Strongly Hermitian Einstein-Maxwell Solutions on Ruled Surfaces

Joint work with Caner Koca

Christina Tønnesen-Friedman

Conference of Differential Geometry
LeBrun Fest
CRM, Montreal, 2016

In this talk we look at the existence of Strongly Hermitian Einstein-Maxwell (SHEM) Solutions on (higher genera) Ruled Surfaces. We will adapt the constructions of extremal Kähler metrics on same surfaces to the SHEM equations.

We got the inspiration and courage to do the constructions from Claude LeBrun.

Plan:

1. Scalar Curvature
2. Einstein-Maxwell Equations
3. Strongly Hermitian Einstein-Maxwell Equations
4. Constructions, results and non-results
5. The Einstein-Hilbert Functional (if time)

Scalar Curvature of Kähler metrics:

Let M be a smooth compact manifold of real dimension 4.
(This will be the underlying assumption for the rest of the talk.)
Given a Kähler structure (M, J, g, ω), the Riemannian metric g defines (via the unique Levi-Civita connection ∇)

- the **Riemann curvature tensor** $R : TM \otimes TM \otimes TM \to TM$
- and the trace thereoff, the **Ricci tensor** $r : TM \otimes TM \to C^\infty(M)$
- the **scalar curvature**, $Scal \in C^\infty(M)$, where $Scal$ is the trace of the map $X \mapsto \tilde{r}(X)$ where $\forall X, Y \in TM, g(\tilde{r}(X), Y) = r(X, Y)$.
- If $Scal$ is a constant function, we say that (M, J, g, ω) is a constant scalar curvature Kähler metric (or just **CSC**).
- Not all complex manifolds (M, J) admit CSC Kähler structures, but...
- from the famous work of Yamabe, Trudinger, Aubin, and Schoen we do know that, for any Riemannian metric g,
  there always exists some positive smooth function $f : M \to \mathbb{R}^+$ such that the metric $h = f^{-2}g$ has constant scalar curvature.

Einstein-Maxwell Equations

Consider a compact Riemannian 4-manifold (M, h). **If** h is part of a solution of the (Euclidean) Einstein-Maxwell equations;

$$\begin{aligned} dF &= 0 \\ d \star F &= 0 \\ [\tilde{r} + F \circ F]_0 &= 0, \end{aligned} \tag{1}$$

where $\tilde{r}$ is the Ricci tensor of h, F is a real 2-form on M, $[\,]_0$ denotes the trace-free part with respect to h, and $F \circ F$ is the composition of F with itself, when we view F as an endomorphism on the tangent bundle TM, **then** the scalar curvature of h must be constant

The converse is not true in general, but....

Strongly Hermitian Einstein-Maxwell Equations

- If (M^4, g, J) is a Kähler manifold and $f > 0$ is a real holomorphic potential on (M, J, g) ($J\,grad\,f$ is killing) such that $h = f^{-2}g$ has constant scalar curvature, then (h, F) solves (1), where F is a unique harmonic 2-form on M with self-dual part equal to the Kähler form ω. Since in that case both h and F are J invariant, such Einstein-Maxwell solutions are called Strongly Hermitian.
(LeBrun and Apostolov, Calderbank and Gauduchon)
- Any SHEM (i.e. Einstein-Maxwell solution (h, F) that is J invariant) must arise this way (from some compatible Kähler structure on the given (M^4, J)), and unless h is a Kähler metric itself we must have that (M^4, J) is rational or ruled.
(LeBrun)
- The above should be seen in light of the fact (due to Shu and Chen, LeBrun, and Weber) that as long as M^4 is of Kähler type then M admits some Einstein-Maxwell metric.(Most of these can be supplied by CSC Kähler metrics.)

Remember we are assuming compactness here.

