Gross-Kohnen-Zagier style results after Bruinier-Ono

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Outline

- Modular curves and Heegner divisors
- Regularized theta lifts of weakly holomorphic modular forms and Borcherds's proof of the Gross-Kohnen-Zagier theorem
- Twisted Heegner divisors
- Regularized lifts of mock modular forms and Green functions for twisted Heegner divisors
- Scholl-Waldschmidt's result relating transcendence and differentials of the third kind on algebraic curves
- Outline of the proof of the main results in Bruinier-Ono's paper.
- Examples

This talk is a survey or Bruinier-Ono's paper "Heegner divisors, L-functions, and harmonic weak Maass forms" and contains no original results.

I. Heegner divisors and Borcherds lifts

Modular curves and Heegner points

- Let $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} | y > 0\}$ be the complex upper half plane. For a positive integer N, we have the congruence group $\Gamma_0(N)$ and the modular curve

$$X_0(N) = \Gamma_0(N) \backslash (\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{Q})).$$

  This is a projective non-singular curve $X_0(N)$ over $\mathbb{C}$. It admits a natural ("canonical") model over $\mathbb{Q}$: a point of $X_0(N)$ corresponds to a cyclic N-isogeny $E \to E'$ between elliptic curves E and E'.
- These curves come with a large supply of points defined over Hilbert class fields of imaginary quadratic fields; these are called Heegner points. Let's define them.
- Let $D < 0$ be a fundamental discriminant such that $D \equiv x^2$ mod $4N$. Let $r \in \mathbb{Z}/2N\mathbb{Z}$ such that $r^2 \equiv D \pmod{4N}$. Suppose that $\tau \in \mathbb{H}$ is the solution to

$$a\tau^2 + b\tau + c = 0, \quad a, b, c \in \mathbb{Z}, \quad b^2 - 4ac = D,$$
$$a > 0, \quad N | a, \quad b \equiv r \pmod{2N}.$$

  Define

$$P_{[a,b,c]} = [\tau] \in \Gamma_0(N) \backslash \mathbb{H} \subset X_0(N)(\mathbb{C}).$$

Heegner divisors. II

- Let K be the imaginary quadratic field of discriminant D. Then $P_{[a,b,c]}$ is defined over the Hilbert class field H of K. The number of points we obtain is $h(K)$ (class number of K) and they are permuted simply transitively by $\mathrm{Gal}(H/K)$. Thus their sum gives divisors

$$P_{D,r} \in \mathrm{Div}(X_0(N))$$
$$y_{D,r} = P_{D,r} - h \cdot [\infty] \in \mathrm{Div}^0(X_0(N))$$

  defined over K.
- Let $X_0^*(N)$ be the quotient of $X_0(N)$ under the Fricke involution w_N and $J_0^*(N)$ be its Jacobian and consider the image

$$y^*_{D,r} \in J_0^*(N)$$

  of $y_{D,r}$. A rough statement of the Gross-Kohnen-Zagier theorem is that (with an appropriate definition of $y^*_{-n,r}$ for more general n) the generating series

$$\sum_{n \geq 0} y^*_{-n,r} q^n \in J_0^*(N)(\mathbb{Q}) \otimes \mathbb{C}[[q]]$$

  is a modular form of weight $3/2$.

A lattice related to $\Gamma_0(N)$. I

- For our purposes it will be convenient to think of $\Gamma_0(N)$ as a subgroup of the automorphism group $\mathrm{SO}(L)$ of a certain indefinite lattice L.
- Namely, let N be a positive integer. Consider the rational quadratic space
$$V = \{X \in \mathrm{Mat}_2(\mathbb{Q}) | \mathrm{tr}(X) = 0\}$$
with quadratic form $Q(X) = -N \det(X)$. Thus $V_{\mathbb{R}}$ has signature $(2, 1)$.
- The group $\mathrm{GL}_2(\mathbb{Q})$ acts on V via
$$\gamma \cdot X = \gamma X \gamma^{-1}$$
and this induces an isomorphism $\mathrm{PGL}_2 \simeq \mathrm{SO}(V)$.
- Define an even lattice
$$L = \left\{ \begin{pmatrix} b & -a \\ c/N & -b \end{pmatrix} | a, b, c \in \mathbb{Z} \right\} \subset V.$$
Its dual lattice is
$$L^\vee = \left\{ \begin{pmatrix} b/2N & -a \\ c/N & -b/2N \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

A lattice related to $\Gamma_0(N)$. II

- The map

$$L^\vee \to \mathbb{Z}/2N\mathbb{Z}$$
$$\begin{pmatrix} b/2N & -a \\ c/N & -b/2N \end{pmatrix} \mapsto b \mod 2N$$

  identifies the discriminant group $L^\vee/L$ with $\mathbb{Z}/2N\mathbb{Z}$ and the quadratic form on $L^\vee/L$ with $x \mapsto x^2$ on $\mathbb{Z}/2N\mathbb{Z}$.
- The group $\Gamma_0(N)$ preserves the lattice L and the induced action on $L^\vee/L$ is trivial.
- Let $D \in \mathbb{Z}$ and let

$$L_D = \{X \in L^\vee | Q(X) = D/4N\}.$$

  Note that L_D is empty unless D is a square modulo $4N$. Thus given $r \in L^\vee/L \simeq \mathbb{Z}/2N\mathbb{Z}$ with $r^2 \equiv D \mod 4N$ we define

$$L_{D,r} = \{X \in L^\vee | Q(X) = D/4N, \quad X \equiv r \mod L\}.$$

  Note that each $L_{D,r}$ is preserved by $\Gamma_0(N)$ by the above remark. The set of orbits is finite.

A lattice related to $\Gamma_0(N)$. III

- Let
$$\mathrm{Gr}^+(2, V_{\mathbb{R}}) = \{z \subset V_{\mathbb{R}} | \dim z = 2, \quad Q|_z > 0\}$$
be the Grassmannian of negative definite planes in $V_{\mathbb{R}}$. Each vector $\lambda \in L^\vee$ with negative norm defines a point
$$z(\lambda) := \lambda^\perp \in \mathrm{Gr}^+(2, V_{\mathbb{R}}).$$

- We can identify $\mathrm{Gr}^+(2, V_{\mathbb{R}})$ with $\mathbb{H}$ by sending $\tau \in \mathbb{H}$ to the plane spanned by the real and imaginary parts of the norm 0 vector
$$\begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix}.$$

- Under the above bijection $\mathrm{Gr}^-(2, V_{\mathbb{R}}) \simeq \mathbb{H}$, the point $z(\lambda)$ for a matrix $\lambda = \begin{pmatrix} b/2N & -a \\ c/N & -b/2N \end{pmatrix}$ corresponds to the point $\tau \in \mathbb{H}$ satisfying
$$Na\tau^2 + b\tau + c = 0.$$
Hence we can write
$$P^*_{D,r} = \sum_{\lambda \in L_{D,r}/\Gamma_0(N)} z(\lambda) \in \mathrm{Div}(X_0(N))$$

Weakly holomorphic modular forms. I

- Before discussing the results in the Bruinier-Ono paper, we will outline the proof of the GKZ theorem by Borcherds. This proof relates Heegner divisors with regularized theta lifts of weakly holomorphic modular forms.
- We begin by recalling the definition of modular forms of half-integral weight.
- Define a factor of automorphy $j(\gamma, \tau)$ of $\Gamma_0(4)$ by

  $$j(\gamma, \tau) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (c\tau + d)^{1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \quad \tau \in \mathbb{H},$$

  where $\epsilon_d = 1$ if $d \equiv 1 \mod 4$ and $\epsilon_d = i$ if $d \equiv 3 \mod 4$.
- **Definition.** Let N be a positive integer, k an odd integer and χ be a Dirichlet character modulo $4N$ such that $\chi(-1) = 1$. A weakly holomorphic modular form of weight $k/2$ on $\Gamma_0(4N)$ with character χ is a holomorphic function f on $\mathbb{H}$ that is meromorphic at the cusps and satisfies

  $$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d) j(\gamma, \tau)^k f(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N).$$

  Denote the space of all such f by $M_k^!(4N)$.

