Uniqueness of Sasaki-extremal metrics

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Introduction

We will discuss a proof of uniqueness of Sasaki-extremal(S-ext) metrics.
Most of the results are discussed in the article arXiv:1511.09167.

- K. Cho, A. Futaki, H Ono 2007 Proved uniqueness of toric cscS structures. The geodesic equation in the space of metrics is just $\ddot{G} = 0$, in terms of symplectic potential G.
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Definition 1.1

A Riemannian manifold (M, g) is Sasakian if the metric cone $(C(M), \bar{g})$, $C(M) := \mathbb{R}_+ \times M$ and $\bar{g} = dr^2 + r^2 g$, is Kähler, i.e. $\bar{g}$ admits a compatible almost complex structure I so that $(C(M), \bar{g}, I)$ is Kähler.

This is a special metric contact structure on M, (η, ξ, Φ, g), where

- $\eta = d^c \log r^2 = \frac{1}{2} J d \log r^2$ is a contact form,
- $\xi = J r \partial_r$ is the Reeb vector field, a Killing field,
- $\Phi \in \mathrm{End}(TM)$ defined by $\Phi|_{D = \ker \eta} = I$ and $\Phi(\xi) = 0$, and
- $g = \frac{1}{2} d\eta(\cdot, \Phi \cdot) + \eta \otimes \eta$.

We also have the following:

- Φ induces a transversal holomorphic structure on the Reeb foliation $\mathscr{F}_\xi$, which has a transverse Kähler structure, $\omega^T = \frac{1}{2} d\eta$.
- The cone $C(M) \cup \{o\}$, with vertex added, is uniquely a normal affine variety.
- (D, J), $D = \ker \eta$ and $J = I|_D$ is a strictly pseudoconvex CR structure on M.

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Consider the space of Sasakian structures $\mathcal{S}(\xi, J)$ with Reeb vector field ξ and fixed trans. complex structure J on $\mathscr{F}_\xi$.

Given $(\eta, \xi, \Phi, g) \in \mathcal{S}(\xi, J)$ for any other $(\tilde{\eta}, \xi, \tilde{\Phi}, \tilde{g}) \in \mathcal{S}(\xi, J)$:

$$\tilde{\eta} = \eta + 2d^c\phi + d\psi + \alpha, \quad \phi, \psi \in C_b^\infty(M), \alpha \in \mathcal{H}_b^1 \tag{1}$$

- $\tilde{\omega}^T = \omega^T + dd^c\phi$,
- $d\psi$ is given by the gauge transformation $\exp(\psi\xi)$, and
- $\alpha \in \mathcal{H}_b^1 = H^1(M, \mathbb{R})$, basic harmonic 1-form.

The last two components in (1) do not effect the scalar curvature so can be largely ignored.

We define the Mabuchi space

$$\mathcal{H}_{\omega^T} := \left\{ \phi \in C_b^\infty(M) \mid (\omega^T + dd^c\phi)^m \wedge \eta > 0 \right\}$$

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Calabi functional

The Calabi functional

$$\mathrm{Cal}_{\xi,J,\omega^T} : \mathcal{H}_{\omega^T} \to \mathbb{R}$$

$$\mathrm{Cal}_{\xi,J,\omega^T}\,\phi = \int_M (S_\phi - \bar{S})^2\, d\mu_\phi, \quad \bar{S}^T = \frac{2m\pi c_1(\mathscr{F}_\xi) \cup [\omega^T]^{m-1}}{[\omega^T]^m}$$
$$= \int_M (S^T_\phi - \bar{S}^T)^2\, d\mu_\phi,$$

where $d\mu_\phi = (\omega^T_\phi)^m \wedge \eta$ and S_ϕ is the scalar curvature of $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$

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and S^T_ϕ is the scalar curvature of the transversal metric $\omega^T + dd^c\phi$.

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(η, ξ, Φ, g) is Sasaki-extremal if it is a critical point of $\mathrm{Cal}_{\xi,J,\omega^T}$.

