^{2k} (\mathbb{P}_{cr}(\mathbf{0} \leftrightarrow \partial B(n)))^k. \tag{1}$$

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The IPC is infinite, whereas the cluster of $\mathbf{0}$ in critical percolation is finite almost surely, so the IPC is better compared to the *incipient infinite cluster* (IIC):

$$\mathbb{P}_{\mathrm{IIC}}(A) = \lim_{n\to\infty} \mathbb{P}(A \mid \mathbf{0} \leftrightarrow \partial B(n)).$$

Limit exists for events A depending on finitely many edges (Kesten, 1982), and determines a measure on configurations: "critical percolation conditioned on $\mathbf{0} \leftrightarrow \infty$".

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The moments of the size of the intersection of the IIC with $B(n)$ also satisfy (1) (Kesten, 1982.)

Critical percolation and IPC: similarities and differences

Jarai also shows that the distribution of invaded sites around a vertex $|v| \to \infty$ in the invasion converges to the IIC:

$$\lim_{|v|\to\infty} \mathbb{P}(\tau_v K \subset \mathcal{S} \mid v \in \mathcal{S}) = \mathbb{P}_{\mathrm{IIC}}(K \subset \mathcal{C}(\mathbf{0})).$$

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Here we present a different way to see that the IIC and the IPC are different on large scales.

Arm events

Let $\sigma \in \{O, C\}^k$, O for "open" and C for "closed". $|\sigma|_O$ is the number of O entries. $|\sigma|_C$ is the number of closed C entries. Denote

$$|\sigma| = |\sigma|_O + |\sigma|_C = k.$$

Let $A_\sigma(n)$ be the event that there are $|\sigma|_O$ open (resp. invaded) connections and $|\sigma|_C$ closed (resp. non-invaded) connections from $B(|\sigma|)$ to $\partial B(n)$.

In percolation theory, these connections are referred to as "arms".

Arm probabilities in critical percolation

Arm events are central to the study of critical probability. Examples:

1. The size of the cluster of the origin is given by $Cn^2\mathbb{P}(A_{(O)}(n))$.
2. The alternating four-arm event $\sigma = (O, C, O, C)$ is related to number of pivotal edges for a crossing. It features prominently in Kesten's work.

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Using SLE, Schramm-Lawler-Werner (2002) and Smirnov-Werner (2001) show

$$\mathbb{P}_{cr}(A_{(O)}(n)) = n^{-5/48+o(1)},$$

and

$$\mathbb{P}_{cr}(A_\sigma(n)) = n^{-(|\sigma|^2-1)/12+o(1)},$$

provided the sequence σ contains both O and C.

Results

For a given color sequence σ, define the reduced sequence $\tilde{\sigma}$ by replacing any run of more than two Cs by CC.

Example: if $\sigma = (O, C, C, C, O)$, then $\tilde{\sigma} = (O, C, C, O)$.

Theorem (Damron, Hanson, S.)

There is a constant C such that if $|\sigma|_O \geq 1$,

$$\mathbb{P}_{cr}(A_{\tilde{\sigma}}(n)) \leq C\mathbb{P}(A_\sigma(n)).$$

If $|\sigma|_O \geq 2$, then

$$\mathbb{P}(A_\sigma(n)) \leq C\mathbb{P}_{cr}(A_{\tilde{\sigma}}(n)).$$

$\mathbb{P}$: invasion measure, $\mathbb{P}_{cr}$: percolation at p_c.

Consequences

1. If $|\sigma|_O \geq 2$ and $\sigma = \tilde{\sigma}$, the probabilities for $A_\sigma(n)$ in invasion and critical percolation are comparable

$$\frac{1}{C}\mathbb{P}_{cr}(A_\sigma(n)) \leq \mathbb{P}(A_\sigma(n)) \leq C\mathbb{P}_{cr}(A_\sigma(n)).$$

2. If $\sigma \neq \tilde{\sigma}$, then by Reimer's inequality

$$\mathbb{P}_{cr}(A_\sigma(n)) \leq n^{-\epsilon}\mathbb{P}_{cr}(A_{\tilde{\sigma}}(n)),$$

so

$$n^{\epsilon}\mathbb{P}_{cr}(A_\sigma(n)) \leq \mathbb{P}(A_\sigma(n)).$$

Results: the case $|\sigma|_O = 1$.

For the next result, we consider arms sequences of the form

$$\sigma_k = (O, \underbrace{C, \ldots, C}_{k\text{times}}).$$

(If $|\sigma|_O > 1$, the previous theorem gives matching upper and lower bounds.)

Theorem (Damron, Hanson, S.)