Weakly holomorphic modular forms. II

- At the cusp $i\infty$, we have the Laurent expansion
$$f(\tau) = \sum_{n \gg -\infty} a(n) q^n, \quad q = e^{2\pi i \tau}.$$
We denote by
$$P_{f,i\infty}(\tau) = \sum_{n \leq 0} a(n) q^n \in \mathbb{C}[q^{-1}]$$
its principal part.
- If the principal parts $P_{f,c}$ for all the cusps c vanish, we say that f is a modular form. If moreover the constant terms are all zero, we say that f is a cusp form. Sometimes one denotes weakly holomorphic modular forms by $f^!$ to emphasize the possibility of poles at the cusps.
- Note that cusps forms are exponentially decreasing on any fundamental domain of $\Gamma_0(4N)$, but weakly holomorphic forms with non-zero principal parts are exponentially *increasing.*
- This means that to define the theta lift of a weakly holomorphic modular form $f^!$ one needs to use some regularization since the integral will usually diverge.

Theta series I. Positive definite lattices

- Let L^+ be a positive definite lattice of rank $n \geq 1$. Thus $L \simeq \mathbb{Z}^n$ and there is a bilinear form
$$(\cdot, \cdot) : L \times L \to \mathbb{Z}$$
such that $(x, x) > 0$ for every non-zero $x \in L$.
- For $\tau \in \mathbb{H}$, define
$$\vartheta_{L^+}(\tau) = \sum_{\lambda \in L^+} e^{\pi i \tau (\lambda, \lambda)}.$$
Note that the sum converges (very rapidly!) since $e^{\pi i \tau (\lambda, \lambda)}$ is exponentially decreasing.
- One can show that ϑ_{L^+} is a modular form of weight $n/2$.
- Example: Let $L^+ = \mathbb{Z}$ with $(x, y) = 2xy$. Then ϑ_{L^+} is the classical Jacobi theta function
$$\vartheta_{L^+}(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} \in M_{1/2}(\Gamma_0(4)).$$
- Example: assume that L is even (that is, $(x, x) \in 2\mathbb{Z}$ for every $x \in L$) and unimodular ($L^\vee = L$). Then Poisson summation shows that $\vartheta_{L^+}(-1/\tau) = \tau^{n/4} \vartheta_{L^+}(\tau)$ and hence $\vartheta_{L^+}(\tau) \in M_{n/4}(\mathrm{SL}_2(\mathbb{Z}))$.

Theta series II. Indefinite lattices

- Now consider the lattice L defined above. It has signature $(2,1)$ and so is indefinite.
- Problem: ϑ_L diverges if L is not positive-definite.
- Solution: change the negative signs! Namely:
  1. Pick orthogonal decomposition $V_{\mathbb{R}} = z^{\perp} \oplus z$, with $(\cdot,\cdot)$ pos. def. on z and neg. def. on $z^{\perp}$.
  2. For $\lambda \in V$, write $\lambda = \lambda_z + \lambda_{z^{\perp}}$ accordingly. Define a new, *positive-definite* bilinear form

$$(\lambda,\lambda)_z := (\lambda_z,\lambda_z) - (\lambda_{z^{\perp}},\lambda_{z^{\perp}}).$$

  3. Replace $\vartheta_L(\tau)$ by the *Siegel theta function*

$$\vartheta_L(\tau,z) = y^{1/2} \sum_{\lambda\in L} e^{\pi i x(\lambda,\lambda)} e^{-\pi y\cdot(\lambda,\lambda)_z}, \quad \tau = x+iy.$$

- Note that we have introduced a new variable $z \in \mathrm{Gr}^{+}(2,V)$. Since L is invariant under $\Gamma_0(N)$, for fixed $\tau \in \mathbb{H}$ we have

$$\vartheta_L(\tau) \in \mathcal{C}^{\infty}(\Gamma_0(N)\backslash \mathrm{Gr}^{+}(2,V)) = \mathcal{C}^{\infty}(X_0(N)).$$

Theta series III. Siegel theta series and theta lifts

- The series $\vartheta_L(\tau, z)$ behaves like a modular form in the τ variable. To see this, we will introduce a slightly more general ("shifted") version of ϑ_L.
- **Definition.** Let $\alpha, \beta \in L^\vee/L \simeq \mathbb{Z}/2N\mathbb{Z}$. Define

$$\vartheta_L(\tau, z; \alpha, \beta) = \sum_{\lambda \in \beta + L} e^{\pi i \tau(\lambda_z, \lambda_z) + \pi i \overline{\tau}(\lambda_{z^\perp}, \lambda_{z^\perp})} e^{-2\pi i(\lambda - r/2, \alpha)}.$$

- Then Poisson summation shows that

$$\vartheta_L(-1/\tau, z; -\beta, \alpha) = (\tau/i)^{1/2} |L^\vee/L|^{-1/2} \vartheta_L(\tau, z; \alpha, \beta).$$

- This implies that each $\vartheta_L(\tau, z; \alpha, \beta)$ is a modular form of weight $1/2$.
- It is easy to see that ϑ_L is of moderate growth in τ. Let f be a cusp form of weight $1/2$. For a small enough congruence subgroup $\Gamma' \subset \Gamma_0(4)$ define the theta lift

$$(f, \vartheta_L(z; \alpha, \beta)) = \int_{\Gamma' \backslash \mathbb{H}} \overline{f(\tau)} \vartheta_L(\tau, z; \alpha, \beta) y^{1/2} \frac{dxdy}{y^2} \in \mathcal{C}^\infty(X_0(N)).$$

  This function is called a theta lift of f. Note that it is well defined since the integrand is rapidly decreasing and invariant under Γ'.

