Weak Maass forms: classification and examples

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Montreal-Toronto Workshop on Number Theory
Mock modular forms and their relatives
CRM, Montreal
December 9, 2016

Introduction

This talk is just a series of remarks.

- Growth conditions: moderate/immoderate.
- A representation theoretic classification of harmonic weak Maass forms (joint work with Kathrin Bringmann).
- Some examples of mock modular forms which are not harmonic.
- A mock modular form of weight $3/2$ and genus 2 arising from arithmetic 0-cycles.
  (old joint work with Michael Rapoport and Tonghai Yang).

Growth conditions

Let (ρ, V) be a finite dimensional complex representation of $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

Harmonic weak Maass forms are smooth functions

$f : \mathfrak{H} \longrightarrow V$ satisfying the conditions:

$$f(\gamma\tau) = (c\tau + d)^k \rho(\gamma) f(\tau)$$

$$\Delta_k f = 0 \qquad \text{(harmonicity)}$$

$$f(\tau) = O(e^{Bv}), \qquad \text{(\textit{immoderate} growth).}$$

Denote the space of these by $H_k^{\mathrm{img}}(\Gamma, \rho)$.

Note that the moderate growth condition is $f(\tau) = O(v^B)$.

Restrict to the scalar case, i.e., $\dim V = 1$, and omit ρ. The Fourier expansion has the form

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n)\, q^n$$

$$+ c_f^-(0)\, w_k(0, v) + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n)\, W_k(2\pi n v)\, q^n,$$

where

$$W_k(x) = \Gamma(1-k, -2x) + \begin{cases} \frac{(-1)^{1-k}\pi i}{(k-1)!} & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

$$w_k(0, v) = \begin{cases} v^{1-k} & \text{for } k \neq 1, \\ -\log v & \text{for } k = 1. \end{cases}$$

There are subspaces:

$$H_k^\flat(\Gamma) = \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^-(n) = 0 \text{ for } n > 0\},$$
$$H_k(\Gamma) = \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^-(n) = 0 \text{ for } n \geq 0\}$$
$$H_k^\sharp(\Gamma) = \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^+(n) = 0 \text{ for } n < 0\},$$
$$H_k^{\mathrm{mg}}(\Gamma) = H_k^\flat(\Gamma) \cap H_k^\sharp(\Gamma) = \text{ functions of moderate growth}$$
$$= \text{ automorphic forms.}$$

Note

$$f \in H_k^\flat(\Gamma) \iff \xi_k f \in M_{2-k}(\Gamma)$$
$$f \in H_k(\Gamma) \iff \xi_k f \in S_{2-k}(\Gamma)$$
$$f \in H_k^\sharp(\Gamma) \iff \text{only constant prin. parts}$$

Fundamental Fact: (Bruinier-Funke)

$$H_k^\sharp(\Gamma) \cap H_k(\Gamma) = M_k(\Gamma).$$

(e.g. $\equiv 0$ if $k < 0$.)

Proof: For $\xi_k f$ cuspidal,

$$(\xi_k f, \xi_k f)_{\mathrm{Pet}} = -(4\pi)^{k-1} \sum_{n>0} c_f^+(-n)\overline{c_f^-(-n)}n^{k-1}.$$

Consequences:

$$\text{constant prin. parts + cuspidal } \xi_k f \quad \Longrightarrow \quad \xi_k f = 0.$$

and so,

$$\xi_k f = \text{a cusp form} \neq 0 \quad \Longrightarrow \quad f \text{ is immoderate.}$$

Hence such an f is never an automorphic form.

Classification

A rough classification of harmonic Maass forms can be made using a little representation theory.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $\mathfrak{g} = \mathrm{Lie}\,(G)_{\mathbb{C}}$, $K = \mathrm{SO}(2)$.

