Basic Background on Mock Modular Forms and Weak Harmonic Maass Forms

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1 Introduction

These notes mainly derive from Ken Ono's exposition "Harmonic Maass Forms, Mock Modular Forms, and Quantum Modular Forms" ([3]), in particular his section 4. No originality is claimed.

The subject of mock modular forms begins with Ramanujan's last letter to Hardy. In it, he presents a vague definition of a mock $\theta-$function, and 17 examples. Ramanujan typically thought of modular forms and these new mock $\theta-$functions as $q-$series; one of his examples is

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

This is also an example of an *Eulerian series*, i.e. a combinatorial formal power series constructed from basic hypergeometric series. His description of mock $\theta-$functions is roughly as follows: take an Eulerian series f, and assume it has exponential singularities at infinitely many roots of unity. If it is *not possible* to write f as a sum of a weakly holomorphic modular form with weight $k \in \frac{1}{2}\mathbb{Z}$ on some $\Gamma_1(N)$, and a function which is $O(1)$ at all roots of unity, then we call f a mock $\theta-$function. Ramanujan admitted that he could not prove that any of his examples satisfied his definition; a gap which was filled in by Griffin, Rolen, and Ono in 2013 ([2]).

As it turns out, Ramanujan's examples are all holomorphic parts of certain weight 1/2 harmonic Maass forms (which will be defined). However as shown by Rhoades in [4], there exist functions which satisfy Ramanujan's definition of mock $\theta-$function, but are not holomorphic parts of any weight 1/2 harmonic Maass forms.

We end the introduction with a historical note. After Ramanujan's death, 50 years passes with little progress on his mysterious mock $\theta-$functions. Then in 1976, mathematician G.E. Andrews found Ramanujan's lost notebook: papers collected by his family after his death, which had been sent to England. The papers had remained in the Trinity College Library archives, somehow forgotten for over 55 years. A remarkable note is Ramanujan listed 5 more examples of his mock $\theta-$functions, three of which had been discovered by G.N. Watson, but the other 2 were new.

2 Definitions

For N a positive integer, recall the following congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$

$$\begin{aligned}
\Gamma_0(N) :=& \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},\\
\Gamma_1(N) :=& \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } c \equiv 0 \pmod{N} \right\},\\
\Gamma(N) :=& \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},
\end{aligned}$$

In this paper, we will be considering smooth functions in $z = x + iy$ on $\mathbb{H}$, where $x, y \in \mathbb{R}$. For a real number k, define the weight k hyperbolic Laplacian Δ_k operator by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = -4y^2\frac{\partial^2}{\partial z \partial \overline{z}} + 2iky\frac{\partial}{\partial \overline{z}}.$$

For d an odd integer, define

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Definition 2.1. Let N be a positive integer, $k \in \frac{1}{2}\mathbb{Z}$, and if $k \notin \mathbb{Z}$ assume that $4 \mid N$. Then a *weight k harmonic (weak) Maass form* on $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ such that:

1) For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$ we have

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k f(z) & \text{if } k \in \mathbb{Z} \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k f(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}, \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the Jacobi symbol, and $\sqrt{z}$ is the principal branch of the square root function. Note that d is necessarily odd if k is not an integer (since N is thus even), whence the above expressions are well-defined.

2) $\Delta_k f = 0$.

3) There exists a polynomial $P_f = \sum_{n \leq 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that

$$f(z) - P_f(z) = O(e^{-\epsilon y})$$

as $y \to +\infty$ for some $\epsilon > 0$ ($q = e^{2\pi i z}$). We have analogous conditions at all other cusps.

We call P_f the *principal part* of f at ∞.

Definition 2.2. Let N be a positive integer, χ a Dirichilet character modulo N, $k \in \frac{1}{2}\mathbb{Z}$, and if $k \notin \mathbb{Z}$ assume that $4 \mid N$. Then define a *level N weight k harmonic Maas form with Nebentypus χ* to be as above, excpet with $\Gamma = \Gamma_0(N)$ and the transformation property 1) replaced by it twisted by χ: For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $z \in \mathbb{H}$ we have

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} \chi(d)(cz+d)^k f(z) & \text{if } k \in \mathbb{Z} \\ \chi(d)\left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k f(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}, \end{cases}$$

Remark 2.3. For simplicity we will drop the "weak" part and refer to forms as harmonic Maass forms.