Regularized theta lifts I

- Now suppose that $f^!$ is weakly holomorphic modular form of weight $1/2$. One cannot use the integral above to define a theta lift since now the integrand is rapidly increasing.
- Borcherds (and, previously, Harvey-Moore) understood that one can define a theta lift

$$(f^!, \vartheta_L(z;\alpha,\beta)) = \int_{\Gamma'\backslash\mathbb{H}}^{\mathrm{reg}} \overline{f^!(\tau)}\vartheta_L(\tau,z;\alpha,\beta)y^{1/2}\frac{dxdy}{y^2}$$

  where $\int^{\mathrm{reg}}$ stands for a regularized integral.
- The regularization is not difficult: it suffices to assume that $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$. Let

$$\mathcal{F} = \{\tau \in \mathbb{H} | -1/2 \leq \mathrm{Re}(\tau) \leq 1/2,\ |\tau| \geq 1\}$$

  be the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. For $T \geq 1$, define its truncation $\mathcal{F}_T = \{\tau \in \mathcal{F} | \mathrm{Im}(\tau) \leq T\}$. Thus $\mathcal{F}_T$ is compact and the integral

$$\int_{\mathcal{F}_T} F(\tau)\frac{dxdy}{y^2}$$

  converges for any smooth function F on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

Regularized theta lifts. II

- Borcherds shows that
$$I(s) := \lim_{T\to\infty} \int_{\mathcal{F}_T} \overline{f(\tau)}\vartheta_L(\tau, z; \alpha, \beta) y^{1/2} y^{-s} \frac{dxdy}{y^2}$$
converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$. He defines
$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \overline{f^!(\tau)}\vartheta_L(\tau, z; \alpha, \beta) y^{1/2} \frac{dxdy}{y^2} := \mathrm{CT}_{s=0} I(s)$$
to be the constant term of the Laurent expansion of $I(s)$ at $s = 0$.
- The resulting function of z is no longer smooth, but has logarithmic singularities along certain Heegner divisors. To describe this precisely, it is useful to switch to the language of modular forms valued in the Weil representation.
- Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$. This can be realized as the set of pairs $(g, \phi_g(\tau))$ where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi_g(\tau) : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that $\phi_g(\tau)^2 = c\tau + d$. The multiplication is then given by
$$(g, \phi_g(\tau) \cdot (h, \phi_h(\tau)) = (gh, \phi_g(h\tau)\phi_h(\tau)).$$

Regularized theta lifts. IV

- We denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image in $\mathrm{Mp}_2(\mathbb{R})$ of the full level congruence group $\mathrm{SL}_2(\mathbb{Z})$. This group is generated by the elements
$$T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \; S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right), \; Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right).$$

- Let $\mathbb{C}[L^\vee/L]$ be the group algebra of $L^\vee/L$ and denote by $[\mu]$ the standard basis element corresponding to $\mu \in L^\vee/L$. The Weil representation ρ_L associated with L is the representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L^\vee/L]$ determined by
$$\rho_L(T)([\mu]) = e^{\pi i(\mu,\mu)}[\mu]$$
$$\rho_L(S)([\mu]) = \frac{e^{-\pi i/4}}{|L^\vee/L|^{1/2}} \sum_{\mu' \in L^\vee/L} e^{-2\pi i(\mu,\mu')}[\mu']$$
$$\rho_L(Z)([\mu]) = e^{-2\pi i/4}[-\mu].$$

- **Definition.** Let $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$ and write $f = \sum_\mu f_\mu[\mu]$. We say that f is a weakly modular form of weight k valued in the Weil representation ρ_L if each f_μ is meromorphic at the cusps and
$$f((g,\phi_g)\tau) = \phi_g(\tau)^{2k}\rho_L(g,\phi_g)f(\tau), \quad (g,\phi_g) \in \mathrm{Mp}_2(\mathbb{Z}).$$

Regularized theta lifts V

- Let $f^!$ be a weakly holomorphic modular form of weight $1/2$ valued in ρ_L. Define

$$(f^!(\tau), \vartheta_L(\tau, z)) = \sum_{\mu \in L^\vee/L} \overline{f^!_\mu(\tau)} \vartheta_L(\tau, z; 0, \mu)$$

and a regularized theta lift

$$\vartheta(f^!)(z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} (f^!(\tau), \vartheta_L(\tau, z)) y^{1/2} \frac{dxdy}{y^2}.$$

- We can now describe the singularities of the regularized lift $\vartheta(f^!)$ precisely. Namely, write

$$f^!(\tau) = \sum_{m \gg -\infty, \mu} a(m, \mu) q^m [\mu]$$

for the Laurent expansion of $f^!$ along the cusp $i\infty$ and define

$$Z(f^!) = \sum_{m<0, \mu} a(m, \mu) y^*_{m,\mu} \in \mathrm{Div}^0(X_0(N)).$$

Borchersds's proof of GKZ

- **Theorem (Borcherds).** Let $f \in M^!_{1/2,\rho_L}$. Assume that $a(0,0) = 0$ and the coefficients $a(m,\mu)$ are integers when $m < 0$. Then some non-zero multiple $nZ(f^!)$ is principal, and we have

$$n\vartheta(f^!)(z) = -4 \log |\Psi_{f^!}(z)|$$

  for some meromorphic function $\Psi_{f^!}(z)$ on $X_0(N)$ whose divisor is $nZ(f^!)$ plus some divisor supported at the cusps.
- It follows that the divisor $Z(f^!)$ vanishes in $J(X_0^*(N)) \otimes \mathbb{Q}$. The Gross-Kohnen-Zagier theorem is now an easy consequence using Serre duality.
- **Theorem. (GKZ)** For any fixed r, the generating series

$$\sum_{n \geq 0, r} y^*_{-n,r} q^n [r] \in J(X_0^*(N)) \otimes \mathbb{C}[[q]]$$

  is a modular form of weight $3/2$ valued in $J(X_0^*(N))$.

II. Twisted Heegner divisors

$\Gamma_0(N)$-classification of binary quadratic forms. I

- The result of Bruinier-Ono is related to twisted Heegner divisors. To define these we need to study the orbits of the action of $\Gamma_0(N)$ on binary quadratic forms a bit closer.
- Given $a, b, c \in \mathbb{Z}$, we write $[a, b, c]$ for the quadratic form $[a, b, c](x, y) = ax^2 + bxy + cy^2$. We say that $[a, b, c]$ is primitive if $(a, b, c) = 1$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on such forms by

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

- Given a discriminant Δ (i.e. an integer $\equiv 0, 1 \bmod 4$), define

$$\mathcal{Q}_\Delta = \{[a, b, c] | b^2 - 4ac = \Delta\}.$$

- Let r be an integer mod $2N$ and Δ be a discriminant with $\Delta \equiv r^2 (\mathrm{mod}\ 4N)$. Define

$$\mathcal{Q}_{N,\Delta,r} = \{[Na, b, c] \in \mathcal{Q}_\Delta | a \in \mathbb{Z},\ \ b \equiv r\ (mod\ 2N)\}.$$

  and let $\mathcal{Q}^0_{N,\Delta,r} = \{[Na, b, c] \in \mathcal{Q}_{N,\Delta,r} | (a, b, c) = 1\}$.

$\Gamma_0(N)$-classification of binary quadratic forms. II

- We are interested in describing the $\Gamma_0(N)$-orbits on $\mathcal{Q}_{N,\Delta,r}$ as well as certain $\Gamma_0(N)$-invariant functions arising from genus characters.
- We have a $\Gamma_0(N)$-invariant bijection

$$\mathcal{Q}_{N,\Delta,r} = \bigcup_{l^2|\Delta} \bigcup_{\substack{\rho \ (2N) \\ \rho^2 \equiv \Delta/l^2 \ (4N) \\ l\rho \equiv r \ (2N)}} l \cdot \mathcal{Q}^0_{N,\Delta/l^2,\rho}$$

  and so it suffices to understand the $\Gamma_0(N)$-orbits on $\mathcal{Q}^0_{N,\Delta,r}$.
- We have already observed that the discriminant Δ of $[Na, b, c]$ and the greatest common divisor (a, b, c) are $\Gamma_0(N)$-invariants. There is an additional invariant

$$m = (N, r, \tfrac{r^2-\Delta}{4N}) \in \mathbb{N}.$$

- Fix a form $Q = [a, b, c] \in \mathcal{Q}^0_{N,\Delta,r}$ and define $m_1 = (N, b, a)$ and $m_2 = (N, b, c)$. Then $(m_1, m_2) = 1$ and $m_1 m_2 = m$. Choose a decomposition $N = N_1 N_2$ into coprime factors with $(N_1, m_2) = (N_2, m_1) = 1$. Then the form $\tilde{Q} = [aN_1, b, cN_2]$ belongs to $\mathcal{Q}^0_\Delta$, and the orbit $\mathrm{SL}_2(\mathbb{Z})\tilde{Q}$ depends only on $\Gamma_0(N)Q$.

