Mock modular forms of weight one and CM elliptic curves

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Goal: to convince you that there are interesting mock modular forms...

Outline:

1. (In)coherent Eisenstein series
2. Kudla-Rapoport-Yang: "On the derivative of an Eisenstein series of weight one"
3. Refinements: Duke-Li, Ehlen

Eisenstein series

- Fix $\chi \colon \mathbb{A}^\times/\mathbb{Q}^\times \;\to\; \mathbb{C}^\times$ quadratic
- $\mathbf{B} := \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset \mathbf{SL}_2$,
- For $\mathbf{b} = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in \mathbf{B}(\mathbb{A})$, set:

$$\chi(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = \chi(a) \qquad |\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)| := |a|_{\mathbb{A}}$$

For $s \in \mathbb{C}$

$$I(s,\chi) = Ind_{\mathbf{B}(\mathbb{A})}^{\mathbf{SL}_2(\mathbb{A})}(\chi|\cdot|^{s+1})$$

i.e. $I(s,\chi) =$ set of functions $\Phi(g,s)$ on $\mathbf{SL}_2(\mathbb{A})$ s.t.
$\Phi(\mathbf{b}g,s) = \chi(\mathbf{b})|\mathbf{b}|^{s+1}\Phi(g,s)$ for all $\mathbf{b} \in B(\mathbb{A})$

- Note $I(s,\chi) = \otimes'_v I_v(s,\chi_v)$

Eisenstein series

$$I(s,\chi) = Ind_{\mathbf{B}(\mathbb{A})}^{\mathbf{SL}_2(\mathbb{A})}(\chi|\cdot|^{s+1})$$

- A family $\Phi(\cdot, s) \in I(s,\chi)$ is:
  - **smooth** if $\Phi(g,s)$ holomorphic in s (where g fixed)
  - **standard** if $\Phi(k,s)$ is indep of s for all $k \in SO(2) \cdot \mathbf{SL}_2(\widehat{\mathbb{Z}}) =: \mathbf{K}$

$$\mathcal{E}(g,s;\Phi) \ := \sum_{\gamma \in \mathbf{B}(\mathbb{Q}) \backslash \mathbf{SL}_2(\mathbb{Q})} \Phi(\gamma g, s)$$

  converges for $Re(s) \gg 0$, admits meromorphic continuation to all $s \in \mathbb{C}$ and functional eqn $s \leftrightarrow -s$

- Rmk: since $\mathbf{SL}_2(\mathbb{A}) = \mathbf{B}(\mathbb{A})\mathbf{K}$, a std section is determined by its value at any fixed $s_0 \implies$ suffices to describe $I(0,\chi)$

- Define $\Phi_\infty^{(n)} \in I_\infty(s, \chi_\infty)$ "weight n" vector by:

$$\Phi_\infty^{(n)}(k_\theta, s) = e^{in\theta} \qquad \forall k_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in SO(2)$$

[here $\chi_\infty(-1) = (-1)^n$]

Classicalize: Fix $\Phi = \Phi_\infty^{(n)} \otimes \Phi_f \rightsquigarrow \mathcal{E}_\Phi(g, s)$ Eisenstein series

- For $\tau = u + iv \in \mathfrak{H}$, set $g_\tau = \left(\left(\begin{smallmatrix} v^{1/2} & u \\ & v^{-1/2} \end{smallmatrix}\right), 1, \dots\right) \in \mathbf{SL}_2(\mathbb{A})$

$$\begin{aligned} E_\Phi(\tau, s) &:= v^{-n/2}\mathcal{E}_\Phi(g_\tau, s) \\ &= v^{(s-n+1)/2} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{\Phi_f(\gamma)}{(c\tau + d)^n |c\tau + d|^{s-n+1}} \end{aligned}$$

Transforms like usual modular form of weight n

Decompose $I(0,\chi)$: the Weil representation

Let $V/\mathbb{Q}$ vector space, $\dim V = 2$ with bilinear form $\langle\cdot,\cdot\rangle$