Constructions on Ruled Surfaces

- On $\mathbb{CP}^1 \times \mathbb{CP}^1$ Lebrun constructed some special cases where g is a (non-CSC) product metric (of one CSC and one S^1-symmetric non-CSC) and f is the "height" of the S^1-symmetric non-CSC one....So "uniqueness of Kähler metrics g with a SHEM in $[g]$ for a fixed Kähler class" is definitely out the window.
- This construction does not work on $S^2 \times \Sigma$ with genus of compact Riemann surface Σ larger than zero (LeBrun).
- It also clearly does not work for any complex structure on the smooth twisted product $S^2 \tilde{\times} \Sigma$ for any genus of Σ.

More constructions

- On Hirzebruch surfaces Lebrun used a Calabi type construction to get examples. Indeed he proved that *Any Kähler class on any Hirzebruch surface may be represented by a $U(2)$-invariant (standard action) Kähler metric which is conformally related to a SHEM. Except for the first Hirzebruch surface this representative is unique (assuming $U(2)$-invariance that is).* These Kähler metrics are actually ambi-Kähler as defined by Apostolov, Calderbank, and Gauduchon.
- The case of the first Hirzebruch surface complex structure is rather unusual...and not just because it hosts the "role model" Einstein Page metric. Later...

On $S^2 \times \Sigma$ and $S^2 \tilde{\times} \Sigma$, where Σ is a compact Riemann surface, we can set up the construction in general.

- This is a special case of the more general (admissible) constructions defined by/organized by Apostolov, Calderbank, Gauduchon, and T-F.
- Credit goes to LeBrun, Koiso, Sakane, Pedersen, Poon, Hwang, Singer, Guan, and others.

Admissible/generalized Calabi constructions

Consider the total space of a projective bundle $S_n = \mathbb{P}(\mathbb{1} \oplus L_n) \to \Sigma$, where

- Σ is a compact Riemann surface of any genus.
- ω_Σ is a primitive integral Kähler form of a CSC Kähler metric.
- $\mathbb{1} \to \Sigma$ is the trivial complex line bundle.
- $L_n \to \Sigma$ is a holomorphic line bundle.
- $c_1(L_n) = [n\,\omega_\Sigma]$
- $n \in \mathbb{Z} \setminus \{0\}$....WLOG $n \in \mathbb{Z}^+$.
- Note that the fiber is $\mathbb{CP}^1$.
- S_n is called **admissible**, or an **admissible manifold**.
- S_n is a special type of ruled surface that generalizes the Hirzebruch surfaces.
- Note that S_n is diffeomorphic to $S^2 \times \Sigma$, if n is even, and $S^2 \tilde{\times} \Sigma$, if n is odd.

- Let $D_1 = [\mathbb{1} \oplus 0]$ and $D_2 = [0 \oplus L_n]$ denote the "zero" and "infinity" sections of $S_n \to \Sigma$.
- Let x be a real number such that $0 < x < 1$.
- By the Leray-Hirsch theorem, Kähler classes on S_n, Ω looks up to scale like
$$\Omega = \frac{[\omega_{\Sigma,n}]}{x} + \Xi,$$
where Ξ is Poincaré dual to $2\pi[D_1 + D_2]$ and $\omega_{\Sigma,n} = 2\pi n \omega_\Sigma$
- So, up to rescaling, the Kähler cone is parametrized by x.
- Another way of writing Ω is
$$\Omega = 4\pi PD(D_1) + \frac{2\pi(1-x)n}{x} PD(C),$$
where C is a fiber and "PD" means "Poincaré Dual".

Small Detour

- Or, if E denotes the section of $S_n \to \Sigma$ which has self-intersection zero, in the case of n being even, and self-intersection one in the case of n being odd,

$$\Omega = 4\pi(PD(E) + p\,PD(C)),$$

with

$$x = \begin{cases} \frac{k}{p}, & \text{when } n = 2k \text{ is even} \\ \frac{2k+1}{2p+1}, & \text{when } n = 2k+1 \text{ is odd} \end{cases}$$

- Thus, for a fixed p, the cohomology class $\Omega = 4\pi(PD(E) + p\,PD(C))$ on the smooth manifold $S^2 \times \Sigma$ is an Kähler class on the admissible ruled surface S_{2k} iff $1 \leq k < p$.
- In the case, where the smooth manifold is instead $S^2 \tilde{\times} \Sigma$, Ω is a Kähler class on S_{2k+1} iff $0 \leq k < p$.