$\Gamma_0(N)$-classification of binary quadratic forms. III

- The map $Q \mapsto \tilde{Q}$ shows that

$$|\Gamma_0(N)\backslash\mathcal{Q}^0_{N,\Delta,r}| = 2^{d(m)} \cdot |\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}^0_\Delta|,$$

  where $d(m)$ is the number of divisors of m.
- We can use this bijection to define a $\Gamma_0(N)$ invariant function on $\mathcal{Q}^0_{N,\Delta,r}$ corresponding to a *genus character*.
- Genus theory: let Δ be a discriminant and $D_0|\Delta$ be a *discriminant divisor*. That is, D_0 is a fundamental discriminant and $\Delta/D_0 \equiv 0, 1 \pmod 4$. Define a function

$$\chi_{D_0} : \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}^0_\Delta \to \{\pm 1\}$$

  by setting

$$\chi_{D_0}(Q) = \left(\tfrac{D_0}{n}\right)$$

  where n is any integer prime to D_0 represented by Q.
- Recall that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}^0_\Delta$ has a natural group structure (class group); χ_{D_0} is a homomorphism, and every homomorphism to $\{\pm 1\}$ is of the form χ_{D_0} for some D_0.

Genus characters and functions on $\mathcal{Q}^0_{N,\Delta,r}$

- Using the bijection $Q \mapsto \tilde{Q}$, we can attach a $\Gamma_0(N)$-invariant function on $\mathcal{Q}_{N,\Delta,r}$ to a discriminant divisor D_0 of Δ, under the additional assumption that

$$D_0 \text{ and } \Delta/D_0 \text{ are both squares mod } 4N.$$

- Namely, for $[Na, b, c] \in \mathcal{Q}_{N,\Delta,r}$, we set

$$\chi_{D_0}(Q) = \begin{cases} \left(\frac{D_0}{n}\right), & \text{if } (a, b, c, D_0) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where n is any integer coprime to D_0 represented by $\tilde{Q}$.
- More explicitly, to compute $\chi_{D_0}([Na, b, c])$, find a factorization $D_0 = D_1 D_2$ into discriminants and $N = N_1 N_2$ (with $N_i > 0$) such that $(D_1, N_1 a) = (D_2, N_2 c) = 1$. Then

$$\chi_{D_0}([Na, b, c]) = \left(\frac{D_1}{N_1 a}\right)\left(\frac{D_2}{N_2 c}\right)$$

and $\chi_{D_0}([N, a, b, c]) = 0$ if no such splitting exists.
- The function $\chi_{D_0} : \mathcal{Q}^0_{N,\Delta,r} \to \{\pm 1\}$ is invariant under $\Gamma_0(N)$ and also under the Fricke involution w_N.

Twisted Heegner divisors I

- We come back to our lattice L.
- Let Δ be a fundamental discriminant and r an integer mod $2N$ such that
$$\Delta \equiv r^2 \pmod{4N}.$$
Let m be a negative rational number with
$$m \in \mathbb{Z} + \operatorname{sgn}(\Delta) Q(r/2N)$$
and let
$$d := 4Nm \cdot \operatorname{sgn}(\Delta) \in \mathbb{Z}.$$
Note that d is a discriminant that is a square mod $4N$ and has sign opposite to that of Δ, so we have a $\Gamma_0(N)$-invariant function $\chi_\Delta : \mathcal{Q}_{N,d\Delta,r} \to \{\pm 1\}$.
- **Definition.** Let $h \in L^\vee/L$. The twisted Heegner divisor is
$$Z_{\Delta,r}(m,h) = \sum_{\lambda \in \mathcal{Q}_{N,d\Delta,rh}/\Gamma_0(N)} \chi_\Delta(\lambda) \frac{1}{w(\lambda)} P_\lambda \in \operatorname{Div}(X_0(N))_{\mathbb{Q}}.$$
Here $w(\lambda)$ is the order of the stabilizer of λ in $\Gamma_0(N)$. Define also
$$y_{\Delta,r}(m,h) = Z_\Delta(m,r) - \deg(Z_{\Delta,r}(m,h)) \cdot [i\infty] \in \operatorname{Div}^0(X_0(N))_{\mathbb{Q}}.$$

Twisted Heegner divisors II. Properties

- If $\Delta = 1$ we recover the usual ("untwisted") Heegner divisors.
- By the theory of complex multiplication, $Z_{\Delta,r}(m,h)$ is defined over $\mathbb{Q}(\sqrt{d},\sqrt{\Delta})$.
- Let σ be the non-trivial automorphism of $\mathbb{Q}(\sqrt{d},\sqrt{\Delta})/\mathbb{Q}(\sqrt{d})$. Then

$$\begin{aligned} w_N(Z_{\Delta,r}(m,h)) &= Z_{\Delta,r}(m,-h) \\ Z_{\Delta,r}(m,h)^c &= Z_{\Delta,r}(m,-h) \\ \sigma(Z_{\Delta,r}(m,h)) &= -Z_{\Delta,r}(m,h) \\ Z_{\Delta,r}(m,-h) &= \operatorname{sgn}(\Delta) Z_{\Delta,r}(m,h) \\ Z_{\Delta,r}(m,h) &\text{ is defined over } \mathbb{Q}(\sqrt{\Delta}). \end{aligned}$$

- If $\Delta \neq 1$, we have $Z_{\Delta,r}(m,h) = y_{\Delta,r}(m,h)$ and hence we can think of

$$Z_{\Delta,r}(m,h) \in J(N)(\mathbb{Q}(\sqrt{\Delta})).$$

Green functions for twisted Heegner divisors

- We will now explain how Bruinier-Ono generalize Borcherds's ideas to construct Green functions for twisted Heegner divisors using a regularized theta lift.
- This will involve:
  1. Replacing the Siegel theta function in Borcherds's regularized theta lift by a version that is "twisted" by a genus character χ_Δ.
  2. Using weak Maass forms, rather than weakly holomorphic modular forms, as an "input" for the theta lift.
- Once these objects have been introduced, it is not difficult to define the regularized theta lift. The technical details concerning the regularization are completely analogous to Borcherds's case.