- $S(V_{\mathbb{A}}) := S(V_{\mathbb{R}}) \otimes S(V_{\mathbb{A}_f})$ Schwartz-Bruhat space
  - $S(V_{\mathbb{R}}) = C^\infty$ functions on $V \otimes \mathbb{R}$ of rapid decay. E.g. $\varphi_\infty(x) = e^{-\pi x^2}$
  - $S(V_{\mathbb{A}_f}) =$ locally constant compactly supported fns on $V \otimes \mathbb{A}_f$
    e.g. the char fn of $L \otimes \widehat{\mathbb{Z}}$ for a lattice $L \subset V$

The **Weil representation** $\omega \colon \mathbf{SL}_2(\mathbb{A}) \to \mathrm{Aut}\, S(V_{\mathbb{A}})$ satisfies:

$$\begin{aligned} (\omega(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix})\varphi)(x) &= \psi(b\langle x,x\rangle)\varphi(x) \\ (\omega(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix})\varphi)(x) &= \chi_V(a)\,|a|_{\mathbb{A}} \cdot \varphi(ax) \end{aligned}$$

[$\psi \colon \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ additive character, $\chi_V = (\cdot, -\det V)_{\mathbb{A}}$]

In particular, for all $g \in \mathbf{SL}_2(\mathbb{A})$, and $\mathbf{b} = (\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix})$,

$$(\omega(\mathbf{b}\, g)\varphi)\,(0) = \chi_V(a)\,|a|_{\mathbb{A}} \cdot (\omega(g)\varphi)(0)$$

Remark: also $\omega = \otimes_v' \omega_v$

In particular, if $\chi = \chi_V$, get an intertwining map

$$\lambda_V : S(V) \to I(\chi, 0), \qquad \lambda_V(\varphi)(g) = \omega(g)\varphi(0)$$

Remark: only intertwining at $s = 0$!

- Fact (Kudla-Rallis): $R(V) = im(\lambda_V)$ is an irred summand of $I(\chi, 0)$, and

  $$\bigoplus_V R(V) \hookrightarrow I(0, \chi)$$

  where sum is over binary quadratic spaces with $\chi_V = \chi$

Definition

A standard section $\Phi(g, s)$ (resp. the corresponding Eisenstein series) is **coherent** if $\Phi(g, 0) \in R(V)$ for some V.

Theta functions

Theta functions: for $g \in \mathbf{SL}_2(\mathbb{A})$

$$\Theta(g, \varphi) := \sum_{x \in V} (\omega(g)\varphi)(x)$$

Invariant under left mult by $\mathbf{SL}_2(\mathbb{Q})$ (Poisson summation)
Note also have action of $O(V_{\mathbb{A}})$ on $S(V_{\mathbb{A}})$ via $(h \cdot \varphi)(x) = \varphi(h^{-1}x)$

Siegel-Weil formula

Suppose V anisotropic, $\varphi \in S(V_{\mathbb{A}})$ and $\Phi(g, s)$ corresponding coherent section (i.e. $\Phi(g, 0) = \lambda(\varphi)$). Then

$$\int_{O(V)(\mathbb{A})/O(V)(\mathbb{Q})} \Theta(g; h \cdot \varphi)\, dh = 2\mathcal{E}_{\Phi}(g, 0)$$

Note $\mathcal{E}_{\Phi}$ is built from local data, but Θ is global.

Incoherent Eisenstein series

Note $\omega = \otimes' \omega_v$ and $I(s, \chi) = \otimes' I(s, \chi_v)$

- If V_ν local quadratic space (i.e. over $\mathbb{Q}_\nu$ or $\mathbb{R}$) such that $\chi_\nu = (\cdot, -\det V_\nu)_\nu$, have local version

$$\lambda_{V_\nu} \colon S(V_\nu) \to I_\nu(0, \chi_\nu), \qquad \lambda_{V_\nu}(\varphi_\nu)(g_\nu) = (\omega_\nu(g_\nu)\varphi_\nu)(0)$$