Back to the main road

- In each Kähler class there is a **canonical** Kähler metric g_c:
- $g_c|_{fiber}$ = the Fubini-Study metric.
- Let K be the vector field generating the fiberwise canonical effective S^1-action on S_n.
- The metric g_c is such that K is Killing and Hamiltonian.
- So we have a moment map $\mathfrak{z} : S_n \to \mathbb{R}$, with $K = J\nabla_{g_c}\mathfrak{z}$.
- I.e. $\omega_c(K, \cdot) = -d\mathfrak{z}$.
- WLOG the image of $\mathfrak{z}$ is $[-1, 1]$, $\mathfrak{z}^{-1}(1) = D_1$, and $\mathfrak{z}^{-1}(-1) = D_2$.

- On $S_{n0} = \mathfrak{z}^{-1}(-1,1)$, g_c and its Kähler form ω_c may be written as

$$
\begin{aligned}
g_c &= \tfrac{1+x\mathfrak{z}}{x} g_{\Sigma,n} + \tfrac{d\mathfrak{z}^2}{\Theta_c(\mathfrak{z})} + \Theta_c(\mathfrak{z})\theta^2 \,, \\
\omega_c &= \tfrac{1+x\mathfrak{z}}{\mathfrak{z}} \omega_{N_n} + d\mathfrak{z} \wedge \theta \,,
\end{aligned}
\tag{2}
$$

- θ is a connection 1-form $(\theta(K) = 1)$
- $\Theta_c(\mathfrak{z}) = 1 - \mathfrak{z}^2$

Admissible metrics

- The function $\Theta_c(\mathfrak{z}) = 1 - \mathfrak{z}^2$
  1. is smooth on $[-1, 1]$
  2. positive on $(-1, 1)$
  3. satisfies the boundary conditions $\Theta_c(\pm 1) = 0$ and $\Theta_c'(\pm 1) = \mp 2$.
- The idea of the moment map construction/generalized Calabi construction/admissible construction is that we may obtain other S^1-invariant Kähler metrics g if we use the same expression for g_c above but replace Θ_c by any smooth function Θ on $[-1, 1]$ satisfying 1., 2. and 3.
- $$g = \frac{1 + x\mathfrak{z}}{x} g_{\Sigma,n} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1 + x\mathfrak{z}}{x}\omega_{\Sigma,n} + d\mathfrak{z} \wedge \theta, \quad (3)$$

  on S_{n0}
- g and ω extend smoothly to all of S_n.
- A metric of this type (up to scale) on S_n is called **admissible** (in the ACGT program).

Complex versus symplectic point of view

Varying the function Θ while keeping x, $\mathfrak{z}$, and θ fixed

- Varies the admissible metric g
- Fixes the Kähler form ω
- So the complex structure varies.

However

- By using pull-backs by appropriate S^1-equivariant diffeomorphisms, we may fix a canonical complex structure J, and change the Kähler form instead.
- Thus, different Θs give rise to different Kähler forms.
- The Kähler class is still fixed.