Harmonic weak Maass forms. I

- Let N be a positive integer. A harmonic weak Mass form of half-integral weight k on $\Gamma_0(4N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying
  1. $f|_k\gamma = f, \quad \gamma \in \Gamma_0(4N)$,
  2. $\Delta_k f = 0$, where Δ_k is the weight k Laplacian defined by
  $$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$
  3. There is a polynomial $P_f = \sum_{n\le 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that
  $$f(\tau) - P_f(\tau) = O(e^{-\epsilon y}) \text{ as } y \to \infty.$$

  The polynomial P_f is unique. It is called the principal part of f at $i\infty$.
- We write $\mathcal{H}_k(4N)$ for the space of harmonic weak Maass forms of weight k on $\Gamma_0(4N)$. Any $f \in \mathcal{H}_k(4N)$ has a Fourier expansion
  $$f(\tau) = \sum_{n\gg -\infty} c^+(n)q^n + \sum_{n<0} c^-(n)W_k(2\pi ny)q^n,$$
  where $W_k(x) = \Gamma(1-k, 2|x|)$.
- Note that $M_k^!(4N)$ is a subspace of $\mathcal{H}_k(4N)$.

Harmonic weak Maass forms. II

- We will also need to consider harmonic weak Maass forms valued in ρ_L, namely functions $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$ satisfying $\Delta_k f = 0$,

$$f(g\tau) = \phi_g(\tau)^{2k} \rho_L(g, \phi_g) f(\tau), \quad \text{for } (g, \phi_g) \in \mathrm{Mp}_2(\mathbb{Z}),$$

  and such that there is a polynomial

$$P_f(\tau) = \sum_{n \leq 0, \mu} c^+(n, \mu) q^n [\mu] \in \mathbb{C}[q^{-1}] \otimes \mathbb{C}[L^\vee/L]$$

  with $f(\tau) - P_f(\tau) = O(e^{-\epsilon y})$ as $y \to \infty$.
- Denote the space of such f by $\mathcal{H}_{k,\rho_L}$. We have a decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{n \gg -\infty, \mu} c^+(n, \mu) q^n [\mu],$$

$$f^-(\tau) = \sum_{n \leq 0, \mu} c^-(n, \mu) W(2\pi n y) q^n [\mu].$$

Twisted Siegel theta series. I

- We now define the twisted version of the Siegel theta series that we will use as the kernel for the theta lift.
- **Definition.** Let Δ be a fundamental discriminant and r an integer such that $\Delta \equiv r^2 \mod 4N$. For $h \in L^\vee/L$, define

$$\vartheta_{\Delta,r,h}(\tau,z) = y^{1/2} \sum_{\substack{\lambda \in rh+L \\ Q(\lambda) \equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\lambda) e\left(\frac{1}{|\Delta|} Q(\lambda_z)\tau + \frac{1}{|\Delta|} Q(\lambda_{z^\perp})\overline{\tau}\right).$$

and

$$\vartheta_{\Delta,r}(\tau,z) = \sum_{h \in L'/L} \vartheta_{\Delta,r,h}(\tau,z)[\mu].$$

- Main point: for $m \in \mathbb{Z} + \operatorname{sgn}(\Delta)Q(h)$, let $d = 4Nm\operatorname{sgn}(\Delta)$. Then the m-th Fourier coefficient of $\vartheta_{\Delta,r,h}$ is given by the sum

$$y^{1/2} \sum_{\substack{\lambda \in rh+L \\ Q(\lambda) = m|\Delta|}} \chi_\Delta(\lambda) e^{-\pi \frac{y}{|\Delta|}(\lambda,\lambda)_z}$$

$$= y^{1/2} \sum_{\lambda \in L_{d\Delta,hr}} \chi_\Delta(\lambda) e^{-\pi \frac{y}{|\Delta|}(\lambda,\lambda)_z},$$

and $Z_{\Delta,r}(m,h)$ is also defined as a sum over $L_{d\Delta,hr}$.

Twisted Siegel theta series. II

- Note that in the variable z, the function $\Theta_{\Delta,r}(\tau,z)$ is $\Gamma_0(N)$-invariant.
- In the variable τ it has modular behaviour. To see this, note that we can write

$$\vartheta_{\Delta,r,h}(\tau,z) = \sum_{\substack{\alpha\in L^\vee/\Delta L \\ \alpha\equiv rh\ (L) \\ Q(\alpha)\equiv \Delta Q(h)\ (\Delta)}} \chi_\Delta(\alpha)|\Delta|^{-1/2}\vartheta_L(|\Delta|\tau, z; 0, \alpha/|\Delta|)$$

  as a linear combination of usual Siegel theta series, using that $\chi_\Delta(\lambda)$ only depends on $\lambda \in L^\vee$ modulo ΔL. With a bit more work, one shows the following.
- **Theorem. (B-O)** In the variable τ, the theta function $\Theta_{\Delta,r}(\tau,z)$ is a (non-holomorphic) modular form of weight $1/2$ valued in $\mathbb{C}[L^\vee/L]$. It transforms with

$$\tilde{\rho}_L = \begin{cases} \rho_L, & \text{if } \Delta > 0, \\ \overline{\rho_L}, & \text{if } \Delta < 0. \end{cases}$$

Twisted Heegner divisor attached to a weak Maass form

- Let $f \in \mathcal{H}_{1/2,\tilde{\rho}_L}$. Recall that we have $f = f^+ + f^-$; we denote the corresponding coefficients by $c^\pm(m,h)$, $h \in L^\vee/L$. Note that

$$c^\pm(m,h) = 0 \text{ unless } m \in \mathbb{Z} + \mathrm{sgn}(\Delta)Q(h).$$

- Using the Fourier coefficients of the principal part P_f, define twisted Heegner divisors

$$Z_{\Delta,r}(f) = \sum_{m<0,h} c^+(m,h) Z_{\Delta,r}(m,h) \in \mathrm{Div}(X_0(N))_{\mathbb{R}},$$

$$y_{\Delta,r}(f) = \sum_{m<0,h} c^+(m,h) y_{\Delta,r}(m,h) \in \mathrm{Div}^0(X_0(N))_{\mathbb{R}}.$$

- We will only consider f such that the coefficients $c^+(m,h)$ for negative m are rational; in this case $Z_{\Delta,r}(f) \in \mathrm{Div}(X_0(N))_{\mathbb{Q}}$ and $y_{\Delta,r}(f) \in \mathrm{Div}^0(X_0(N))_{\mathbb{Q}}$.

Regularized theta lifts of weak Maass forms

- For $f \in \mathcal{H}_{1/2,\tilde{\rho}_L}$, define

$$(f(\tau), \Theta_{\Delta,r}(\tau,z)) = \sum_h \overline{f_h(\tau)}\vartheta_{\Delta,r,h}(\tau,z).$$

- Consider the regularized theta lift

$$\Phi_{\Delta,r}(z,f) = \int_{\mathrm{SL}_2(\mathbb{Z}\backslash\mathbb{H}}^{\mathrm{reg}} (f(\tau), \Theta_{\Delta,r}(\tau,z)) y^{1/2} \frac{dxdy}{y^2}.$$

- **Theorem.** The integral $\Phi_{\Delta,r}(z,f)$ defines a $\Gamma_0(N)$-invariant function on $\mathbb{H}$ that induces a function on $Y_0(N) - Z_{\Delta,r}(f)$ with a logarithmic singularity along $-4Z_{\Delta,r}(f)$. We have

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_{\Delta,r}(z,f) = \left(\frac{\Delta}{0}\right) c^+(0,0).$$

  Here $\left(\frac{\Delta}{0}\right) = 1$ if $\Delta = 1$ and $\left(\frac{\Delta}{0}\right) = 0$ otherwise.
- In particular, if $\Delta \neq 1$ or $\Delta = 1$ and $c^+(0,0) = 0$, the function $\Phi_{\Delta,r}(z,f)$ is harmonic.

