2016 Montreal-Toronto workshop in number theory:
Mock modular forms

Toward a p-adic theory of mock modular forms?

old ideas

Henri Darmon

Montreal, December 9, 2016


LEONARD COHEN
OLD IDEAS
1. GOING HOME
2. AMEN
3. SHOW ME THE PLACE
4. DARKNESS
5. ANYHOW
6. CRAZY TO LOVE YOU
7. COME HEALING
8. BANJO
9. LULLABY
10. DIFFERENT SIDES
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8 8697-98671-2 3

The theme of this lecture

Fourier coefficients of mock modular forms encode interesting arithmetic quantities:

- Logarithms of algebraic numbers (Kudla-Rapoport-Yang, Duke-Li, Ehlen, Viazovska, as described in Siddarth's lecture);

- Heegner points on a *varying collection* of quadratic twists, (the Gross-Kohnen-Zagier style results of Bruinier-Ono, as described in Luis's lecture);

- Arithmetic intersections of naturally occuring collections of cycles on Shimura varieties (e.g., of orthogonal or unitary type, in the spirit of the Kudla program);

Thesis: these results should admit p-adic counterparts.

Goal: to describe two results supporting this thesis.

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Weak harmonic Maass forms

Definition

A *weak harmonic Maass form* of weight k, level N, and character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a real analytic function

$$F : \mathcal{H} \longrightarrow \mathbb{C}$$

satisfying

1. $F|_k\gamma = \chi(\gamma)F \quad$ for all $\gamma \in \Gamma_0(N)$;
2. F has at most linear exponential growth at all the cusps;
3. $\Delta_k(F) = 0$,

where

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is the weight k hyperbolic Laplacian.

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Differential operators

The hyperbolic Laplacian Δ_k factors as

$$\Delta_k = \xi_{2-k} \circ \xi_k,$$

where

$$\xi_k F(z) = 2iy^k \overline{\partial_{\bar{z}} F(z)}.$$

This operator sends weak Harmonic Maass forms of weight k to holomorphic cusp forms of weight $2-k$: i.e., $f := \xi_k F$ is holomorphic on $\mathcal{H}$, and vanishing at the cusps.

$$0 \longrightarrow M_k^{\dagger} \longrightarrow H_k \xrightarrow{\xi_k} S_{2-k} \longrightarrow 0.$$

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Fourier expansions

A weak harmonic Maass form F has a fourier expansion of the form

$$F(z) = \left(\sum_{n \geq n_0} c^+(n) q^n \right) - \left(\sum_{n>0} c^-(n) \beta_k(n,y) q^{-n} \right),$$

where

$$\beta_k(0,y) = \frac{y^{1-k}}{k-1}, \qquad \beta_k(n,y) = \int_y^\infty e^{-4\pi n t} t^{-k} dt.$$

The coefficients $c(n)$ of the non-holomorphic part of $F(z)$ are just the fourier coefficients of

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Mock modular forms

Definition

A *mock modular form* is the holomorphic part of a weak harmonic Maass form.

Definition

If $\tilde{f} := \sum_{n \geq n_0} c^+(n) q^n$ is the holomorphic part of a WHMF F, then the cusp form $f := \xi_k F$ is called the *shadow* of $\tilde{f}$.

- Any two $\tilde{f}$ with the same shadow differ by a classical weakly holomorphic modular form.

Expectation: Fourier expansions of mock modular forms give rise to generating series for interesting arithmetic functions.

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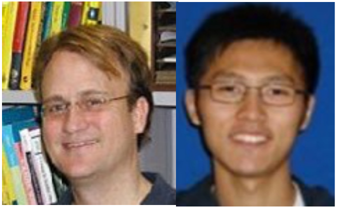
Part I: A KRY/Duke-Li/Ehlen/Viazovska style theorem



Mock modular forms of weight one

As we saw in Siddarth's lecture, Kudla-Rapoport-Yang, Duke-Li, Ehlen, Viazovska, suggest

Principle. The fourier coefficients of mock modular forms of weight one encode the logarithms of interesting algebraic numbers.