Remark 2.4. There are other slight variants of this definition (with part 3 changed), for example $E_2^*(z) = E_2(z) - \frac{3}{\pi y}$ is not a harmonic Maass form for $\mathrm{SL}(2,\mathbb{Z})$ under our definition, but it is under a more general defintion given in [1].

Definition 2.5. The definition of a harmonic Maass form can be rephrased in terms of the slash operator. To define this, let $k \in \frac{1}{2}\mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ ($\Gamma_0(4)$ if $k \notin \mathbb{Z}$), and define

$$j(\gamma, z) = \begin{cases} \sqrt{cz+d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{cz+d} & \text{if } k \notin \mathbb{Z}, \end{cases}$$

where $\sqrt{z}$ is again taken on the principal branch. The appearance of the $\left(\frac{c}{d}\right)\epsilon_d^{-1}$ factor is just a correction term introduced to fix the problem of defining the square root function. If $f : \mathbb{H} \to \mathbb{C}$ is a function, we define the slash operator by

$$(f|_k\gamma)(z) := j(\gamma, z)^{-2k} f(\gamma z).$$

Thanks to the Cauchy Riemann equations, one sees that a weakly holomorphic modular form is also a harmonic Maass form (the transformation law for non-integral weights corresponds to the one found in [5]).

As such, let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a congruence subgroup, and we have the following list of forms ordered by inclusion:

$$\begin{aligned} S_k(\Gamma) &:= \text{ weight } k \text{ cusp forms on } \Gamma, \\ M_k(\Gamma) &:= \text{ weight } k \text{ holomorphic modular forms on } \Gamma, \\ M_k^!(\Gamma) &:= \text{ weight } k \text{ weakly holomorphic modular forms on } \Gamma, \\ H_k(\Gamma) &:= \text{ weight } k \text{ harmonic Maass forms on } \Gamma. \end{aligned}$$

If χ is a Dirichlet character modulo N, we also have:

$$\begin{aligned} S_k(N,\chi) &:= \text{ level } N \text{ weight } k \text{ cusp forms with Nebentypus } \chi, \\ M_k(N,\chi) &:= \text{ level } N \text{ weight } k \text{ holomorphic modular forms with Nebentypus } \chi, \\ M_k^!(N,\chi) &:= \text{ level } N \text{ weight } k \text{ weakly holomorphic modular forms with Nebentypus } \chi, \\ H_k(N,\chi) &:= \text{ level } N \text{ weight } k \text{ harmonic Maass forms with Nebentypus } \chi. \end{aligned}$$

3 Fourier Expansions

Definition 3.1. Let $s \in \mathbb{C}$ have $\mathrm{Re}(s) > 0$, and let $x \in \mathbb{R}$. Define the (upper) incomplete Gamma function $\Gamma(s,x)$ by

$$\Gamma(s;x) = \int_x^\infty e^{-t} t^{s-1} dt.$$

This has a meromorphic continuation in s to all of $\mathbb{C}$. Furthermore, integration by parts implies that

$$\Gamma(s,x) = x^{s-1}e^{-x} + (s-1)\Gamma(s-1,x) \tag{3.1}$$

We can now define the generalization of the $q-$expansion to harmonic Maass forms, where $q := e^{2\pi i z}$ as normal.

Proposition 3.2. *Let N be a positive integer and let $f(z) \in H_{2-k}(\Gamma_1(N))$, where $k \in \frac{1}{2}\mathbb{Z}$. Then the Fourier expansion of f takes the form*

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, -4\pi n y) q^n$$

where $z = x + iy \in \mathbb{H}$, $x, y \in \mathbb{R}$.

Definition 3.3. With reference to Proposition 3.2, define

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

to be the *holomorphic part* of $f(z)$, and define

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, -4\pi n y) q^n$$

to be the *non-holomorphic part* of $f(z)$.
We refer to $f^+(z) = \sum_{n \gg -\infty} c_f^+(n) q^n$ as a "mock modular form".