A Convenient Notation:

Define a function $F(\mathfrak{z})$ by the formula

$$\Theta(\mathfrak{z}) = \frac{F(\mathfrak{z})}{(1 + x\mathfrak{z})} \tag{4}$$

Since $(1 + x\mathfrak{z})$ is positive for $-1 < \mathfrak{z} < 1$, conditions on $\Theta(\mathfrak{z})$ imply the following equivalent conditions on $F(\mathfrak{z})$:

$$\boxed{\begin{array}{l} (i)\ F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ \\ (ii)\ F(\pm 1) = 0, \\ \\ (iii)\ F'(\pm 1) = \mp 2(1 \pm x). \end{array}} \tag{5}$$

Some useful formulas

(Thanks Abreu and Apostolov, Calderbank, and Gauduchon.)
For a metric as in (3) we have

- The scalar curvature is given by

$$Scal(g) = \frac{2s_{\Sigma,n}x}{1+x\mathfrak{z}} - \frac{F''(\mathfrak{z})}{1+x\mathfrak{z}}, \tag{6}$$

  where $s_{\Sigma,n} = 2(1-\mathfrak{g})/n$ with $\mathfrak{g}$ being the genus of Σ.
- If $p(\mathfrak{z})$ is a smooth function of $\mathfrak{z}$, then

$$\Delta p = -[F(\mathfrak{z})p'(\mathfrak{z})]'/(1+x\mathfrak{z}), \tag{7}$$

  where Δ is the Laplacian associated to g.

Let $h = (\mathfrak{z} + b)^{-2} g$, where we require that $|b| > 1$. Using the conformal change formula for scalar curvature and the formulas (6) and (7) above, we calculate that the scalar curvature of h is given by

$$Scal(h) = \frac{-(\mathfrak{z} + b)^2 F''(\mathfrak{z}) + 6(\mathfrak{z} + b)F'(\mathfrak{z}) - 12F(\mathfrak{z}) + 2s_{\Sigma,n} x(\mathfrak{z} + b)^2}{(1 + x\mathfrak{z})}.$$

The equations we get from setting $Scal(h)$ equal to some constant:

$$F(\mathfrak{z}) = (1-\mathfrak{z}^2)\left((1+x\mathfrak{z}) - c(1-\mathfrak{z}^2)\right) \tag{8}$$

with

$$\boxed{\begin{aligned} c &= \tfrac{-1+3bx-s_{\Sigma,n}x}{2(3b^2-1)} \\ &+ \tfrac{3x\left(xb^2-2b+x\right)\left((s_{\Sigma,n}x-2)b^2+2bx-s_{\Sigma,n}x\right)}{2(3b^2-1)((b^2-3)x^2+4bx+(1-3b^2))} \end{aligned}} \tag{9}$$

$$\boxed{\left(xb^2-2b+x\right)\left((s_{\Sigma,n}x-2)b^2+2bx-s_{\Sigma,n}x\right)=0.} \tag{10}$$

So here is the construction job in a nutshell:

- For a given $0 < x < 1$, solve (10) for $|b| > 1$.
- For any solution check if F from (8) with the associated c from (9) satisfies (i) of (5).

Another detour:

(Apropos Maschler's talk)
The equation (8) with c given by (9) guarantees that $Scal(h)$ is an affine function of $\mathfrak{z}$.

Following the work by Apostolov and Maschler we have an analogue of the **extremal polynomial**,

and see that the left hand side of (10) is a Futaki invariant,

which in an appropriate sense is an obstruction to SHEMs similar to the usual Futaki invariant being an obstruction to Kähler CSC.

End of detour.

Assuming $\mathcal{G} \geq 1$ here is what we found:

- For a given $0 < x < 1$ and $s_{\Sigma,n} \leq 0$ we have only one solution:

$$b = \frac{1 + \sqrt{1 - x^2}}{x} \tag{11}$$

  (which indeed gives us $b > 1$ since $0 < x < 1$).
- For $s_{\Sigma,n} = 0$ (i) of (5) is satisfied and hence we have our desired metrics.
- For $s_{\Sigma,n} < 0$, there exists a unique value $x_{s_\Sigma,2} \in (0,1)$ such that for $0 < x < x_{s_\Sigma,2}$, (i) of (5) is satisfied and hence we have our desired metrics and if $x_{s_\Sigma,2} < x < 1$, (i) of (5) is not satisfied and hence we do not have our special types of metrics.
- Very rough estimate: $x_{\Sigma,n} > \frac{1}{s_{\Sigma,n}^2 + 2}$.