For $f \in H_k^{\mathrm{img}}(\Gamma, \rho)$, define $\widetilde{f} \in C^\infty(G, V)$ by

$$\widetilde{f}(g) = j(g,i)^{-k} f(g(i)).$$

Then

$$\widetilde{f}(\gamma g k_\theta) = \widetilde{f}(g)\, e^{ik\theta}, \qquad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$C\widetilde{f} = (k^2 - 2k)\,\widetilde{f}, \qquad C = \text{Casimir op.}$$

$$|\widetilde{f}(g)| = O(e^{B||g||}), \qquad \text{immoderate growth.}$$

Classification

Let $C^\infty(G,V)^o = K$-finite vectors in $C^\infty(G,V)$.

The right action of G on $C^\infty(G,V)$ makes $C^\infty(G,V)^o$ into a $(\mathfrak{g},K)$-module.

Problem: Given $f \in H_k^{\text{img}}(\Gamma)$, describe the $(\mathfrak{g},K)$-submodule $\Pi(\widetilde{f})$ of $C^\infty(G,V)^o$ generated by $\widetilde{f}$.

Answer: (Bringmann-K.)

- There are 9 possibilities for the $(\mathfrak{g},K)$-module $\Pi(\widetilde{f})$.
- All 9 cases occur for vector valued Maass forms.

That there are 9 possibilities results from an elementary exercise in Lie theory.

That all 9 cases actually arise is shown by explicit examples.

Classification

Recall that: on $f \iff$ on $\widetilde{f}$

$$\text{raising} = R_r \iff X_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

$$\text{lowering} = L_r \iff X_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

$$\Delta_k f = R_{k-2}L_k f = 0 \iff X_+X_-\widetilde{f} = 0.$$

Writing $j = k \pm 2r$, let

$$\widetilde{f}_j = X_\pm^r \widetilde{f}.$$

Key point:

f harmonic $\implies$ the K-types in $\Pi(\widetilde{f})$ occur with multiplicity 1

$\implies$ the $\widetilde{f}_j$'s span $\Pi(\widetilde{f})$.

Recall that the basic building blocks are the irreducible constituents of the principal series $I(s, \chi)$.

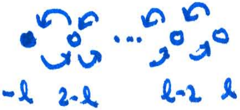
Since the Casimir eigenvalue is fixed on $\Pi(\widetilde{f})$ all of its constituents must be pieces of the same principal series.

Since $X_+\widetilde{f}_{k-2} = 0$, (harmonicity) the irreducible principal series do not occur.

Here are the building blocks:

Classification

$FD(\ell+1)$

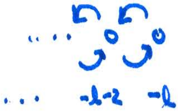


DS^+_k $(k \geq 2)$

LDS^+ $(k=1)$

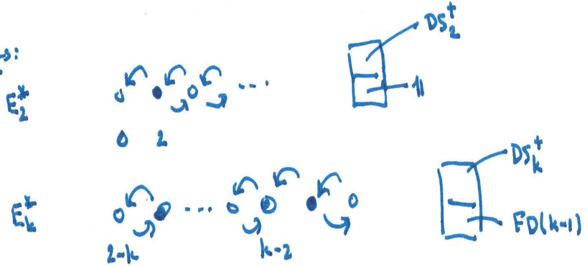
DS^-_ℓ $(\ell \geq 2)$

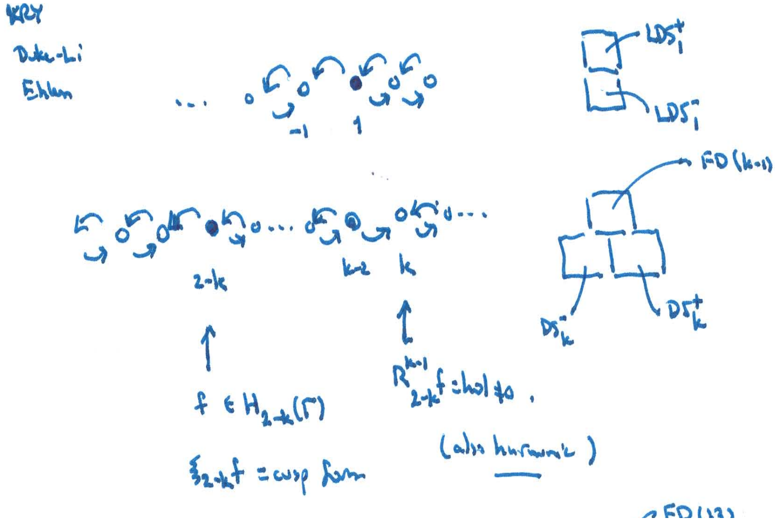
LDS^- $(\ell = 1)$



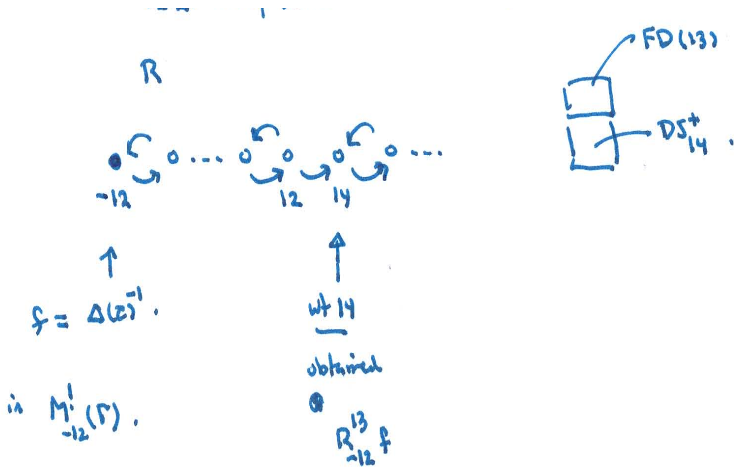
Classification

examples:




KRY
Duke-Li
Ehlen
-1
1
LDS$_1^+$
LDS$_1^-$
FD(k-1)
2-k
k-2
k
DS$_k^-$
DS$_k^+$
$f \in H_{2-k}(\Gamma)$
$\xi_{2-k} f$ = cusp form
$R^{k-1}_{2-k} f$ = hol ≠ 0,
(also harmonic)

Classification



Back to growth conditions

So far $G = \mathrm{SL}_2(\mathbb{R})$.

For groups of higher rank, there is a
Conjecture of Miatello-Wallach: Suppose

G a real semi-simple Lie group of split rank ≥ 2,

$\Gamma \subset G$ a discrete subgroup, $\mathrm{vol}(\Gamma\backslash G) < \infty$,

$f \in C^\infty(\Gamma\backslash G)^o$, K-finite,

f is annihilated by an idea of finite codimension in $\mathfrak{z}(\mathfrak{g})$.

Then f is of moderate growth.

Like the Koecher principle, this says there will be no immoderate Maass forms for higher rank groups.

Back to growth conditions

- **Example:** (Miatello-Wallach)
  $F/\mathbb{Q}$ totally real, $|F:\mathbb{Q}| = d > 1$,
  V = isotropic inner product space over F,

$$\begin{aligned} \mathrm{sig}(V) &= ((n,1),\dots,(n,1)), \quad n \geq 2 \\ L &\subset V, \quad O_F\text{-lattice}, \\ G &= R_{F/\mathbb{Q}}\mathrm{SO}(V)(\mathbb{R}) \simeq \mathrm{SO}(n,1)^d \\ \Gamma &\subset \Gamma_L \quad \text{finite index.} \end{aligned}$$

  Then the conjecture is true.
- There are recent results of Miller and Trinh for $\mathrm{SL}_3(\mathbb{Z}) \subset \mathrm{SL}_3(\mathbb{R})$.

A Siegel mock modular form

A mock modular form of genus 2 arises as a generating series for arithmetic 0-cycles.

Here is the setup:

$$B = \text{indefinite quaternion algebra over } \mathbb{Q}$$

$$D = D(B) = \text{prod of ramified primes}$$

$$O_B = \text{maximal order} \quad \Gamma = O_B^\times$$

$$M^B = \text{Shimura curve over } \mathbb{Q}, \quad M^B(\mathbb{C}) = \Gamma\backslash\mathfrak{H}^\pm$$

$$\mathcal{M}^B = \text{moduli space over } \operatorname{Spec}\mathbb{Z}$$

for abelian surfaces with

$$\iota : O_B \longrightarrow \operatorname{End}(A)$$

and the 'balanced condition'