Euler-Lagrange equation for $\mathrm{Cal}_{\xi,J,\omega^T}$ show that this is equivalent to $\partial^{\#} S_g$ being transversally holomorphic.

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Calabi functional

The Calabi functional

$$\mathrm{Cal}_{\xi,J,\omega^T} : \mathcal{H}_{\omega^T} \to \mathbb{R}$$

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where $d\mu_\phi = (\omega^T_\phi)^m \wedge \eta$ and S_ϕ is the scalar curvature of $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$

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Uniqueness

Theorem 3.1

Suppose $(\eta_0, \xi, \omega_0^T), (\eta_1, \xi, \omega_1^T) \in \mathcal{S}(\xi, J)$ *are two Sasaki-extremal structures. Then there is a* $g \in \mathrm{Fol}(\mathscr{F}_\xi, J)$ *with* $g^*\omega_1^T = \omega_0^T$.

Consider now just the transversely holomorphic foliation $(\mathscr{F}, J)$ on M.

Corollary 3.2

Suppose $(\eta_0, \xi_0, \omega_0^T), (\eta_1, \xi_1, \omega_1^T)$ *are two Sasaki-extremal structures with Reeb foliation* $(\mathscr{F}, J)$*, then there is a* $g \in \mathrm{Fol}(\mathscr{F}_\xi, J)$ *and* $a > 0$ *so that* $g^* a\omega_1^T = \omega_0^T, g_*\xi_0 = a^{-1}\xi_1$.

In other words,

$$g^*(a\eta_1, a^{-1}\xi_1, a\omega_1^T) = (\hat{\eta}_0, \xi_0, \omega_0^T),$$

with $\hat{\eta}_0 = \eta_0 + \alpha,\ \alpha \in \mathcal{H}_b^1 = H_b^1(M, \mathbb{R}) = H^1(M, \mathbb{R})$ is harmonic.

We have uniqueness up to harmonic 1-forms, and homotheties

$$\begin{aligned} (\eta, \xi, \Phi, g) \quad &\mapsto \quad (a\eta, a^{-1}\xi, \Phi, g_a) \\ &\qquad\ g_a = ag + (a^2 - a)\eta \otimes \eta. \end{aligned}$$

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K-energy

Given a Sasakian manifold M the K-energy is a functional on $\mathcal{H}_{\omega^T}$:

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X. X. Chen 2000 rewrote this formula to extend $\mathcal{M}$ to weak $C^{1,1}$ structures

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Convexity of K-energy

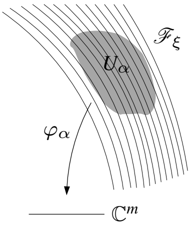

Figure: Transversally complex foliation

The *transversely holomorphic structure* on a foliation $\mathscr{F}_\xi$ is given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M

- $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^m$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

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- There is a Kähler structure ω_α on $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^m$.

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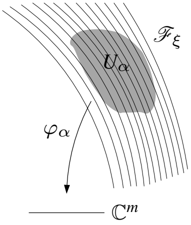

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Analysis is done on the foliation charts.

Let T_α be a closed degree (k,k) current defined on V_α so that $g^*_{\alpha\beta}T_\alpha = T_\beta$.

$$\mathrm{PSH}(M,\omega) := \{\phi \mid \phi \text{ u.s.c. inv. under } \xi \text{ and plurisubharmonic on each chart} V_\alpha\}$$

Given $\phi_1,\ldots,\phi_{m-k} \in \mathrm{PSH}(M,\omega)$, in each V_α we define (E. Bedford and B. Taylor 1976):

$$\omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T_\alpha$$

a positive Radon measure on V_α, and we take the product measure on each chart which is easily seen to be invariant of the chart by Fubini's theorem, defining

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Convexity of K-energy

The following will be useful

Proposition 4.1

Let $\phi \in \mathrm{PSH}(M,\omega) \cap C^0(M)$. *Then there exists a sequence* $\phi_i \in \mathrm{PSH}(M,\omega) \cap C^\infty(M)$ *with* $\phi_i \searrow \phi$ *as* $i \to \infty$.