- An **incohorent collection** $\mathcal{C} = (V_\nu)_{\nu \leq \infty}$ with character χ is a collection of local spaces s.t.
    1. $\chi_\nu = (\cdot, -\det V_\nu)_\nu$ for all ν
    2. for every global $W/\mathbb{Q}$, have $W_\nu \simeq V_\nu$ almost everywhere, but **not everywhere** (colloquially, $\mathcal{C} \neq W \otimes \mathbb{A}$)

String together local maps to get $\lambda_{\mathcal{C}} \colon S(\mathcal{C}) \to I(0, \chi)$, set $R(\mathcal{C}) = im(\lambda_{\mathcal{C}})$

Theorem (Kudla-Rallis)

1. $$I(0,\chi) = \left(\bigoplus_{\mathcal{C}} R(\mathcal{C})\right) \oplus \left(\bigoplus_{V} R(V)\right)$$

   where $\mathcal{C}$ and V range over (incoherent collections of) binary quadratic spaces of character χ

2. *Suppose $\Phi(g,0) \in R(\mathcal{C})$ for some incoherent collection and $\mathcal{E}_{\Phi}(g,s)$ the corresponding* **incoherent** *Eisenstein series. Then*

   $$\mathcal{E}_{\Phi}(g,0) = 0.$$

The Eisenstein series of Kudla-Rapoport-Yang

Fix $q \equiv 3 \pmod 4$ prime, $\chi = (\cdot, -q)_{\mathbb{A}}$

- $\mathbf{k} := \mathbb{Q}(\sqrt{-q})$
- Two binary quadratic spaces: $V^{\pm} = (\mathbf{k}, \pm Nm)$
- Schwartz-Bruhat functions: $\varphi^{\pm} = \varphi^{\pm}_{\infty} \otimes \varphi^{\pm}_{f} \in S(V^{\pm}_{\mathbb{R}}) \otimes S(V^{\pm}_{\mathbb{A}_f})$

$$\varphi^{+}_{\infty}(x) = e^{-2\pi Nm(x)} \in S(V^{+}_{\mathbb{R}}), \qquad \varphi^{+}_{f}(x) = \text{char fn of } \widehat{o}_{\mathbf{k}} \in S(V^{+}_{\mathbb{A}_f})$$

$$\varphi^{-}_{\infty}(x) = e^{-2\pi Nm(x)} \in S(V^{-}_{\mathbb{R}}), \qquad \varphi^{-}_{f}(x) = \text{char fn of } \widehat{o}_{\mathbf{k}} \in S(V^{-}_{\mathbb{A}_f})$$

- "same" function but different spaces! In particular, $\varphi^{\pm}_{\infty}$ has weight ± 1

Obtain **theta functions** $\Theta^{\pm}(g) := \Theta(g, \varphi^{\pm})$ of weight ± 1. Classicalize:

$$\theta^{+}(\tau) = \sum_{x \in o_{\mathbf{k}}} e^{2\pi i N(x)\tau}, \qquad \theta^{-}(\tau) = v \sum_{x \in o_{\mathbf{k}}} e^{-2\pi i N(x)\overline{\tau}} = v\,\overline{\theta^{+}(\tau)}$$

This data also gives two coherent Eisenstein series $\mathcal{E}^{\mathrm{coh}}_{\pm 1}(g, s)$ [classicalized: $E^{coh}_{\pm 1}(\tau, s)$]

Siegel-Weil: integrating Θ^{+} over $[O(V^{+})]$ gives

$$E^{coh}_{1}(\tau, 0) = 1 + \frac{1}{h_{\mathbf{k}}} \sum_{\mathfrak{a} \subset o_{\mathbf{k}}} e^{2\pi i N(\mathfrak{a})\tau} \;=\; \frac{1}{h_{k}} \cdot \sum_{[\mathfrak{a}] \in Cl(\mathbf{k})} \theta_{\mathfrak{a}}(\tau)$$

where $\theta_{\mathfrak{a}}(\tau) = \sum_{x \in \mathfrak{a}} e^{2\pi i N(x)\tau / N(\mathfrak{a})}$