Main goal of this first part:

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The result of Kudla-Rapoport-Yang

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \pm 1$ be an odd Dirichlet character of prime conductor N, let $E_1(1,\chi)$ be the associated weight one Eisenstein series, and let $\tilde{E}_1(1,\chi)$ be the associated mock modular form.

Theorem (Kudla-Rapoport-Yang)

For all $n \geq 2$ with $\gcd(n, N) = 1$,

$$a_n(\tilde{E}_1(1,\chi)) \sim_{\mathbb{Q}^\times} \frac{1}{2} \sum_{\ell | n} \log(\ell) \cdot (\mathrm{ord}_\ell(n) + 1) \cdot a_{n/\ell}(E_1(1,\chi)).$$

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Theta series of imaginary quadratic fields

Let H be the Hilbert class field of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-p})$ with $p \equiv 3 \pmod 4$, let

$$\psi : \mathrm{Gal}(H/K) \longrightarrow \mathbb{C}^\times$$

be a class group character, and let

$$\theta_\psi := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \psi(\mathfrak{a}) q^{\mathfrak{a}\bar{\mathfrak{a}}}$$

be the associated theta series of weight one, level p and character χ_K. We are interested in a mock modular form

$$\tilde{\theta}_\psi = \sum_{n \geq n_0} c_\psi^+(n) q^n$$

having θ_ψ as its shadow.

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The fourier coefficients of $\tilde{\theta}_\psi$

The following was obtained independently by Duke-Li, Ehlen, and Viazovska.

Theorem

There exists a mock modular form $\tilde{\theta}_\psi$ having θ_ψ as shadow, and for which

1. *If $\chi_K(n) = 1$ or $n < -\frac{p+1}{4}$, then $c_\psi^+(n) = 0$;*
2. *The coefficients $c_\psi^+(n)$ are of the form*

$$c_\psi^+(n) \sim_{\mathbb{Q}^\times} \sum_{\mathfrak{a} \in Cl(K)} \psi^2(\mathfrak{a}^{-1}) \log |u(n)^{\sigma_\mathfrak{a}}|,$$

where the $u(n)$ are algebraic numbers in H (which are units when $n < 0$), and $\sigma_\mathfrak{a}$ is the image of $\mathfrak{a}$ under the Artin map.

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- The proofs rest on the fact that the coefficients $c_\psi^+(n)$ are related to (twisted) traces of singular moduli, hence the theory of complex multiplication plays an essential role in all the proofs.
- But the theory of complex multiplication is not essential for the statements, which generalise in a natural way.

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General weight one forms

Let f be *any* classical newform of weight one, associated to an odd, irreducible, two-dimensional Artin representation

$$\rho_f : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Conjecture (Bill Duke–Yingkun Li)

The fourier coefficients of the mock modular form $\tilde{f}$ are simple linear combinations with algebraic coefficients of logarithms of algebraic numbers in H—more precisely, in the field which is cut out by $\mathrm{Ad}(\varrho_f)$.

Duke and Li give some experimental evidence for this statement, for an octahedral newform f of level 283.

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The Duke-Li conjecture for real dihedral newforms

Let K be a real quadratic field, let

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be any character of K of mixed signature, and let $\theta_\psi \in S_1(N, \chi)$ be Hecke's theta series of weight one attached to ψ, and let $c_\psi^+(n)$ be the nth fourier coefficient of $\tilde{\theta}_\psi$.

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For all rational primes ℓ, the real and imaginary parts of $c_\psi^+(\ell)$ are logarithms of elements of $\mathcal{O}_K[1/\ell]_1^\times$.

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Explicit Class Field Theory

The Duke-Li conjecture suggests a potential appraoch towards explicit class field theory.

Kronecker-Weber: All abelian extensions can be generated by roots of unity: values of the function $e^{2\pi i z}$ at rational arguments.

Complex multiplication: If K is a quadratic imaginary field, all its abelian extensions can be generated (essentially) by roots of unity and values of the modular function j at arguments in K.