We have weak continuity of the Monge-Ampère measure.

Given decreasing sequences $\phi_1^i \to \phi_1, \ldots, \phi_{m-k}^i \to \phi_{m-k}$ in $\mathrm{PSH}(M,\omega)$ we have

$$\omega_{\phi_1^i} \wedge \cdots \wedge \omega_{\phi_{m-k}^i} \wedge T \wedge \eta \to \omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T \wedge \eta$$

weak convergence of Radon measures.

Convexity of K-energy

The following will be useful

Proposition 4.1

Let $\phi \in \mathrm{PSH}(M, \omega) \cap C^0(M)$*. Then there exists a sequence* $\phi_i \in \mathrm{PSH}(M, \omega) \cap C^\infty(M)$ *with* $\phi_i \searrow \phi$ *as* $i \to \infty$*.*

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Let $D \subset \mathbb{C}$ then we have the Homogeneous Monge-Ampère equation

$$(\pi^*\omega + dd^c U_\tau)^{m+1} = 0 \quad \text{for } U_\tau \in \mathrm{PSH}(M \times D, \pi^*\omega),$$

P. Guan and X. Zhang 2012 solved it for $D = \{\tau \in \mathbb{C} \mid 1 \leq |\tau| \leq e\}$ and $U(\cdot,1) = \phi_0, U(\cdot,e) = \phi_1 \in C_b^\infty(M)$ on ∂D, and showed U is weak $C^{1,1}$, meaning

$$\pi^*\omega + dd^c U_\tau \geq 0 \quad \text{is } L^\infty(M \times D).$$

Then

$$\omega + dd^c u_t \geq 0 \text{ is weak } C^{1,1} \text{ geodesic connecting } \omega_{\phi_0}, \omega_{\phi_1}, 0 \leq t \leq 1.$$

$t = \log \tau$.

Proposition 4.2

If $u \in \mathrm{PSH}(M,\omega) \cap C^0$ then the first variations of the functionals $\mathcal{E}$ and $\mathcal{E}^{\mathrm{Ric}}$ are

$$d\mathcal{E}|_u = (m+1)\omega_u^m \wedge \eta, \quad d\mathcal{E}^{\mathrm{Ric}}|_u = m\omega_u^{m-1} \wedge \mathrm{Ric}_\omega \wedge \eta.$$

And second variations

$$d_\tau d_\tau^c \mathcal{E}(U_\tau) = \int_M (\pi^*\omega + dd^c U_\tau)^{m+1} \wedge \eta \quad d_\tau d_\tau^c \mathcal{E}^{\mathrm{Ric}}(U_\tau) = \int_M (\pi^*\omega + dd^c U_\tau)^m \wedge \pi^* \mathrm{Ric}_\omega \wedge \eta.$$

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Theorem 4.3

Let u_τ be a weak $C^{1,1}$ geodesic connecting two points in $\mathcal{H}_{\omega^T}$. Then $\mathcal{M}(u_\tau)$ is subharmonic with respect to $\tau \in D$. Thus $\mathcal{M}(u_t),\ 0 \leq t \leq 1$, $t = \log \tau$, is convex.

$\omega_{u_\tau}^m$ defines a singular metric e^Ψ on the transversal canonical bundle $\mathbf{K}_{\mathscr{F}_\xi}$,
The second variation is the current

$$d_\tau d_\tau^c \mathcal{M}(U_\tau) = \int_M T, \quad T := dd^c(\Psi(\pi^*\omega + dd^c U)^m) \wedge \eta$$

But the main problem is to show that T defines a non-negative current on $M \times D$, i.e. a Radon measure.