For all $k \geq 1$,

$$\mathbb{P}_{cr}(A_{\tilde{\sigma}_k}(n))n^{\epsilon} \leq C\mathbb{P}(A_{\sigma_k}(n)).$$

For $k = 1, 2$, there exists $\epsilon > 0$ such that

$$\mathbb{P}(A_{\sigma_k}(n))n^{\epsilon} \leq C\mathbb{P}_{cr}(A_{\hat{\sigma}_k}(n)),$$

where

$$\hat{\sigma}_k = (\underbrace{C, \ldots, C}_{k \text{ times}}).$$

Consequences

1. If $k \geq 3$, then $\sigma_k \neq \tilde{\sigma}_k$, and the previous theorem shows

$$\mathbb{P}(A_{\sigma_k}(n)) \asymp \mathbb{P}(A_{\sigma_2}) \leq Cn^{-\epsilon}\mathbb{P}(A_{(OCC)}(n)).$$

2. If an exponent α_k existed such that $\mathbb{P}(A_{\sigma_k}(n)) = n^{-\alpha_k + o(1)}$, then

$$\alpha_2 > \beta(CC),\ \beta(OC) > \alpha_1,\ \text{ and } \beta(OCC) > \alpha_k, k \geq 2,$$

so all "monochromatic" invasion exponents are between 0 and β_3:

$$0 < \alpha_1 < \alpha_3 \leq \alpha_3 \leq \ldots < \beta_3.$$

Ideas of the proof: correlation length

Let us present a sketch of how to prove the inequality

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Russo-Seymour-Welsh says:

$$c(\alpha) \leq \mathbb{P}_{cr}(\text{exists an open crossing of } [-\alpha n, \alpha n] \times [-n, n]) \leq 1 - c(\alpha),$$

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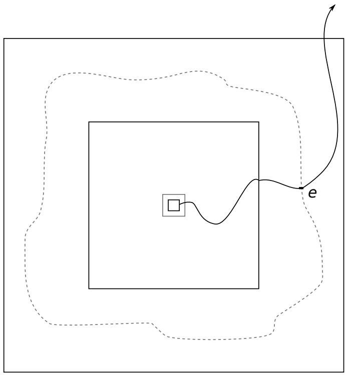
For $\epsilon > 0$, define the *correlation length*

$$L(p, \epsilon) = \min\{n \geq 1 : \mathbb{P}_p(\text{horizontal open crossing of } [-n, n]^2) > 1 - \epsilon\}.$$

Then for any n such that $\max(n, \alpha n) \leq L(p)$

$$c(\alpha) \leq \mathbb{P}_p(\text{exists an open crossing of } [-\alpha n, \alpha n] \times [-n, n]) \leq 1 - c(\alpha).$$

Step 1: Outlet



Let

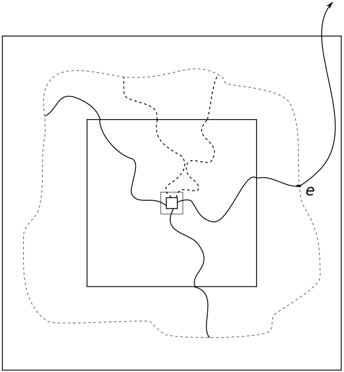
$$p_n = \min\{p : L(p) \leq n\}.$$

By a gluing construction, we construct an *outlet*: an edge $e \in B(2n) \setminus B(n)$ with weight $\omega_e \in (p_c, p_n)$, part of a p_n-closed circuit, connected to $\mathbf{0}$ by a p_c-open path, and ∞ by a p_n-open path.

e will be invaded eventually, and after that the invasion remains outside the circuit.

Constant probability cost.

Step 2: Arms inside the circuit



We construct σ arms inside the circuit. The open arms are p_c-open. The closed arms are p_n-closed. The invasion never enters the region delimited by p_n-closed paths.

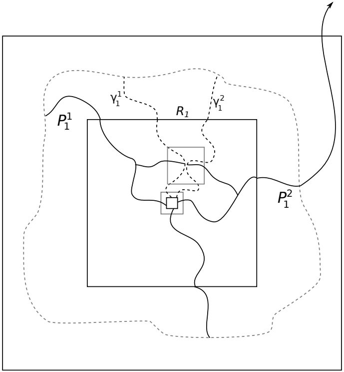
To evaluate the probability cost of this construction, we use the following well-known result

Theorem

Let $p, q \in [p_c, 1]$, and $n \leq \min\{L(p), L(q)\}$. Let $A_\sigma(n, p, q)$ be the probability that there are σ arms to distance n, with the open arms p-open and the closed arms q-closed. Then

$$\mathbb{P}(A_\sigma(n, p, q)) \asymp \mathbb{P}(A_\sigma(n, p_c, p_c)).$$

Step 3: 6-arm event



Any path in the region between the closed arms and the circuit will remain non-invaded, so we can obtain more arms there, provided there is enough space. We want to show that requiring that the closed arms are separated by some fixed distance costs constant probability.