and $E^{coh}_{-1}(\tau, 0) = v\overline{E^{coh}_{1}(\tau, 0)}$

The incoherent Eisenstein series of KRY

Recall $V^{\pm} = (\mathbf{k}, \pm Nm)$

- Let $\mathcal{C} = (V_v)$ incoherent collection: $V_\infty = V_\infty^+$, $V_v = V_v^-$
- $\varphi^{inc} = \varphi_\infty^+ \otimes \varphi_f^- \in S(\mathcal{C}) \rightsquigarrow \Phi^{inc} \in I(s, \chi)$ standard section

$$\mathcal{E}_1^{inc}(g, s) \text{ weight 1 incoherent } \Longrightarrow \mathcal{E}_1^{inc}(g, 0) = 0$$

Classicalization: $E_1^{inc}(\tau, s)$

We consider:

$$E_1'(\tau, 0) \ := \ \frac{d}{ds} E_1^{inc}(\tau, s)\Big|_{s=0}$$

Goal: show $E_1'(\tau, 0)$ is **harmonic**, and study holomorphic+non-holomorphic coeffs.

Take $\chi_\infty(-1) = -1$, then

$$I_\infty(s, \chi_\infty) = \bigoplus_{n \text{ odd}} \mathbb{C} \cdot \Phi_\infty^{(n)} \qquad \text{weight } n \text{ vector}$$

- Let $X_\pm = \left(\begin{smallmatrix} 1 & \pm i \\ \pm i & -1 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$. Infinitesimal action:

$$\omega(X_\pm)\Phi_\infty^{(n)}(g, s) = (s + 1 \pm n)\Phi_\infty^{(n\pm 2)}(g, s)$$

- Under classicalization, action of $X_\pm$ on weight n vector translates to Maass raising and lowering ops:

$$R = -2i\frac{\partial}{\partial \tau} + \frac{n}{v}, \qquad L = 2i\frac{\partial}{\partial \overline{\tau}}$$

- $X_- \Phi_\infty^{(1)}(g,s) = s\,\Phi_\infty^{(-1)}(g,s) \qquad X_+ \Phi_\infty^{(-1)}(g,s) = s\,\Phi_\infty^{(1)}(g,s)$

$$L(E_1^{inc}(\tau,s)) = s \cdot E_{-1}^{coh}(\tau,s) \qquad \text{and} \qquad R(E_{-1}^{coh}(\tau,s)) = s \cdot E_1^{inc}(\tau,s)$$

Consequences at $s = 0$ (compare Laurent exp.):

1. $R(E_{-1}(\tau,0)) = 0 \implies v^{-1}E_{-1}(\tau,0)$ is **anti-holomorphic** in τ
2. $LE_1'(\tau,0) = E_{-1}(\tau,0)$

Note $\xi_1(f) = v^{-1}\overline{L(f)}$ by defn, so

$$\xi_1\left(E_1'(\tau,0)\right) = v^{-1}\overline{E_{-1}^{coh}(\tau,0)} = E_1^{coh}(\tau,0) \in M_1(\Gamma_0(q),\chi)$$

i.e. $E_1'(\tau,0)$ is a harmonic form of weight 1 with shadow $E_1^{coh}(\tau,0)$

Fourier coefficients of $E_1'(\tau, 0)$

$$E_1'(\tau,0) \;=\; \sum_{n\leq 0} c^-(n)\,\beta_1(2\pi|n|v)\,q^n \;+\; \sum_{n\geq 0} c^+(n)q^n$$

Non-holomorphic coefficients governed by shadow:

$$\xi E_1'(\tau,0) = E_1^{coh}(\tau,0) = 1 + \frac{1}{h_{\mathbf{k}}} \sum_{\mathfrak{a}\subset o_{\mathbf{k}}} e^{2\pi i N(\mathfrak{a})\tau}$$

so $h_{\mathbf{k}} \cdot c^-(n) = \rho(|n|) = \#$ of integral ideals of norm $|n|$