This is done as in the Kähler case with a local Bergman kernel approximation as in

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Some ideas of the proof

Bergman kernel for holomorphic functions on the ball $B \subset \mathbb{C}^m$ with weight ϕ.

$$\beta_k = \frac{m!}{k^m} K_{k\phi} e^{-k\phi}$$

$$K_{k\phi}(x) = \sup_{s \in H^0(B, K_B)} \frac{s \wedge \bar{s}(x)}{\int_B s \wedge \bar{s} e^{-k\phi}}.$$

$$\beta_k \to (dd^c \phi)^m \quad \text{in total variation.}$$

Choose local psh Φ so that $dd^c \Phi = \pi^* \omega + dd^c U$, $\phi_\tau = \Phi(\cdot, \tau)$. Define

$$T_k = dd^c \Psi_k \wedge (dd^c \Phi)^m \wedge \eta, \quad \Psi_k = \log \beta_k.$$

Then $\lim_{k \to \infty} T_k = T$.

(B. Berndtsson 2006) Plurisubharmonic variation of Bergman kernels

$$dd^c \log K_{k\phi_\tau} \geq 0 \quad \text{on } B \times D$$

So

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Consequences of convexity

One easy application of convexity is the subslope inequality:

For $\phi_0, \phi_1 \in \mathcal{H}_{\omega^T}$ we have

$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq -d(\phi_1, \phi_0)\big(\mathrm{Cal}(\phi_0)\big)^{\frac{1}{2}},$$

where d is the distance function of the Mabuchi metric on $\mathcal{H}$.

- If $\omega^T_{\phi_0}$ cscS, then $\mathcal{M}$ achieves its minimum at ϕ_0.
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Proof of uniqueness

Define

$$\tilde{\mathcal{H}}_{\omega^T} := \{\phi \in \mathcal{H}_{\omega^T} \mid \mathcal{E}(\phi) = 0\}$$

isomorphic to the space of transversal Kähler metrics.

Suppose (η, ξ, Φ, g) is Sasaki-extremal.

- $G = \mathrm{Aut}(\eta, \xi, \Phi, g)_0 \subset H$ is maximal compact.
- Define $P := N_H(G)_0$, normalizer of G in H.
- Define $\mathcal{H}^G$ and $\tilde{\mathcal{H}}^G$ to be the G-invariant potentials.

Proposition 5.1

Let (η, ξ, Φ, g) be Sasaki-extremal. Then its orbit $\mathcal{O} = P/G$ is a symmetric space with Riemannian structure induced by $\mathcal{O} \subset \tilde{\mathcal{H}}^G_{\omega^T}$.

The exponential mapping

$$\exp : \mathfrak{p}/\mathfrak{g} \to \mathcal{O}$$

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Relative K-energy

We consider the relative Mabuchi K-energy on $\mathcal{H}^G$.

Define

$$\mathcal{E}^V(\phi) := \int_0^1 \int_M \dot{\phi}_t h^V_{\phi_t} \, d\mu_{\phi_t} dt,$$

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$$\mathcal{M}^V(\phi) := \mathcal{M}(\phi) + \mathcal{E}^V(\phi), \quad \phi \in \mathcal{H}^G,$$

which extends to $\mathcal{H}^G; = \mathrm{PSH}(M, \omega^T) \cap C^{1,1}_w$.

Proposition 5.2

Let $\{u_t \mid 0 \geq t \geq 1\}$ be a weak $C^{1,1}_w$ geodesic between $u_0, u_1 \in \mathcal{H}^G$. Then $\mathcal{E}^V(u_t)$ is linear in t.

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S^G_ϕ is the reduced scalar curvature which is zero precisely when $\phi \in \mathcal{H}^G$ gives an extremal structure.

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Thus the relative K-energy $\mathcal{M}^V$ is convex and we have a subslope inequality:

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We do not have strict convexity for $\mathcal{M}$ or $\mathcal{M}^V$ along weak geodesics.
So we deform it with a strictly convex term.