Question: It it possible to generate class fields of other number fields from values of concrete transcendental functions at explicit arguments?

E.g.: K is a real quadratic field.

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Kronecker-Weber: All abelian extensions can be generated by roots of unity: values of the function $e^{2\pi iz}$ at rational arguments.

Complex multiplication: If K is a quadratic imaginary field, all its abelian extensions can be generated (essentially) by roots of unity and values of the modular function j at arguments in K.

Question: It it possible to generate class fields of other number fields from values of concrete transcendental functions at explicit arguments?

E.g.: K is a real quadratic field.

A more traditional approach: Stark's conjecture

Let K be a real quadratic field,

$$\psi : \mathrm{Gal}(H/K) \longrightarrow L^\times \subset \mathbb{C}^\times$$

a finite order character of *mixed signature*.

Conjecture (Stark)

The L-function $L(K, \psi, s)$ has a simple zero at $s = 0$ and

$$L'(K, \psi, 0) = \log |u(\psi)|,$$

where $u(\psi) \in (\mathcal{O}_H^\times \otimes L)^\psi$.

One can construct explicit units in H by exponentiating the values of $L'(K, \psi, 0)$.

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The p-adic Gross-Stark conjecture

Conjecture (Gross)

Let $\psi : \mathrm{Gal}(H/F) \longrightarrow L^\times$ be a totally odd character of a totally real field F, and suppose that $\psi(\mathfrak{p}) = 1$ for some prime $\mathfrak{p}$ of F above p. Then there exists $u_p(\psi) \in (\mathcal{O}_H[1/p])^\times \otimes L)^\psi$ satisfying

$$L_p'(F, \psi, 0) \sim \log_p \mathrm{Norm}_{F_\mathfrak{p}/\mathbb{Q}_p}(u_p(\psi)).$$

Dasgupta, Pollack, D + Kevin Ventullo (2011): This conjecture is true.

The proof uses p-adic deformations and congruences with families of Eisenstein series, following the pioneering approach of Ribet, Mazur-Wiles.

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A p-adic Duke-Li theorem

Let ψ be a totally odd ring class character of K.

Lemma (Tate): There exists a ray class character ψ_0 of K of mixed signature, satisfying $\psi_0/\psi_0' = \psi$, and hence

$$\mathrm{Ad}(\mathrm{Ind}_K^{\mathbb{Q}} \psi_0) = \mathrm{Ind}_K^{\mathbb{Q}} \psi \oplus 1 \oplus \chi_K.$$

Because ψ_0 has mixed signature, θ_{ψ_0} is a classical, holomorphic modular form of weight one.

What is the p-adic analogue of $\tilde{\theta}_{\psi_0}$?

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Overconvergent generalised eigenforms

The p-adic counterpart of the mock modular form $\tilde{\theta}_{\psi_0}$ whose shadow is θ_{ψ_0} is an *overconvergent generalised eigenform* attached to θ_{ψ_0}.

"overconvergent" means the usual thing (cf. Francesc's lecture): a special kind of well-behaved p-adic modular form.

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The Coleman-Mazur eigencurve at θ_{ψ_0}

Assume that θ_{ψ_0} is *regular* at p, i.e., its two (ordinary) p-stabilisations are distinct, and replace θ_{ψ_0} by one such of these:

$$U_p\theta_{\psi_0} = \alpha\theta_{\psi_0}.$$

Theorem (Cho-Vatsal, Bellaiche-Dimitrov, Adel Betina)

The Coleman-Mazur eigencurve is smooth at the the classical weight one point x_{ψ_0} attached to θ_{ψ_0}, but it is not étale *above weight space at this point.*

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Proof: *Both the tangent space and the relative tangent space of the fiber above weight 1 at x_{ψ_0} are one-dimensional.* The proof uses the fact that the three irreducible constituents of

$$\mathrm{Ad}(\mathrm{Ind}_K^{\mathbb{Q}}\psi_0) = 1 \oplus \mathrm{Ad}^0(\mathrm{Ind}_K^{\mathbb{Q}}\psi_0) = 1 \oplus \chi_K \oplus \mathrm{Ind}_K^{\mathbb{Q}}\psi$$

occur with multiplicities $(0,1,0)$ in $\mathcal{O}_H^\times \otimes \mathbb{C}$.