Proof of Proposition 3.2. Fix y and consider the function $f_y : \mathbb{R} \to \mathbb{C}$ given by $f_y(x) = f(x+iy)$. The transformation law gives us $f_y(x+1) = f_y(x)$, and so we take the Fourier expansion of f_y to be

$$f_y(x) = \sum_{n=-\infty}^{\infty} a_n(y) q^n = \sum_{n=-\infty}^{\infty} a_n(y) e^{-2\pi n y} e^{2\pi i n x},$$

which converges on $\mathbb{H}$ (and the $a_n(y)$ are smooth in y). We get 0 by applying Δ_{2-k} to $f(x+iy) = f_y(x)$, so using the above formula we get

$$
\begin{aligned}
0 = \Delta_{2-k}(f) = &\sum_{n=-\infty}^{\infty} -y^2\Big(-4\pi^2 n^2 a_n(y) + 4\pi^2 n^2 a_n(y) - 4\pi n \frac{da_n}{dy} + \frac{d^2 a_n}{dy^2}\Big) q^n + \\
&\sum_{n=-\infty}^{\infty} i(2-k)y\Big(2\pi i n a_n(y) - 2\pi i n a_n(y) + i\frac{da_n}{dy}\Big) q^n \\
= &\sum_{n=-\infty}^{\infty} \Big(\big(4\pi n y^2 + (k-2)y\big)\frac{da_n}{dy} - y^2 \frac{d^2 a_n}{dy^2}\Big) q^n,
\end{aligned}
$$

whence

$$\frac{d^2 a_n}{dy^2} = \left(4\pi n + \frac{k-2}{y}\right)\frac{da_n}{dy}.$$

Solving this differential equation, for $n \neq 0$ we first get

$$\frac{da_n}{dy} = C(n) y^{k-2} e^{4\pi n y}$$

for some constant $C(n)$. From this it quickly follows that we can write

$$a_n(y) = c_f^+(n) + c_f^-(n)\Gamma(k-1, -4\pi n y)$$

for some constants $c_f^+(n), c_f^-(n)$ and all $n \neq 0$. For $n = 0$ and $k \neq 1$, we get $a_0(y) = c_f^+(0) + c_f^-(0)y^{k-1}$. For $n = 0, k = 1$ we have $a_0(y) = c_f^+(0) + c_f^-(0)\log(y)$. Thus for $k \neq 1$ we get

$$f(z) = \sum_{n=-\infty}^{\infty} c_f^+(n) q^n + \sum_{n\neq 0} c_f^-(n)\Gamma(k-1, -4\pi n y) q^n + c_f^-(0) y^{k-1}.$$

Finally, we invoke condition 3) of Definition 2.1, which said that there exists a $P_f = \sum_{n\leq 0} c(n)q^n \in \mathbb{C}[q^{-1}]$ such that

$$f(z) - P_f(z) = O(e^{-\epsilon y})$$

as $y \to +\infty$ for some $\epsilon > 0$. This immediately implies that the first sum is for $n \gg -\infty$, all terms with $n > 0$ in the second sum are 0, and $c_f^-(0) = 0$ (the case $k = 1$ gives the same result), which finishes the proof. □

4 The xi Operator

The ξ operator provides a non-trivial way to relate harmonic Maass forms to modular forms. More explicitly, for $w \in \mathbb{R}$ define

$$\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}} = iy^w \overline{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)} = iy^w \left(\overline{\frac{\partial}{\partial x}} - i\overline{\frac{\partial}{\partial y}}\right).$$

Proposition 4.1. *Let N be a positive integer, let χ be a Dirichlet character modulo N, and let $f(z) \in H_{2-k}(N, \chi)$ where $k \in \frac{1}{2}\mathbb{Z}$. Then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \to S_k(N, \overline{\chi})$$

is surjective. Moreover, following the notation of Proposition 3.2, we have

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n.$$

We call the cusp form $\xi_{2-k}(f)$ the "shadow" of f.