Differentials on non-singular projective curves I. Definitions

Let X be a non-singular projective curve over $\mathbb{C}$ and ω be a meromorphic differential on X, that is, a meromorphic section of Ω^1_X.
Let $P \in X$ and z be a local coordinate around P. Then we can write

$$\omega = \sum_{n \gg -\infty} a_n z^n dz.$$

and we define the residue of ω at P to be

$$\mathrm{Res}_P\, \omega := a_{-1} \in \mathbb{C}.$$

- ω is *of the first kind* if it is holomorphic everywhere, i.e. $\omega \in \mathrm{H}^0(X, \Omega^1_X)$.
- ω is *of the second kind* if it is meromorphic and $\mathrm{Res}_P\, \omega = 0$ for every $P \in X$.
- ω is *of the third kind* if all its poles are simple and the residues are all integers. For such a differential, define

  $$\mathrm{Res}\, \omega = \sum_P \mathrm{Res}_P(\omega) \cdot [P] \in \mathrm{Div}^0(X).$$

  Note that $\mathrm{Res}(\omega)$ has degree zero by the residue theorem.

Differentials on compact Riemann surfaces II. Examples

- Let $\tau \in \mathbb{H}$ and consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. The holomorphic differential dz on $\mathbb{C}$ descends to a differential of the first kind
$$dz \in \mathrm{H}^0(E_\tau, \Omega^1).$$
- Let
$$X = \mathbb{P}^1 = \mathbb{C}^2 - \{(0,0)\}/\mathbb{C}^\times.$$
We can write $X = \mathbb{C} \cup \{\infty\}$, where the map $\mathbb{C} \to X$ sends $z \mapsto [z : 1]$ and we write $\infty = [1 : 0]$. Writing z for the standard coordinate on $\mathbb{C}$, we obtain a meromorphic differential $z^{-2}dz$ of the second kind on X.
- Let X be arbitrary and f be a meromorphic function on X. Then
$$\psi := \frac{df}{f}$$
is a differential of the third kind on X, with divisor
$$\mathrm{Res}(\psi) = \mathrm{div}(f).$$

Differentials III. Canonical differentials of the third kind

Let $W(X)$ be the additive group of differentials of the third kind on X. The map $\omega \mapsto \mathrm{Res}\,\omega$ induces a short exact sequence

$$0 \to \mathrm{H}^0(X, \Omega^1) \to W(X) \to \mathrm{Div}^0(X) \to 0.$$

(Surjectivity follows from Riemann-Roch). Thus given a divisor D of degree zero, there are several differentials $\omega \in W(X)$ with $\mathrm{Res}\,\omega = D$. We will now show that there is a canonical choice. For this, let $\gamma_1, \ldots, \gamma_{2g} \in \mathrm{H}_1(X - D, \mathbb{Z})$ whose images span $\mathrm{H}_1(X, \mathbb{Z})$. By Riemann's relations, the map

$$\mathrm{H}^0(X, \Omega^1) \to \mathbb{R}^{2g}$$
$$\omega \mapsto \left(\mathrm{Re} \int_{\gamma_j} \omega\right)_{1 \le j \le 2g}$$

is an isomorphism. So given a differential of the third kind ψ, there is a unique $\omega \in \mathrm{H}^0(X, \Omega^1)$ such that

$$\mathrm{Re} \int_{\gamma_j} (\psi - \omega) = 0, \quad 1 \le j \le 2g.$$

Differentials IV. Canonical differentials of the third kind

Definition. Let $D \in \mathrm{Div}^0(X)$. We denote by ψ_D the unique differential of the third kind such that $\mathrm{Res}\ \psi_D = D$ and

$$\mathrm{Re} \int_{\gamma_j} \psi_D = 0, \quad 1 \leq j \leq 2g.$$

Note that $\mathrm{H}_1(X - D, \mathbb{Z})$ is generated by the cycles $\gamma_1, \ldots, \gamma_{2g}$ and one small loop c_i for each $P_i \in D$. We have

$$\int_{c_i} \psi_D = 2\pi \mathrm{i}\ \mathrm{Res}_{P_i}\ \psi_D \in 2\pi i\mathbb{Z}.$$

Hence the period $\int_\gamma \psi_D$ is purely imaginary for each $\gamma \in \mathrm{H}_1(X - D, \mathbb{Z})$. Here is another characterization of ψ_D: write

$$df = \psi_D$$

where f is multivalued. Since ψ_D has imaginary periods, the function

$$g_D = \mathrm{Re}(f)$$

is harmonic and single-valued on X. This gives an alternative characterization of ψ_D.

Differentials V. A transcendence result

- **Proposition.** Let $D \in \mathrm{Div}^0(X)$. The canonical differential ψ_D is the unique differential of the third kind with $\mathrm{Res}\ \psi_D = D$ of the form $\partial g_D = \psi_D$ with g_D harmonic.
- **Example.** If $D = \mathrm{div}(f)$ is principal, then we have $\psi_D = f^{-1} df$ and
$$g_D = \log |f|^2.$$

- From now on, assume that the curve is defined over a number field $F \subset \mathbb{C}$.
- **Theorem. (Scholl, Waldschmidt)** If some non-zero multiple of D is principal, then ψ_D is defined over F. Otherwise, ψ_D is not defined over $\overline{\mathbb{Q}}$.
- We will apply this theorem when X is the modular curve $X_0(N)$, which is defined over $\mathbb{Q}$. If ψ is a differential of the third kind on $X_0(N)$, then we have a q-expansion at the cusp $i\infty$:
$$\psi = \sum_{n \geq 0} a(n) q^n \frac{dq}{q}.$$
Given a number field F, the q-expansion principle says that
$$\psi \text{ is defined over } F \Leftrightarrow a(n) \in F, \quad n \geq 0.$$

Differentials VI. A transcendence result

Combining all the above, we obtain the following.

Theorem. Let $D \in \mathrm{Div}^0(X_0(N))$. Assume that D is defined over F. Let

$$\psi_D = \sum_{n \geq 0} a(n) q^n \frac{dq}{q}$$

be the corresponding canonical differential. Then:

- If some non-zero multiple of D is principal, then $a(n) \in F$ for all $n \geq 0$.
- Otherwise, some $a(n)$ is transcendental.

Canonical differentials for twisted Heegner divisors

- From now on, assume that $\Delta \neq 1$ (for $\Delta = 1$ most of the results were already proved by Borcherds).
- Recall that we have defined:
  1. twisted Heegner divisors $Z_{\Delta,r}(m,h)$,
  2. Given a weak Maass form $f \in \mathcal{H}_{1/2,\tilde{\rho}_L}$ whose coefficients $c^+(m,\mu)$ for negative m are rational, a divisor
  $$Z_{\Delta,r}(f) = y_{\Delta,r}(f) = \sum_{m<0,h} c^+(m,h) y_{\Delta,r}(m,h) \in \mathrm{Div}^0(X_0(N))_{\mathbb{Q}}$$
  that is defined over $\mathbb{Q}(\sqrt{\Delta})$.
  3. Given f as above, a harmonic function $\Phi_{\Delta,r}(f)$ that has a logarithmic singularity along $-4Z_{\Delta,r}(f)$.
- Hence the differential
  $$\eta_{\Delta,r}(z,f) = -\frac{1}{2}\partial\Phi_{\Delta,r}(z,f)$$
  has residue divisor $Z_{\Delta,r}(f)$. By our characterization of canonical differentials, $\eta_{\Delta,r}(z,f)$ is the canonical differential for $Z_{\Delta,r}(f)$.
- Upshot: we can combine the Scholl-Waldschmidt theorem and the q-expansion principle to relate the non-vanishing of $Z_{\Delta,r}(f)$ in the Jacobian to transcendence properties of the coefficients $c^+(m,\mu)$.