This is the same reason why the Stark conjecture for $L(\mathrm{Ad}(\mathrm{Ind}_K^{\mathbb{Q}}\psi_0), s)$—and the Duke-Li conjecture for θ_{ψ_0}—fail to produce units in the ring class field of K cut out by ψ!

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Corollary

The natural inclusion

$$M_1^{p,oc}(N,\chi)[\theta_{\psi_0}] \hookrightarrow M_1^{p,oc}(N,\chi)[[\theta_{\psi_0}]]$$

is not surjective.

Definition

A modular form θ'_{ψ_0} in $M_1^{p,oc}(N,\chi)[[\theta_{\psi_0}]]$ which is not classical (i.e., not an eigenvector) is called an *overconvergent generalised eigenform* attached to θ_{ψ_0}. This generalised eigenform is said to be *normalised* if $a_1(\theta'_{\psi_0}) = 0$.

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The main theorem

Theorem (Alan Lauder, Victor Rotger, D)

The normalised generalised eigenform θ'_{ψ_0} attached to θ_{ψ_0} can be scaled in such a way that, for all primes $\ell \nmid N$ with $\chi_K(\ell) = -1$,

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where

$$u_\ell(\psi) = \sum_{\sigma \in \mathrm{Gal}(H/K)} \psi^{-1}(\sigma) u_\ell^\sigma, \quad u_\ell \in \mathcal{O}_H[1/\ell]^\times.$$

More generally, for all $n \geq 2$ with $\gcd(n, N) = 1$,

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Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \pm 1$ *be an odd Dirichlet character of prime conductor* N*, and let* $E_1(1,\chi)$ *be the associated weight one Eisenstein series. For all* $n \geq 2$ *with* $\gcd(n,N) = 1$,

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Remarks on [DLR] vs KRY/Duke-Li/Ehlen/Viazovska.

• The techniques in [DLR] are fundamentally p-adic in nature, relying only on p-adic deformations, and some simple class field theory for H. They are less deep than KRY, Duke-Li, Ehlen, Viazovska.

• The theory of complex multiplication or singular moduli plays no role in [DLR], while it is crucial in the archimedean setting. It would be interesting to give an independent, more analytic construction of θ'_{ψ_0}, closer in spirit to what is done in the archimedean setting.

• Can one interpret the fourier coefficients of θ'_{ψ_0} as arithmetic intersection numbers on a Shimura variety? **Problem**: $spec(\mathcal{O}_H)$ is not an integral Shimura variety, when H is a ring class field of a real quadratic field!

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Part II: A Bruinier-Ono style result

Half-integral weight forms

We place ourselves in the general framework of Bruinier-Ono, as described in Luis's lecture.

$$f = \sum a_n q^n \in S_{2k}(\Gamma_0(N)) \qquad (N \text{ an odd prime}).$$

$S^+_{k+1/2}(4N)$:= Kohnen's plus-space.

- Forms in $S^+_{k+1/2}(4N)$ transform like $\theta(\tau)^{2k+1}$ under $\Gamma_0(4N)$.
- If $g = \sum_D c(D) q^D \in S_{k+1/2}(4N)$, then $c(D) = 0$ unless

$$D^* := (-1)^k D \equiv 0, 1 \pmod 4.$$

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The Shimura-Kohnen correspondence

If $f := \sum a_n q^n \in S_{2k}^{new}(N)$, there is a

$$g = \sum c(D) q^D \in S^+_{k+1/2}(4N),$$

unique up to scaling, for which

$$T_{\ell^2} g = a_\ell \cdot g, \text{ for all } \ell \nmid 2N.$$

Waldspurger, Kohnen: $|c(D)|^2 \sim D^{k-1/2} L(f, \chi_D, k)$ for all fundamental D satisfying

$$\left(\frac{D^*}{N}\right) = w_N \qquad (\text{where } W_N f = w_N f.)$$

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The theorem of Bruinier-Ono

Take $k = 1$, and assume that $f \in S_2^{new}(N)$ is associated to an elliptic curve over conductor N.