$$\det(T - \iota(b)|\mathrm{Lie}\,(A)) = (T-b)(T-b')) \in \mathcal{O}_S[T].$$

A Siegel mock modular form

The space of special endomorphisms is the $\mathbb{Z}$-lattice

$$V(A,\iota) = \{x \in \mathrm{End}_{O_B}(A) \mid \mathrm{tr}(x) = 0\},$$

with positive definite quadratic form

$$x^2 = -Q(x)\, 1_A.$$

For $t \in \mathbb{Z}_{>0}$, $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$

$$\mathcal{Z}(t) = \text{locus of } (A,\iota,x),\ Q(x) = t,$$

$$\mathcal{Z}(T) = \text{locus of } (A,\iota,\mathbf{x}),\ \mathbf{x} = [x_1, x_2],\ Q(\mathbf{x}) = \frac{1}{2}((x_i, x_j)) = T,$$

are special cycles on $\mathcal{M}^B$ = arithmetic surface.

A Siegel mock modular form

For any quaternion algebra C over $\mathbb{Q}$, let

$$V^C = \{x \in C \mid \mathrm{tr}(x) = 0\}.$$

For $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$, let

$$C_T = \text{unique } C \text{ s.t. } V^C \text{ represents } T.$$

Let

$$\begin{aligned} \mathrm{Diff}(T, B) &= \{p \mid \mathrm{inv}_p(B) = -\mathrm{inv}_p(C_T)\} \\ |\mathrm{Diff}(T, B)| &= \text{odd, since} \\ \mathrm{inv}_\infty(B) &= -\mathrm{inv}_\infty(C_T). \end{aligned}$$

A Siegel mock modular form

The special cycle $\mathcal{Z}(T)$ is then

$$\begin{aligned}
\mathcal{Z}(T) &= \text{empty, if } |\mathrm{Diff}(T,B)| > 1 \\
\mathcal{Z}(T) &= \text{supported in } \mathcal{M}_p^B, \text{ if } \mathrm{Diff}(T,B) = \{p\}. \\
\mathcal{Z}(T) &= \text{0-cycle, if } p \nmid D(B). \\
&= \mathrm{Spec}\, R(T).
\end{aligned}$$

In the last case, we say T is <u>very good</u> and let

$$\widehat{\deg}\mathcal{Z}(T) := \log |R(T)|.$$

Our <u>Siegel mock modular form</u> is then

$$\phi_2^B(\tau) := \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z})_{>0} \\ T = \underline{\text{good}}}} \widehat{\deg}\, \mathcal{Z}(T)\, q^T, \quad \tau \in \mathfrak{H}_2, q^T = e(\mathrm{tr}(T\tau)).$$

A Siegel mock modular form

One of the main theorems in our book is:

There is a Siegel-Eisenstein series $\mathcal{E}(\tau, s; B)$ of weight $\frac{3}{2}$ such that

$\tau = u + iv$,

$$\mathcal{E}'(\tau, 0; B) = \phi_2^B(\tau) + \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z}) \\ T \text{ not good}}} c^B(T, v)\, q^T.$$

This is the genus 2 analogue of the KRY-IMRN form of weight 1 discussed earlier.

There are explicit formulas for all coefficients and an Arakelovian 'description' of each.

A Siegel mock modular form

Here is what the non-holomorphic part looks like.

For $T \in \mathrm{Sym}_2(\mathbb{Z})$ with $\mathrm{sig}(T) = (1,1)$ or $(0,2)$,

$$\mathcal{E}'(\tau, 0; B)_T = q^T \cdot \nu_\infty(T, v) \cdot \frac{1}{2}\,\mathrm{Rep}(T; L^B)$$

where $L_B = O_B \cap V^B$,

$$\mathrm{Rep}(T; L^B) = \left| \Gamma \backslash \{\, \mathbf{x} \in L_B^2 \mid \; Q(\mathbf{x}) = T \} \right|$$

and

$$q^T \cdot \nu_\infty(T, v) = \frac{\partial}{\partial s}\left\{ W_{T,\infty}(\tau, s, \Phi^{\frac{3}{2}}) \right\}\Big|_{s=0}$$

is the 'central derivative' Shimura's confluent hypergeometric function of matrix argument.