Holomorphic coefficients

Suppose $\Phi = \otimes \Phi_v$ factors, $E_\Phi(\tau, s) = \sum A_m(\tau, s, \Phi)$ Fourier expansion

- $A_m(\tau, s, \Phi)$ can be expressed as integral over $N(\mathbb{Q})\backslash N(\mathbb{A})$, where $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\}$
- Unfolding gives product formula (e.g. for $m \neq 0$)

$$A_m(\tau, s, \Phi) = W_{m,\infty}(\tau, s, \Phi_\infty) \prod_{p<\infty} W_{m,p}(s, \Phi_p)$$

where e.g.

$$W_{m,p}(s, \Phi_p) = \int_{\mathbb{Q}_p} \Phi_p\left(\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right), s\right) \psi_p(-mb)db$$

For $m = 0$ have analogous expression:

$$A_0(\tau, s, \Phi) = \Phi(\tau, s) \; + \; W_{0,\infty}(\tau, s, \Phi_\infty) \prod_{p<\infty} W_{0,p}(s, \Phi_p)$$

For our weight one incoherent Eis series: $\Phi = \Phi_\infty^+ \otimes \Phi_f^-$ and $\chi = (\cdot, -q)_{\mathbb{A}}$

$$E_1^{inc}(\tau, s) = \sum A_m(\tau, s), \quad A_m(\tau, s) = W_{m,\infty}(\tau, s, \Phi_\infty^{(1)}) \prod_{p<\infty} W_{m,p}(s)$$

For $m \neq 0$

1. $W_{m,\infty}(\tau, 0) = q^m$ if $m > 0$ and $W_{m,\infty}(\tau, 0) = 0$ if $m < 0$
2. If $p \neq q$:

$$W_{m,p}(s) = L_p(s+1, \chi)^{-1} \cdot \sum_{r=0}^{ord_p(m)} (\chi_p(p) p^{-s})^r$$

3. If $p = q$ then $W_{m,q}(s) = i\sqrt{q}\left(1 - \chi_q(m) q^{-s(ord_q(m)+1)}\right)$
4. In all cases if m is not a norm in V_p^-, then $W_{m,p}(0) = 0$.

Note: if $m > 0$, $\exists p$ s.t. m is not a norm in V_p^- (Hasse principle)
$\implies E_1^{inc}(\tau, 0) = 0$

$$E_1^{inc}(\tau, s) = \sum A_m(\tau, s), \qquad A_m(\tau, s) = W_{m,\infty}(\tau, s) \prod_{p<\infty} W_{m,p}(s)$$

$$A'_m(\tau, 0) = W'_{m,\infty}(\tau, 0) \prod_{p<\infty} W_{m,p}(0) + \left(\sum_{p<\infty} W'_{m,p}(0) \cdot \prod_{v \neq p} W_{m,v}(0) \right) \cdot q^m$$

Recall for $m > 0$: $\exists p$ s.t. $W_{m,p}(0) = 0$, and $W_{m,\infty}(\tau, 0) = q^m$

Let $\mathit{Diff}(m) := \{p \mid m \text{ not a norm in } V_p^-\}$;

- If $\#\mathit{Diff}(m) > 1$ then $c^+(m) = A'_m(\tau, 0) q^{-m} = 0$
- If $\mathit{Diff}(m) = \{p\}$ then

$$\begin{aligned} c^+(m) &= W'_{m,p}(0) \cdot \prod_{v \neq p} W_{m,v}(0) \\ &= \frac{1}{h_{\mathbf{k}}} (ord_p(m) + 1) \log p \cdot \begin{cases} \rho(m/p) & p \text{ inert} \\ \rho(m) & p = q \end{cases} \end{aligned}$$

[$\rho(m) = \#$ of $o_{\mathbf{k}}$ ideals of norm m]