Let $g \in S^+_{3/2}(4N)$ be the newform whose Shimura lift is f.

Let $\tilde{g}$ be the mock modular form of weight $1/2$ associated to g.

Theorem (Bruinier, Ono)

Let D be a positive *fundamental discriminant for which $\left(\frac{D}{N}\right) = w_N$. The coefficient $c^+(\tilde{g}, D)$ is transcendental if and only if the following equivalent conditions are satisfied:*

1. $L'(f, \chi_D, 1) \neq 0$;
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2. *The Heegner point $P_D \in E^{(D)}(\mathbb{Q})$ is of infinite order.*

The theorem of Bruinier-Ono

Take $k = 1$, and assume that $f \in S_2^{new}(N)$ is associated to an elliptic curve over conductor N.

Let $g \in S_{3/2}^{+}(4N)$ be the newform whose Shimura lift is f.

Let $\tilde{g}$ be the mock modular form of weight $1/2$ associated to g.

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A p-adic analogue: the set-up

Set $p = N$. Let $f_{2k} \in S_{2k}(\mathbf{SL}_2(\mathbb{Z}))$, $k > 1$ be the weight $2k$ specialisation of the p-adic family specialising to f in weight 2.

Let $g_k \in S^+_{k+1/2}(4)$ be the Shimura-Kohnen correspondent of f_{2k}.

Choose a Δ_0 for which $c(g, \Delta_0) \neq 0$.

Theorem (Hida, Stevens)

There is a p-adic neighbourhood U of $k = 1$ in weight space for which the coefficient

$$c^\flat(D,k) := \frac{\left(1 - \left(\frac{-D}{p}\right) a_p(k)^{-1} p^{k-1}\right) c(D,k)}{\left(1 - \left(\frac{-\Delta_0}{p}\right) a_p(k)^{-1} p^{k-1}\right) c(\Delta_0,k)} = \frac{c(p^2 D, k)}{c(p^2 \Delta_0, k)}$$

extends to a p-adic analytic function of $k \in U$.

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Behaviour of the coefficients $c^\flat(D,1)$ in weight two

For all $D > 0$ with $-D \equiv 0, 1 \pmod 4$,

$$c(D,1) \sim \begin{cases} D^{1/4} L(f, \chi_{-D}, 1)^{1/2} & \text{if } \left(\frac{-D}{N}\right) = w_N; \\ 0 & \text{if } \left(\frac{-D}{N}\right) = -w_N. \end{cases}$$

Theorem (D, Tornaria, 2008)

Suppose that $\left(\frac{-D}{N}\right) = -w_N$. Then there exists a global point $P_D \in E(\mathbb{Q}(\sqrt{-D}))^- \otimes \mathbb{Q}$ satisfying:

1. $\frac{d}{dk} c^\flat(D,k)_{k=1} = \log_{E,p}(P_D)$,
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Bruinier-Ono style generating series

Let ε be an infinitesimal, $\varepsilon^2 = 0$. The modular form $g_{1+\varepsilon}$ of "weight $3/2 + \varepsilon$" has q expansion of the form

$$g_{1+\varepsilon} = \left(\sum_{(-D/N)=w_N} c^\flat(D) q^D \right) + \varepsilon \left(\sum_{(-D/N)=-w_N} \log_{E,p}(P_D) q^D \right).$$

The first order deformation of g encodes the p-adic logarithms of Heegner points defined over a varying collection of imaginary quadratic twists of E, as in the result of Bruinier-Ono.

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Some questions

• Can the p-adic modular generating series of [DLR] and [DT] be interpreted in terms of the p-adic theory of mock modular forms being developed by Candelori-Castella?

• what new insights into explicit class field theory for real quadratic fields, the behaviours of $L'(E,\chi,1)$ as χ varies over a collection of quadratic characters, etc.; can be obtained from [DLR] and [DT]?

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Thank you for your attention!!