A Siegel mock modular form

Here is the underlying representation theory.
(details remain to be checked.)

$$K \subset G = \mathrm{Mp}_2(\mathbb{R}) \longrightarrow \mathrm{Sp}_2(\mathbb{R}) \supset \underline{U(2)} \simeq U(2)$$
$$\Gamma \longrightarrow \Gamma_0(4D(B)^0) \subset \mathrm{Sp}_2(\mathbb{Z}).$$

Let $f(\tau) = \mathcal{E}'(\tau, 0; B)$ and

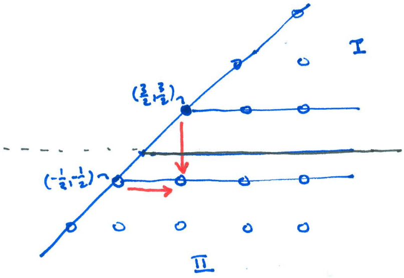
$$\widetilde{f}(g) = j(g, i)^{-3/2} f(g(i \cdot 1_2)).$$

Let

$$\Pi(\widetilde{f}) \subset C^\infty(\Gamma\backslash G)^o$$

be the $(\mathfrak{g}, K)$-module generated by $\widetilde{f}$.

The relevant principal series has the following picture:



A Siegel mock modular form

The open dots represent the various K-types $\sigma_{(a,b)}$, indexed by their highest weight $(a, b) \in -(\frac{1}{2}, \frac{1}{2}) + 2\mathbb{Z}^2$ with $a \geq b$.

They have multiplicity 1.

The K-types are linked by various raising and lowering operators coming from $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

Transitions crossing the black line are 0, all others are nonzero.

The regions labeled I and II are the irreducible summands Π_I and Π_{II}. They are generated by the scalar K-types $\sigma_{(\frac{3}{2},\frac{3}{2})} = \det^{\frac{3}{2}}$ and $\sigma_{(-\frac{1}{2},-\frac{1}{2})} = \det^{-\frac{1}{2}}$ respectively.

Π_I is a lowest weight representation (i.e., occurs for holomorphic Siegel modular forms of weight $\frac{3}{2}$).

A Siegel mock modular form

The $(\mathfrak{g}, K)$-module $\Pi(\widetilde{f})$ is a non-split extension with the same K-types as the principal series.

It has Π_{II} as a submodule and Π_I as quotient:

$$0 \longrightarrow \Pi_{II} \longrightarrow \Pi(\widetilde{f}) \longrightarrow \Pi_I \longrightarrow 0.$$

The downward red arrow is a <u>non-zero</u> lowering operator carrying $\sigma_{(\frac{3}{2},\frac{3}{2})}$ to $\sigma_{(\frac{3}{2},-\frac{1}{2})}$.

It corresponds to the classical lowering operator

$$L_{\frac{3}{2}} := \frac{\partial}{\partial \bar{\tau}} = \begin{pmatrix} \frac{\partial}{\partial \bar{\tau}_{11}} & \frac{1}{2}\frac{\partial}{\partial \bar{\tau}_{12}} \\ \frac{1}{2}\frac{\partial}{\partial \bar{\tau}_{12}} & \frac{\partial}{\partial \bar{\tau}_{22}} \end{pmatrix}.$$

It cannot be inverted.

A Siegel mock modular form

So the shadow $L_{\frac{3}{2}} f$ of $f = \mathcal{E}'(\tau, 0; B)$ corresponds to the 'not-holomorphic representation Π_{II}.

It can be linked via the horizontal red arrow to a Siegel-Eisenstein series of weight $-\frac{1}{2}$ associated to the space V^B of signature $(1, 2)$, hence the appearance of the indefinite representation numbers $\mathrm{Rep}(T; L^B)$ in the non-holomorphic Fourier coefficients of $\mathcal{E}'(\tau, 0; B)$.

It seems unlikely that there will be an analogue of the Duke-Li and Ehlen examples in higher genus, at least not realizable in some simple function space.

Perhaps one needs to further expand the context to spaces of distributions.