APROPOS THE EXTREMAL METRICS CONSTRUCTIONS!

So what is going on for the bad classes?

- ???
- Perhaps an appropriate notion of K-polystability - as developed in the toric case by Apostolov and Maschler - fails.
- Is it possible that there is no ω-compatible Kähler metric which is conformal to an Einstein-Maxwell metric with the conformal factor $(\mathfrak{z}+b)^{-2}$?
- Can we develop (and prove) a notion of uniqueness?
- Beware of the $\mathcal{G}=0$ case.

The $\mathcal{G} = 0$ Case (due to LeBrun)

- There is a family as above with $b = \frac{1+\sqrt{1-x^2}}{x}$ and here (i) of (5) is satisfied for any $x \in (0, 1)$.
- When $s_{\Sigma,n} = 2$ (i.e. $n = 1$ and we have the first Hirzebruch surface) and $x \geq 4/5$ we also get solutions from $(s_\Sigma x - 2)b^2 + 2bx - s_\Sigma x = 0$, namely

$$b = \frac{x \pm \sqrt{x(5x-4)}}{2(1-x)}.$$

  - For $x = 4/5$ this gives one extra solution (with (i) of (5) satisfied).
  - For $x > 4/5$ there are two solutions for b...BUT $c = (1-x)/2$ in both cases so we get only one F from (8) and hence only one Kähler form conformal to two different SHEMs!!!!

The Einstein-Hilbert functional

The (normalized) Einstein-Hilbert functional evaluated for a Riemannian metric on a compact 4-manifold M is defined by

$$\mathfrak{S} := \frac{\int_M Scal\, d\mu}{\sqrt{\int_M d\mu}}.$$

Note that this is invariant under re-scaling. Restricting $\mathfrak{S}$ to the conformal class $[g]$ of a Riemannian metric yields the **Yamabe Functional**.

From Yamabe, Trudinger, Aubin, and Schoen we have:

- The **Yamabe Constant** $Y_{[g]} := Min \mathfrak{S} |_{[g]}$ exists and any (Yamabe) minimizer has constant scalar curvature.
- $Y_{[g]} \leq 8\sqrt{6}\pi$ with "$<$ "unless $(M, [g])$ is conformal to the 4-sphere.
- If $Y_{[g]} \leq 0$, then the Yamabe minimizer is unique in $[g]$.
- If $Y_{[g]} > 0$, then uniqueness might not hold and further, not every metric with constant scalar curvature is a minimizer.

When g is admissible and $h \in [g]$ is SHEM:

It follows from an estimate due to LeBrun that

$$Y_{[h]} = Y_{[g]} \overset{LeBrun}{\leq} \frac{4\pi c_1 \cdot \Omega}{\sqrt{\Omega^2/2}} = \frac{4\pi(2 + 2s_{\Sigma,n}x)\sqrt{n}}{\sqrt{2x}}.$$

Equality would only hold if $g = h$ and a Yamabe minimizer (hence CSC)....so definitely not here.

Incorporating the SHEM metric $h \in [g]$ and calculating

$$Scal(h)Vol(h)^{1/2}$$

$$= \quad \frac{12\pi\sqrt{n}\left(1-6b^2+b^4+2bx+2b^3x-s_{\Sigma,n}x+b^4s_{\Sigma,n}x\right)}{3b^2-1}\sqrt{\frac{2(3b^2-4bx+1)}{3x(b^2-1)^3}},$$

we spotted some non-Yamabe minimizer examples as well as some cases with improvements of the LeBrun estimate.

Example

Assume that $n = 1$ and the genus of Σ, $\mathfrak{g} = 2$. Then $s_{\Sigma,n} = -2$ and one may check that numerically $x_{\Sigma,2} \approx 0.97367$.Recall that $0 < x < x_{\Sigma,2}$ is where we succeeded in producing a SHEM metric h in $[g]$.