Proof. One can check that $\xi_{2-k}(q^n) = 0$ and $\xi_{2-k}(\Gamma(k-1, -4\pi ny)) = iy^{2-k}(i(-4\pi n)^{k-1}y^{k-2}e^{4\pi ny}) = -(-4\pi n)^{k-1}e^{4\pi ny}$, so applying this to the q-expansion of f gives us

$$\begin{aligned}\xi_{2-k}(f) &= \sum_{n<0} \overline{c_f^-(n)}(-(-4\pi n)^{k-1}e^{4\pi ny})\overline{q}^n \\ &= -(4\pi)^{k-1}\sum_{n<0}\overline{c_f^-(n)}(-n)^{k-1}q^{-n} = -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_f^-(-n)}n^{k-1}q^n.\end{aligned}$$

This proves the formula given, and shows that $\xi_{2-k}(f)$ is holomorphic on $\mathbb{H}$. By direct calculation using the modular transformation property of f, one can show that $\xi_{2-k}(f)$ retains the transformation with Nebentypus $\overline{\chi}$ instead, whence it is a weakly holomorphic modular form. For example, if $k \in \mathbb{Z}$ we have

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{2-k}f(z),$$

so applying ξ_{2-k} gives

$$\frac{2i\mathrm{Im}(z)^{2-k}}{(cz+d)^2}\overline{\frac{\partial f}{\partial \overline{z}}}\left(\frac{az+b}{cz+d}\right) = \overline{\chi(d)}(cz+d)^{2-k}\xi_{2-k}f(z).$$

Using $\mathrm{Im}(\frac{az+b}{cz+d}) = \frac{\mathrm{Im}(z)}{(cz+d)^2}$, the left hand side becomes $\xi_{2-k}f(\frac{az+b}{cz+d})(cz+d)^{2-2k}$, from which the transformation follows.
Our formula shows that at the cusp ∞ it is holomorphic, with no constant term. We also have that condition 3) of Definition 2.1 holds at all cusps, and so similar formulae hold showing that $\xi_{2-k}f$ is in fact a cusp form, which shows that ξ_{2-k} does map $\xi_{2-k} : H_{2-k}(N, \chi) \to S_k(N, \overline{\chi})$.

The surjectivity is more difficult; see [1]. □

Remark 4.2. If $k \leq 0$, then the image is 0, hence harmonic Maass forms of weight $2-k$ are just weakly holomorphic modular forms of weight $2-k$. With a more general definition of harmonic Maass form, the ξ operator maps to weakly holomorphic modular forms and not just cusp forms, so this statement no longer holds.

From the above proposition we observe that up to a factor and conjugation, the coefficients of the cusp form $\xi_{2-k}(f)$ are the same as the coefficients of f^-. Is the coefficients of the holomorphic part of f, f^+, which are most mysterious.

It is important to note that we can recover the non-holomorphic part of $f \in H_{2-k}(N, \chi)$ given $\xi_{2-k}(f)$ via a "period integral". Indeed, consider

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi n\tau}}{(-i(\tau+z))^{2-k}}d\tau = \int_{2iy}^{i\infty}\frac{e^{2\pi in(\tau-z)}}{(-i\tau)^{2-k}}d\tau = i(2\pi n)^{1-k}\Gamma(k-1, 4\pi ny)q^{-n}.$$

Taking $g(z) = \sum_{n=1}^{\infty} a(n)q^n$ to be a weight k cusp form, this gives

$$Per(g(z)) := \int_{-\overline{z}}^{i\infty}\frac{g(\tau)}{(-i(\tau+z))^{2-k}}d\tau = i(2\pi)^{1-k}\sum_{n=1}^{\infty}a(n)n^{1-k}\Gamma(k-1, 4\pi ny)q^{-n}.$$

In particular, letting $g(z) = \xi_{2-k}(f)(z)$, we can use this to calculate that

$$f^-(z) = i2^{1-k}Per(\overline{g(-\overline{z})}).$$

5 Raising and Lowering Operators

Definition 5.1. Define the differential operator D as

$$D := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}.$$

Also, define the raising and lowering operators as

$$\begin{aligned} R_k =& 2i\frac{\partial}{\partial z} + \frac{k}{y} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{k}{y} \\ L_k =& -2iy^2\frac{\partial}{\partial \overline{z}} = -iy^2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$

The following properties of these operators can be checked:

$$R_k(f|_k\gamma) = (R_k f)|_{k+2}\gamma, \tag{5.1}$$

$$L_k(f|_k\gamma) = (L_k f)|_{k-2}\gamma, \tag{5.2}$$

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k. \tag{5.3}$$

Thanks to this, we are able to compare eigenfunctions of the Laplacian operator of differing weights. Indeed, if $\Delta_k f = \lambda f$, then

$$\Delta_{k+2}(R_k f) = (\lambda + k)(R_k f), \tag{5.4}$$

$$\Delta_{k-2}(L_k f) = (\lambda - k + 2)(L_k f). \tag{5.5}$$

This justifies calling them raising and lowering operators.