Moduli of CM elliptic curves

- $\mathbf{k} = \mathbb{Q}(\sqrt{-q})$, $o_{\mathbf{k}}$ ring of integers
- Let $\mathcal{M} \to \mathrm{Spec}(o_{\mathbf{k}})$ moduli stack of CM elliptic curves
- i.e. for $S/\mathrm{Spec}(o_{\mathbf{k}})$,

$$\mathcal{M}(S) \;=\; \{(E,\iota) \mid E/S \text{ ell curve, } \iota\colon o_{\mathbf{k}} \to End(E)\}$$

  where action is normalized to coincide with structural morphism $o_{\mathbf{k}} \to \mathcal{O}_S$ on $Lie(E)$
- E.g. fix $\mathbf{k} \subset \mathbb{C}$:

$$\mathcal{M}(\mathbb{C}) \;=\; \text{cpx ell curves with CM by } o_{\mathbf{k}} \;\simeq\; Cl(\mathbf{k})$$

  where $\mathfrak{a}$ corresponds to $E_{\mathfrak{a}} := \mathbb{C}/\mathfrak{a}$
- Coarse moduli space is $M = \mathrm{Spec}(O_H)$, where $H =$Hilbert class field

Special endomorphisms and special cycles

Let $m \in \mathbb{Z}, m > 0$.

Special cycles

Define moduli space $Z(m)$ whose S points parametrize (E, ι, y) where

- $(E, \iota) \in \mathcal{M}(S)$
- $y \in End(E)$ such that
  1. $y \circ \iota(a) = \iota(\bar{a}) \circ y \quad \forall a \in o_{\mathbf{k}}$
  2. $y^2 = -m$

Forgetful morphism $Z(m) \to \mathcal{M}$ defines a divisor. Define

$$\widehat{\deg}\, Z(m) = \sum_{\mathfrak{p} \subset o_{\mathbf{k}}} \sum_{x \in Z(m)(\overline{\mathbb{F}_p})} \left(\operatorname{length} \mathcal{O}_{Z(m),x}\right) \cdot \log N(\mathfrak{p})$$

Theorem (Kudla-Rapoport-Yang)

$$\widehat{\deg} Z(m) = h_{\mathbf{k}} \cdot c^{+}(m)$$

Support of $Z(m)$

Q: in which char's does $Z(m)$ have geom points?
Let $B_0 = (\frac{-m,-q}{\mathbb{Q}})$ denote the quaterion algebra over $\mathbb{Q}$ generated by i, j,

$$i^2 = -m, \quad j^2 = -q, \quad ij = -ji$$

Let $\mathbb{F}$ algebraically closed field, $(E, \iota, y) \in Z(m)(\mathbb{F})$

$$\implies B_0 \hookrightarrow End(E) \otimes \mathbb{Q}$$

From classification of end algebras of ell curves: if E has CM by $\mathbf{k}$, then

- if $char(\mathbb{F}) = 0$ then $End(E) \otimes \mathbb{Q} = \mathbf{k}$
- if $char(\mathbb{F}) = p$, then

$$End(E) \otimes \mathbb{Q} = \begin{cases} \mathbf{k} & \textbf{[ordinary]} \\ \mathbb{B}_{(p,\infty)} = \text{ quat alg ramified at } p \text{ and } \infty & \textbf{[supersing.]} \end{cases}$$

with **[ordinary]** iff p splits in $\mathbf{k}$.

So $Z(m)(\overline{\mathbb{F}_p}) \neq 0 \implies B_0 = (\frac{-m,-q}{\mathbb{Q}}) \simeq \mathbb{B}_{(p,\infty)}$
There is **at most** one p for which this can happen, and in this case

$$-m \notin N(k_p), \qquad \text{but} \qquad -m \in N(k_l) \text{ for all } l \neq p$$

More precisely:

Let
$Diff(m) = \{p \text{ non-split s.t. } -m \notin N(k_p)\} = \{p \mid m \text{ not a norm in } V_p^-\}$

- If $\#Diff(m) \neq 1$, then $Z(m) = \emptyset$
- If $Diff(m) = \{p\}$ then $|Z(m)|$ is supported at supersingular points in char. p