Let μ be a smooth G-invariant strictly positive volume form, and define

$$\mathcal{F}^\mu(\phi) := \int_M \phi\, d\mu - c_\mu \mathcal{E}(\phi),$$

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And we define

$$\mathcal{M}^{V,t\mu}(\phi) := \mathcal{M}^V(\phi) + t\mathcal{F}^\mu(\phi), \quad \phi \in \mathcal{H}^G$$

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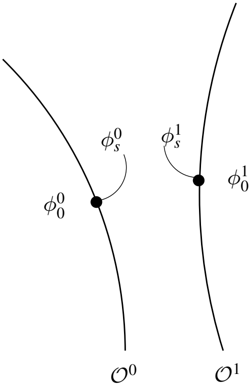

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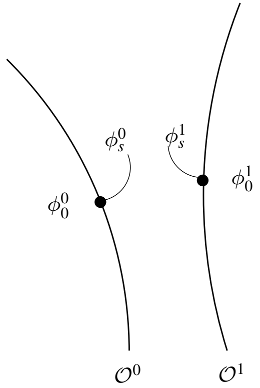

Example

We consider some simple examples of Sasaki-extremal manifolds.

Proposition 6.1

Let (M, η, ξ, Φ, g) be a compact Sasakian manifold. The following are equivalent:

1. *$C(M) = \mathbb{C}^m$, as an affine variety.*
2. *The CR manifold (M, D, J) has a CR embedding into $\mathbb{C}^m$, as a hypersurface.*
3. *(M, η, ξ, Φ, g) is a transversal deformation of the weighted Sasakian structure $(S^{2m-1}, \xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}),\ \mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m_{>0}$, denoted $\mathcal{S}^{2m-1}_{\mathbf{w}}$.*

The Reeb vector fields of these are induced by $\mathfrak{u}(m)^+ \subset \mathfrak{u}(m)$, the positive cone in the unitary Lie alg.

$$\mathfrak{u}(m)^+ := \{\zeta \in \mathfrak{u}(m) \mid -\sqrt{-1}\zeta \text{ acts on } R(\mathbb{C}^m) = \mathbb{C}[z_1, \ldots, z_m] \text{ with pos. weights}\}$$

Up to conjugation by $\mathrm{U}(m)$

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The Sasakian structure $\mathcal{S}_{\mathbf{w}}^{2m-1}$ is as follows. Consider the unit sphere

$$S^{2m-1} \subset \mathbb{C}^m, \text{ with coordinates } (z_1, \ldots, z_m) = (x_1 + iy_1, \ldots, x_m + iy_m).$$

The CR structure is $D = \ker \eta_0$ where

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Theorem 6.2 (Boyer, Galicki, Simanca, David, Gauduchon, van Coevering)

We have the following.

1. *The S-ext structures in the proposition are precisely those whose CR structure (M, D, J) is equivalent to the round structure.*
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Note: These S-ext structures are all transversally Bochner flat. Conversely, any compact, simply connected, transversely Bochner flat Sasakian manifold is one of $\mathcal{S}_{\mathbf{w}}^{2m-1}$

Proof.
The Chern-Moser tensor of the CR structure with contact form (M, D, J, η) is equal to the Bochner tensor of the transversal Kähler structure (g^T, ω^T, J) by the Sasakian condition. Since $\mathcal{S}_{\mathbf{w}}^{2m-1}$ has the round CR structure the Chern-Moser tensor vanishes and it is transversely Bochner flat and thus S-ext. It follows from Theorem 3.1 that these are unique for each Reeb v.f. up to biholomorphism. □

Example

Note: These S-ext structures are all transversally Bochner flat. Conversely, any compact, simply connected, transversely Bochner flat Sasakian manifold is one of $\mathcal{S}^{2m-1}_{\mathbf{w}}$

Proof.

The Chern-Moser tensor of the CR structure with contact form (M, D, J, η) is equal to the Bochner tensor of the transversal Kähler structure (g^T, ω^T, J) by the Sasakian condition. Since $\mathcal{S}^{2m-1}_{\mathbf{w}}$ has the round CR structure the Chern-Moser tensor vanishes and it is transversely Bochner flat and thus S-ext. It follows from Theorem 3.1 that these are unique for each Reeb v.f. up to biholomorphism. □

Thank you