For $n \in \mathbb{Z}^+$, define

$$R_k^n := R_{k+2(n-1)} \circ \cdots R_{k+2} \circ R_k,$$

and define R_0^k to be the identity.

Lemma 5.2. *[Bol's Identity]*

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1}.$$

Theorem 5.3. *Let k be an integer with $k \geq 2$, $f \in H_{2-k}(N)$. Then*

$$D^{k-1}(f) \in M_k^!(N).$$

Furthermore, we have

$$D^{k-1}f = D^{k-1}f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n.$$

Proof. First, consider the transformation behaviour of $D^{k-1}f$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$f(\gamma z) = (cz+d)^{2-k} f(z),$$

so after applying D we get

$$Df(\gamma z) = \frac{c(2-k)}{2\pi i}(cz+d)^{3-k} f(z) + (cz+d)^{4-k} Df(z).$$

Note that the exponents of $cz + d$ are getting larger, and as they started off as non-positive integers (where $k \geq 2$ is an integer is crucial), it can be shown that after applying D^{k-1} they end up disappearing and we are left with $D^{k-1}f(\gamma z) = (cz + d)^k D^{k-1} f(z)$ as desired. If $k \leq 1$ or k was not an integer, by applying D repeatedly we would not be able to make these terms disappear.

To show $D^{k-1}f$ is holomorphic, by Lemma 5.2 it suffices to show that $L_k R_{2-k}^{k-1} f = 0$. Starting with $\Delta_{2-k} f = 0$, we make repeated use of equation 5.4:

$$\begin{aligned}\Delta_{4-k} R_{2-k} f &= (2-k) R_{2-k} f \\ \Delta_{6-k} R_{2-k}^2 f &= (2-k+4-k) R_{2-k}^2 f \\ \cdots &= \cdots \\ \Delta_{k-2} R_{2-k}^{k-2} f &= ((2-k)+(4-k)+\cdots+(k-4)) R_{2-k}^{k-2} f = (2-k) R_{2-k}^{k-2} f.\end{aligned}$$

Equation 5.3 thus implies that

$$L_k R_{2-k}^{k-1} f = (L_k R_{k-2}) R_{2-k}^{k-2} f = (-\Delta_{k-2} - (k-2)) R_{2-k}^{k-2} f = 0,$$

as required.

To finish the proof, it suffices to prove the series expansion for $D^{k-1}f$, as this shows that $D^{k-1}f$ is meromorphic at the cusps, whence it is a weakly holomorphic modular form (of weight k). Writing $f = f^+ + f^-$, it is clear that $D^{k-1}f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n$. For f^-, we have

$$\begin{aligned}Df^-(z) &= \sum_{n<0} D(c_f^-(n)\Gamma(k-1, -4\pi n y) q^n) \\ &= \sum_{n<0} c_f^-(n) q^n (n\Gamma(k-1, -4\pi n y) - n e^{4\pi n y}(-4\pi n y)^{k-2}) \\ &= (k-2)\sum_{n<0} n c_f^-(n) \Gamma(k-2, -4\pi n y) q^n\end{aligned}$$

where we used the functional equation satistied by Γ, equation 3.1. Repeating this argument $k-1$ times brings a factor of $(k-1)(k-2)\cdots(k-k) = 0$, whence $D^{k-1}f^-(z) = 0$, finishing the theorem.. ☐

The above theorem shows how to obtain the coefficients of a mock modular form of level N with weight a non-positive integer: divide the n^{th} coefficient of certain weakly holomorphic modular forms by n^{k-1}. Thus a natural question is to which forms can we do this for, i.e. what is the image of $D^{k-1}f$ in $M_k^!(N)$? To answer this question, we need to define a regularized inner product.