The Fourier expansion of $\Phi_{\Delta,r}(z,f)$ at the cusp $i\infty$

- The following result gives a formula for $\Phi_{\Delta,r}$ near the cusp ∞ of $X_0(N)$.
- **Theorem.** For $z \in \mathbb{H}$ with $\mathrm{Im}(z) \gg 0$, we have

$$\Phi_{\Delta,r}(z,f) = 2\sqrt{\Delta}c^+(0,0)L(1,\chi_\Delta) - 4\sum_{n\geq 1}\sum_{b\ (\Delta)} \left(\frac{\Delta}{b}\right) c^+\left(\tfrac{|\Delta|n^2}{4N}, \tfrac{rn}{2N}\right) \log|1-e(nz+b/\Delta)|.$$

- To prove this, one gives an explicit expression for the twisted Siegel theta series $\Theta_{\Delta,r}$ as a sum over $\mathrm{Mp}_2(\mathbb{Z})_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})$. The regularized theta lift can be unfolded as in the Rankin-Selberg method, and the integral can be explicitly computed, giving the above formula.
- Applying ∂ term by term we obtain the q-expansion of the canonical differential $\eta_{D,r}(z,f)$.
- **Theorem.** The canonical differential of the third kind $\eta_{D,r}(z,f)$ corresponding to the divisor $Z_{\Delta,r}(f)$ has q-expansion

$$\eta_{\Delta,r}(z,f) = -\mathrm{sgn}(\Delta)\sqrt{\Delta}\sum_{\substack{n\geq 1\\ d|n}} \frac{n}{d}\left(\frac{\Delta}{d}\right) c^+\left(\tfrac{|\Delta|n^2}{4N}, \tfrac{rn}{2N}\right) ne(nz)\cdot 2\pi i dz.$$

When is $y_{\Delta,r}(f)$ torsion in $J(X_0(N))(\mathbb{Q}(\sqrt{\Delta}))$?

- Combining all the above, we obtain the first main result in Bruinier-Ono's paper.
- **Theorem.** The following are equivalent:
  1. A non-zero multiple of $y_{\Delta,r}(f)$ is a principal divisor in $X_0(N)$.
  2. The coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ of f are algebraic for all $n \in \mathbb{N}$.
  3. The coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ of f are rational for all $n \in \mathbb{N}$.
- **Proof:** By Scholl-Waldschmidt, $y_{\Delta,r}(f)$ is principal if and only if $\eta_{D,r}(f)$ is defined over $\mathbb{Q}(\sqrt{\Delta})$. By the q-expansion principle and the above formula, this is equivalent to

  $$c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N}) \in \mathbb{Q}(\sqrt{\Delta}) \text{ for all } n \in \mathbb{N}.$$

  Let σ be the non-trivial automorphism of $\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}$. We have seen that $\sigma(y_{\Delta,r}(f)) = -y_{\Delta,r}(f)$. Hence $\sigma(\eta_{\Delta,r}(f)) = -\eta_{\Delta,r}(f)$ and this shows that σ fixes the coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$.

Generalized Borcherds products. I

- Let $f \in \mathcal{H}_{1/2,\tilde{\rho}_L}$ with real coefficients. Assume that $c^+(n,h) \in \mathbb{Z}$ for every negative n. Recall the expression

$$\Phi_{\Delta,r}(z,f) = 2\sqrt{\Delta}c^+(0,0)L(1,\chi_\Delta) - 4\sum_{n\geq 1}\sum_{b\ (\Delta)} \left(\frac{\Delta}{b}\right) c^+(\tfrac{|\Delta|n^2}{4N}, \tfrac{rn}{2N}) \log|1 - e(nz + b/\Delta)|.$$

- It is tempting to exponentiate this and consider the infinite product

$$\Psi_{\Delta,r}(z,f) := \prod_{n\geq 1}\prod_{b\ (\Delta)} (1 - e((\lambda,z) + b/\Delta))^{\left(\frac{\Delta}{b}\right)c^+(\frac{|\Delta|n^2}{4N},\frac{rn}{2N})}.$$

  For $y \gg 0$ this converges and has meromorphic continuation to $z \in \mathbb{H}$ with divisor $Z_{\Delta,r}(f)$. It satisfies

$$\Phi_{\Delta,r}(z,f) = -4\log|\Psi_{\Delta,r}(z,f)|.$$

- In Borcherds's case ($\Delta = 1$) such a product defined a meromorphic modular form of weight $c^+(0,0)$ on $X_0(N)$ transforming with a unitary character of finite order. In our case ($\Delta \neq 1$) the weight is zero but this character sometimes has infinite order.

Generalized Borcherds products. II

- More precisely, one can show that there exists a unitary character $\sigma : \Gamma_0(N) \to \mathbb{C}^\times$ such that

$$\Psi_{\Delta,r}(\gamma z, f) = \sigma(\gamma)\Psi_{\Delta,r}(z, f), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

  It is interesting to determine when σ is finite.
- **Theorem.** Let $f \in \mathcal{H}_{1/2,\tilde{\rho}_L}$ with real coefficients. Assume that $c^+(n, h) \in \mathbb{Z}$ for every negative n. TFAE:
  1. The character σ has finite order.
  2. The coefficients $c^+(\frac{|\Delta|n^2}{4N}, \frac{rn}{2N})$ of f are rational for all $n \in \mathbb{N}$.
- An interesting special case happens when $f \in M^!_{1/2,\tilde{\rho}_L}$. One can show that for weakly holomorphic f, we have

$$c^+(m, h) \in \mathbb{Q}, \text{ for all } m < 1 \Rightarrow c^+(m, h) \text{ for all } m.$$

  For such an f, the theorem shows that $y_{\Delta,r}(f)$ vanishes in $J(N)(\mathbb{Q}(\sqrt{\Delta}) \otimes \mathbb{Q}$. Using Serre duality gives the following generalization of the GKZ theorem.
- **Theorem.** The generating series $\sum_{n\geq 0,\mu} y_{\Delta,r}(-n, \mu)q^n[\mu]$ is a cusp form of weight $3/2$ valued in $\tilde{\rho}_L$, with values in $J(N)(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{Q}$.