First sign of a miracle:

$$\#Diff(m) \neq 1 \implies \deg Z(m) = 0 = c^+_{E'}(m)$$

Suppose $\mathit{Diff}(m) = \{p\}$, and $\mathfrak{p} \subset o_{\mathbf{k}}$ above p

$$\implies \deg Z(m) \;=\; \sum_{x \in Z(m)(\overline{\mathbb{F}_{\mathfrak{p}}})} \mathrm{length}\left(\mathcal{O}_{Z(m),x}\right) \cdot \log N(\mathfrak{p})$$

Theorem

- *(Gross):* $\mathrm{length}\left(\mathcal{O}_{Z(m),x}\right) \cdot \log N(\mathfrak{p}) = (\mathit{ord}_p(m) + 1) \log p$
- $\#Z(m)(\overline{\mathbb{F}_p}) = \rho(m/p)$ *if p inert*
- $\#Z(m)(\overline{\mathbb{F}_p}) = \rho(m)$ *if $p = q$*

$$\implies h_{\mathbf{k}} \cdot c^{+}(m) = (\mathit{ord}_p(m) + 1) \log p \cdot \begin{cases} \rho(m/p) \\ \rho(m) \end{cases} = \deg Z(m)$$

as required.

Proof of point count

Fix $(E_0, \iota_0) \in \mathcal{M}(\overline{\mathbb{F}_p})$ and an isom $End(E_0)_{\mathbb{Q}} \simeq \mathbb{B} = \mathbb{B}_{(p,\infty)}$

- Can write $\mathbb{B} = \mathbf{k} + \mathbf{k} \cdot \vartheta$ with

$$\mathbf{k} \cdot \vartheta = \{b \in \mathbb{B} \mid b\, a = \bar{a}\, b \;\forall a \in \mathbf{k}\}, \qquad Nrd(\vartheta) = \begin{cases} p, & \text{if } p \text{ inert} \\ 1, & \text{if } p = q \end{cases}$$

- Let $\mathcal{O} = End(E_0) \subset \mathbb{B}$, wolog can assume $\iota_0 \colon o_{\mathbf{k}} \to \mathcal{O}$ optimal
- Serre construction: action of $Cl(\mathbf{k})$ on $M(\overline{\mathbb{F}_p})$ by $\mathfrak{a} \cdot E = E \otimes \mathfrak{a}$ [simply transitive]
- if $\mathfrak{a} \subset o_k$, then have isogeny $E \otimes \mathfrak{a} \to E$ of degree $N(\mathfrak{a})$
- Identifies $End(E \otimes \mathfrak{a}) \simeq \mathfrak{a} \cdot \mathcal{O} \cdot \mathfrak{a}^{-1} \subset \mathbb{B}$

$$
\begin{aligned}
\#Z(m)(\overline{\mathbb{F}_p}) &= \coprod_{(E,\iota)\in\mathcal{M}(\mathbb{F}_p)} \#\{y \in End(E) \mid y \circ i(a) = i(\overline{a}) \circ y,\ y^2 = -m\} \\
&= \coprod_{[\mathfrak{a}]\in Cl(\mathbf{k})} \#\left\{b \in \mathfrak{a}\cdot\mathcal{O}\cdot\mathfrak{a}^{-1} \mid b^2 = -m, b\cdot a = \overline{a}\cdot b\ \forall a \in \mathbf{k}\right\} \\
&= \coprod_{[\mathfrak{a}]} \#\left\{b \in (\mathfrak{a}\cdot\mathcal{O}\cdot\mathfrak{a}^{-1}) \cap \mathbf{k}\cdot\vartheta \mid b^2 = -m\right\} \\
&= \coprod_{[\mathfrak{a}]} \#\left\{t \in \mathbf{k} \mid t\vartheta \in \mathfrak{a}\cdot\mathcal{O}\cdot\mathfrak{a}^{-1},\ Nm(t) = m/Nrd(\vartheta)\right\} \\
&= \coprod_{[\mathfrak{a}]} \#\left\{t \in \mathfrak{a}\cdot(\overline{\mathfrak{a}})^{-1},\ Nm(t) = m/Nrd(\vartheta)\right\}
\end{aligned}
$$