Let N be a positive integer, and $k \geq 2$ a positive integer. For $T > 0$, define

$$\mathcal{F}_T(\Gamma(1)) = \{x + iy \in \mathbb{H} : \ |x| \leq 1/2, \ |x+iy| \geq 1, \text{ and } y \leq T\}$$

to be the truncated fundamental domain for $\mathrm{SL}(2,\mathbb{Z}) = \Gamma(1)$. Extend this to $\Gamma_0(N)$ by defining

$$\mathcal{F}_T(N) \bigcup_{\gamma \in \Gamma_0(N)\backslash\Gamma(1)} \gamma\mathcal{F}_T(\Gamma(1))$$

Definition 5.4. For $g \in M_k(N)$ and $h \in M_k^!(N)$, define the regularized inner product $(g,h)^{reg}$ as the constant term in the Laurent expansion at $s=0$ of the meromorphic continuation in s of

$$\frac{1}{[\Gamma(1):\Gamma_0(N)]} \lim_{T\to\infty} \int_{\mathcal{F}_T(N)} g(z)\overline{h(z)} y^{k-s} \frac{dxdy}{y^2}.$$

It can be shown that this exists, and that if g, h are cusp forms, that $(g,h)^{reg}$ is just the Petersson inner product.

Theorem 5.5. *If $g \in M_k(N)$ and $f \in H_{2-k}(N)$, then*

$$(g, R_{2-k}^{k-1} f)^{reg} = \frac{(-1)^k}{[\Gamma(1):\Gamma_0(N)]} \sum_{\kappa \in \Gamma_0(N)\backslash \mathbb{P}_1(\mathbb{Q})} w_k c_g(0,\kappa) \overline{c_f^+(0,\kappa)},$$

where $c_g(0,\kappa)$ (respectively $c_f^+(0,\kappa)$) is the constant term of the Fourier expansion of g (respectively f^+) at the cusp κ, and w_κ is the width of the cusp κ.

Using this theorem, we can characterize the image of D^{k-1}.

Theorem 5.6. *Let $2 \le k \in \mathbb{Z}$; then the image of $D^{k-1} : H_{2-k}(N) \to M_k^!(N)$ is the set of $h \in M_k^!(N)$ which have constant term 0 at all cusps of $\Gamma_0(N)$ and are orthogonal to all cusp forms with respect to the regularized inner product.*

Proof. From $D^{k-1}f = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n$, we see that the constant term is 0 at all cusps. Theorem 5.5 implies that if g is a cusp form and $f \in H_{2-k}(N)$, then $(g, D^{k-1}f)^{reg} = 0$.

If $h \in M_k^!(N)$ has no constant term at all cusps and is orthogonal to cusp forms, by Lemma 3.11 of [1] there exists an $f \in H_{2-k}(N)$ such that the principal parts of $D^{k-1}f, h$ agree at all cusps up to constant term. However these constants are all 0, whence we see that $h - D^{k-1}f \in S_k(N)$. But $h - D^{k-1}f$ is orthogonal to cusp forms, whence it is 0. □

6 Hecke Operators

We will only briefly touch on the theory of Hecke operators here. Hecke operators acting on classical modular forms can be defined in terms of the $q-$expansion; for example, if

$$f(z) = \sum_{n \gg -\infty} c(n) q^n$$

is a weakly holomorphic modular form on $\Gamma_0(N)$ of integral weight k with Nebentypus χ, then for p prime we have

$$T_p^k(f) = \sum_{n \gg -\infty} (c(pn) + \chi(p) p^{k-1} c(n/p)) q^n.$$

The definition can be expanded to $k \in \frac{1}{2}\mathbb{Z}$, and thanks to the Fourier expansion, we can extend the definition to harmonic Maass forms as well.

Theorem 6.1. *Let $k \in \frac{1}{2}\mathbb{Z}$ satisfy $k > 1$. Suppose $f(z) \in H_{2-k}(N, \chi)$, $p \nmid N$ is a prime such that $\xi_{2-k}(f) \in S_k(N, \overline{\chi})$ is an eigenform of T_p^k with eigenvalue λ_p. Then*
1) If $k \notin \mathbb{Z}$, then

$$T_p^{2-k}(f) - p^{2-2k} \lambda_p f \in M_{2-k}^!(N, \chi).$$

2) If $k \in \mathbb{Z}$, then

$$T_p^{2-k}(f) - p^{1-k} \lambda_p f \in M_{2-k}^!(N, \chi).$$

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