Hecke eigenforms. I

- We will now consider the action of the Hecke algebra of $\Gamma_0(N)$.
- Consider the space of newforms $S_2^{\mathrm{new}}(N)$. It admits a decomposition
$$S_2^{\mathrm{new}}(N) = S_2^{\mathrm{new},+}(N) \oplus S_2^{\mathrm{new},-}(N)$$
according to the sign of the Fricke involution: $S_2^{\mathrm{new},+}(N)$ is the subspace where this sign is -1.
- The spaces $S_{3/2,\rho_L}$ and $S^{\mathrm{new}}_{3/2,\overline{\rho_L}}$ carry a Hecke action. Explicitly, if $g(\tau) = \sum_{n,\mu} a(n,\mu) q^n[\mu]$ and p is a prime coprime to N, then writing
$$g|_{3/2} T(p) = \sum_{n,\mu} b(n,\mu) q^n[\mu],$$
we have
$$b(n,\mu) = a(p^2 n, p\mu) + \left(\frac{4N\sigma n}{p}\right) a(n,\mu) + pa(n/p^2, \mu/p)$$
where $\sigma = 1$ if $\rho = \rho_L$ and $\sigma = -1$ if $\rho = \overline{\rho_L}$.
- The Shimura correspondence gives isomorphisms of vector spaces
$$S^{\mathrm{new}}_{3/2,\rho_L} \simeq S_2^{\mathrm{new},+}(N), \quad S^{\mathrm{new}}_{3/2,\overline{\rho_L}} \simeq S_2^{\mathrm{new},-}(N)$$
together with an explicit correspondence of Hecke eigenvalues.

The ξ operator

- Recall the operator $\xi_{1/2}$ defined by

$$\xi_{1/2}(f) = y^{-3/2}\overline{L_{1/2}f},$$

where $L_{1/2} = -2iy^2\frac{d}{d\overline{\tau}}$ is the Maass lowering operator. It induces a short exact sequence

$$0 \to M^!_{1/2,\rho_L} \to \mathcal{H}_{1/2,\rho_L} \xrightarrow{\xi_{1/2}} S_{3/2,\overline{\rho_L}} \to 0$$

and a pairing $\{\cdot,\cdot\} : S_{3/2,\overline{\rho_L}} \times \mathcal{H}_{1/2,\rho_L} \to \mathbb{C}$ given by

$$\{g, f\} = (g, \xi_{1/2}f)_{\mathrm{Pet}} = \sum_{n\leq 0,h} c_f^+(n,h)c_g(-n,h).$$

This pairing identifies $\mathcal{H}_{1/2,\rho_L}(F)/M^!_{1/2,\rho_L}(F)$ with the dual of $S_{3/2,\overline{\rho_L}}(F)$ for any number field $F \subset \mathbb{C}$.

- Here $S_{3/2,\rho_L}(F)$ is the space of cusp forms with coefficients in F, and $\mathcal{H}_{1/2,\rho_L}(F)$ the space of weak Maass forms whose principal part has coefficients in F.

Hecke eigenforms. II

- From now on, fix a fundamental discriminant Δ and a normalized Hecke eigenform

$$G = \sum_{n \geq 1} a_G(n) q^n \in S_2^{\mathrm{new}, -\mathrm{sgn}(\Delta)}(N).$$

- Let $\rho = \rho_L$ is Δ is positive and $\rho = \overline{\rho_L}$ if Δ is negative.
- Let $g \in S^{\mathrm{new}}_{3/2, \overline{\rho}}$ correspond to G. One can choose g so that all its coefficients are contained in

$$F := F_G = \mathbb{Q}[\{a_G(n) |\ n \geq 1\}].$$

- Let $f \in \mathcal{H}_{1/2,\rho}(F)$ such that

$$\xi_{1/2}(f) = ||g||^{-2} g.$$

Then f is determined up to addition of a weakly holomorphic form $f' \in M^!_{1/2,\rho}(F)$.

Isotypic components of $J(N)(\mathbb{Q}(\sqrt{\Delta}))$. I

- **Theorem.** The divisor $y_{\Delta,r}(f) \in \mathrm{Div}(X_0(N)) \otimes F$ defines a point in the G-isotypic component of $J(N)(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$.
- **Proof sketch:**
  1. Let p be a prime with $(p, N) = 1$. It suffices to show that
  $$T(p)y_{\Delta,r}(f) = \lambda_p(G)y_{\Delta,r}(f).$$
  2. One can show directly that
  $$T(p)y_{\Delta,r}(m,\mu) = y_{\Delta,r}(p^2 m, p\mu) + \left(\frac{4N\sigma m}{p}\right) y_{\Delta,r}(m,\mu) + p y_{\Delta,r}(m/p^2, \mu/p)$$
  and hence that
  $$T(p)y_{\Delta,r}(f) = p y_{\Delta,r}(f|_{1/2}T(p)).$$
  3. We have $\xi_{1/2}(f|_{1/2}T(p) - p^{-1}\lambda_p f) = 0$ and hence
  $$T(p)y_{\Delta,r}(f) = \lambda_p y_{\Delta,r}(f) + y_{\Delta,r}(f')$$
  for some $f' \in M^!_{1/2,\rho}(F)$. The result follows since $y_{\Delta,r}(f')$ is torsion in $J(N)(\mathbb{Q}(\sqrt{\Delta}))$.

Isotypic components of $J(N)(\mathbb{Q}(\sqrt{\Delta}))$. II

- Recall that we have shown that the generating series

$$A_{\Delta,r}(\tau) = \sum_{n>0,\mu} y_{\Delta,r}(-n,\mu) q^n [\mu]$$

  belongs to $S_{3/2,\overline{\rho}} \otimes J(X_0(N))(\mathbb{Q}(\sqrt{\Delta}))$.
- Denote by $y^G_{\Delta,r}(m,h)$ the projection of $y_{\Delta,r}(m,h)$ onto the G-isotypic component of $J(N)$ and set

$$A^G_{\Delta,r}(\tau) = \sum_{n>0,\mu} y^G_{\Delta,r}(-n,\mu) q^n [\mu] \in S_{3/2,\overline{\rho}} \otimes J(X_0(N))(\mathbb{Q}(\sqrt{\Delta})).$$

- **Theorem.** We have $A^G_{\Delta,r}(\tau) = g(\tau) \otimes y_{\Delta,r}(f)$. In particular, the space spanned by the $y^G_{\Delta,r}(-n,\mu)$ is at most one-dimensional and is spanned by $y_{\Delta,r}(f)$.
- **Proof sketch.** As in the previous proof one can show that

$$A_{\Delta,r}|_{3/2} T(p) = T(p) A_{\Delta,r}.$$

  and hence $A^G_{\Delta,r}|_{3/2} T(p) = \lambda_p(G) A^G_{\Delta,r}$. This shows that $A^G_{\Delta,r} = Cg$ for some constant C that one determines by computing $\{A^G_{\Delta,r}, f\}$.

Isotypic components of $J(N)(\mathbb{Q}(\sqrt{\Delta}))$. III

- Combining these results with the Gross-Zagier formula, one obtains the following.
- **Theorem.** Let $f \in \mathcal{H}_{1/2,\rho}(F)$ be a Maass form such that $\xi_{1/2}(f)$ is a newform which maps to $G \in S_2^{\mathrm{new}}(N)$ under the Shimura correspondence. TFAE:
  1. The Heegner divisor $y_{\Delta,r}(f)$ vanishes in $J(X_0(N))(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$.
  2. The coefficient $c^+(\frac{|\Delta|}{4N}, \frac{r}{2N})$ is algebraic.
  3. The coefficient $c^+(\frac{|\Delta|}{4N}, \frac{r}{2N}) \in F$.
  4. $L'(G, \chi_\Delta, 1) = 0$.