Since $h(\mathbf{k})$ is odd, every ideal class has rep of the form $\mathfrak{a}\cdot(\overline{\mathfrak{a}})^{-1}$

$$
\implies \#Z(m)(\overline{\mathbb{F}_p}) = \coprod_{\mathfrak{b}} \#\{t \in \mathfrak{b}, Nm(t) = m/Nrd(\vartheta)\} = \rho(m/Nrd(\vartheta))
$$

Refinements:

Recall $\xi(E_1'(\tau,0)) = E_1^{coh}(\tau,0) = \frac{1}{h_{\mathbf{k}}}\sum_{[\mathfrak{a}]}\Theta_{\mathfrak{a}}(\tau)$
Q: Can we find interesting preimages of the individual $\Theta_{\mathfrak{a}}(\tau)$'s?

Theorem (Duke-Li, Ehlen)

There exists $\widetilde{\Theta}_{\mathfrak{a}} \in H_1^!(\Gamma_0(q),\chi)$ *with shadow* $\Theta_{\mathfrak{a}}$ *and holomorphic part*

$$\sum_{m \gg -\infty} c_{\mathfrak{a}}^+(m) q^m$$

1. $c_{\mathfrak{a}}^+(m) = -\frac{2}{r}\log|u_{\mathfrak{a}}(m)|$ *for some* $u_{\mathfrak{a}}(m) \in O_H$
   [H = Hilbert class field, $r \in \mathbb{Z}_{>0}$ depends only on **k***]*
2. *If* $m \le 0$ *or* $m > 0$ *and* $\#Diff(m) \neq 1$ *then* $u_{\mathfrak{a}}(m) \in O_H^\times$

Theorem (Duke-Li, Ehlen)

There exists $\widetilde{\Theta}_{\mathfrak{a}} \in H_1^!(\Gamma_0(q), \chi)$ *with shadow* $\Theta_{\mathfrak{a}}$ *and holomorphic part*

$$\sum_{m \gg -\infty} c_{\mathfrak{a}}^+(m) q^m$$

1. $c_{\mathfrak{a}}^+(m) = -\frac{2}{r} \log |u_{\mathfrak{a}}(m)|$ *for some* $u_{\mathfrak{a}}(m) \in O_H$
   [H = *Hilbert class field,* $r \in \mathbb{Z}_{>0}$ *depends only on* $\mathbf{k}$]
2. *If* $m \leq 0$ *or* $m > 0$ *and* $\#Diff(m) \neq 1$ *then* $u_{\mathfrak{a}}(m) \in O_H^\times$
3. *If* $Diff(m) = \{p\}$, *let* $\mathfrak{B}_0 \subset o_H$ *unique conj.-invariant prime above* p. *If* $\mathfrak{B}|p$, *write* $\mathfrak{B} = \sigma(\mathfrak{B}_0)$ *for* $\sigma \in Gal(H/\mathbf{k}) \leftrightarrow \mathfrak{b} \in Cl(\mathbf{k})$. *Then*

$$ord_{\mathfrak{B}}(u_{\mathfrak{a}}(m)) = r\,(ord_p(m) + 1)\,\rho(mq/p, [\mathfrak{a}\mathfrak{b}^{-2}])$$

where $\rho(t, [\mathfrak{c}]) = \#$ *of integral ideals of norm* t *in ideal class of* $[\mathfrak{c}]$

Key ideas of proof (following Ehlen):

1. see saw identity relating linear combinations of $c_{\mathfrak{a}}^{+}(m)$'s to log of values of Borcherds products on mod curves (rational meromorphic **functions** i.e. weight zero) at CM points
2. Factorization formula: CM point corresponds to morphism $z\colon \mathrm{Spec}(o_H) \to X_0(q)$; pulling back the Borcherds product along z gives relation between the ideal generated by $\Psi(z)$ and pullback of its divisor
3. Relations between $Z(m)$'s on $\mathrm{Spec}(o_H)$ induced by $div(\Psi)$ are same as relations between $c_{\mathfrak{a}}^{+}(m)$, get a bootstrapping argument from principal part