6-arm event

By an argument of Kesten-Sidoravicius-Zhang, the 5-arm exponent is 2:

$$(1/C)n^{-2} \leq \mathbb{P}(A_{OCCCO}(n)) \leq Cn^{-2}.$$

By Reimer's inequality:

$$\mathbb{P}(A_{OCCOCC}(n)) \leq Cn^{-2}n^{-\epsilon}.$$

So the expected number of 6-arm points to distance K in $B(n) \setminus B(K)$ is $O(K^{-\epsilon})$.

6-arm event

If 2 closed arms come close to each other outside $B(K)$, a 6-arm point appears, so by the bound for the six-arm probability and a gluing argument:

$$\mathbb{P}(A_{\tilde{\sigma}}(K, n, p_c, p_n)) \leq \mathbb{P}(A^*_{\tilde{\sigma}}(K, n, p_c, p_n)),$$

where the starred event includes a separation condition.

Second inequality

The second bound of our first theorem is

$$\mathbb{P}(A_\sigma(n)) \leq C\mathbb{P}_{cr}(A_{\tilde{\sigma}}(n)).$$

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Let $\hat{e}_1$ be the *first outlet* of invasion: the edge e such that

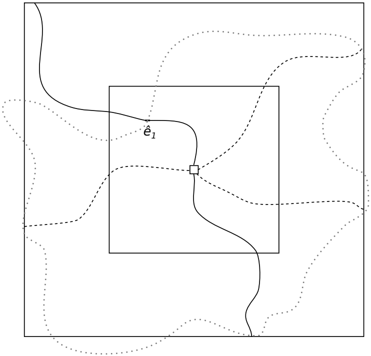
$$\tau_{\hat{e}_1} = \max\{\tau_e : e \in \mathcal{E}(\mathcal{S})\}.$$

We prove the inequality by splitting the left side into two:

$$\mathbb{P}(A_\sigma(n), \hat{e}_1 \in B(n/2)) + \mathbb{P}(A_\sigma(n), \hat{e}_1 \notin B(n/2)).$$

and deal with the two terms separately.

The outlet is near the origin



If $\hat{e}_1$ is near the origin an the invasion reaches to $\partial B(n)$, the outlet must in on one of the open arms. Then there are two disjoint $p_{\tau_{\hat{e}_1}}$-closed arms from $\partial B(n/2)$ to $\partial B(n)$ emanating from $\hat{e}_1$.

If $p_{\tau_{\hat{e}_1}}$ is high, this is very unlikely, if $p_{\tau_{\hat{e}_1}} \approx p_n$, then the entire invasion is contained in the p_n-open infinite cluster.

In each region with $l \geq 1$ consecutive non-invaded arms, we obtain $\min(2, l)$ p_c-closed arms.

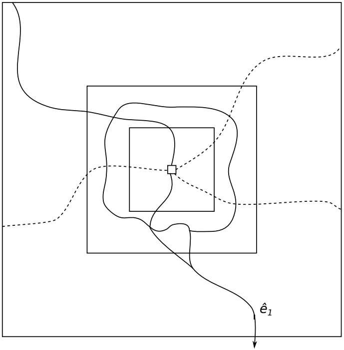
From the above construction, we obtain an upper bound of

$$\mathbb{P}(A_{\tilde{\sigma}}(n, p_c, p_n)),$$

which is comparable to $\mathbb{P}_{p_c}(A_{\tilde{\sigma}}(n))$.

In reality, there is an extra error term (to control the probability that $\tau_{\hat{e}_1}$ is high). We handle this by choosing an appropriate sequence of $p_n(j) \gg p_n$ and using an argument of Jarai.

Outlet is far from $\mathbf{0}$



When $\hat{e}_1$ is outside $B(n/2)$, we can make a construction to ensure the invasion is contained in the p_n-open infinite cluster.

In this case again, we obtain an error coming from the probability cost of the construction unless we use a sequence of $p_n(j) \downarrow p_c$. Then we can conclude by the same trick of Jarai.

Summary

1. We studied arm probabilities for invasion percolation in dimension $d = 2$. These are analogous to well-known quantities in critical percolation.
2. For some color sequences (two open arms or more, at most two consecutive closed arms), the arm probabilities are of the same order as in critical percolation.
3. For arm sequences with one open arm, the invasion and critical probabilities are not comparable.
4. It is possible that to obtain sharp results in case $|\sigma|_O = 1$, one needs to compare to the incipient infinite cluster instead.

Thanks for your attention!