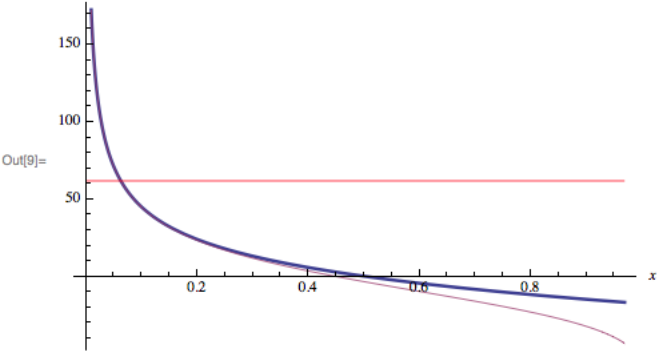
$$Scal(h)Vol(h)^{1/2} = 4\sqrt{6}\pi \left(\sqrt{\frac{1-x^2}{x\left(1+2\sqrt{1-x^2}\right)}} - 2\sqrt{\frac{x}{(1+2\sqrt{1-x^2})}} \right),$$

which reaches negative values by (approximatively) $x = 0.44722$ and safely before $x = x_{s_{\Sigma},2} \approx 0.97367$. When $s_{\Sigma,n} = -2$ and $n = 1$, the LeBrun estimate becomes

$$Y_{[g]} < \frac{8\pi(1-2x)}{\sqrt{2x}}.$$

Note that

$$4\sqrt{6}\pi \left(\sqrt{\frac{1-x^2}{x\left(1+2\sqrt{1-x^2}\right)}} - 2\sqrt{\frac{x}{(1+2\sqrt{1-x^2})}} \right) < \frac{8\pi(1-2x)}{\sqrt{2x}}.$$



Note the interval (approx. $(0.44722, 0.5)$) where $\frac{8\pi(1-2x)}{\sqrt{2x}} > 0$ and $Scal(h)Vol(h)^{1/2} < 0$.

Kähler class/Conformal class generalized:

Assume the orientation of M is fixed.
Let Ω be a fixed cohomology class in $H^2(M, \mathbb{R})$ such that $\Omega^2 > 0$ and let $\mathscr{G}_\Omega$ denote the set of smooth Riemannian metrics h on M for which the harmonic representative ω of Ω is self-dual.

In particular, $\mathscr{G}_\Omega$ will include any Riemannian metric that is Kähler with respect to some complex structure (compatible with the fixed orientation) on M such that its Kähler form belongs to Ω. Obviously if $h \in \mathscr{G}_\Omega$, then $[h] \subseteq \mathscr{G}_\Omega$.

The Einstein-Hilbert functional restricted to $\mathscr{G}_\Omega$:

From LeBrun's work we know:

- The critical points of $\mathfrak{S}|_{\mathscr{G}_\Omega}$ are exactly the Einstein-Maxwell solutions for which the self-dual part F^+ of the 2-form F is in Ω.
- The moduli-space of the Ω-compatible solutions of the Einstein-Maxwell equations is

$$\mathcal{M}_\Omega = \{(h, F) \text{ solves } (1) \mid F^+ \in \Omega\}/[\mathrm{Diff}_H(M) \times \mathbb{R}^+],$$

  where $\mathrm{Diff}_H(M)$ is the group of diffeomorphisms of M acting trivially on $H^2(M, \mathbb{R})$. $\mathbb{R}^+$ acts by rescaling the metric h, but does not change the 2-form F.
- The value of $\mathfrak{S}$ is an invariant of connected components of $\mathcal{M}_\Omega$

By computing the value of the functional $\mathfrak{S}$ for some of the Einstein-Maxwell solutions he found on the Hirzebruch surfaces, LeBrun proved that **on $S^2 \times S^2$ and $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ $\mathcal{M}_\Omega$ has as many connected components as we wish for an appropriate de Rham class Ω on these manifolds.**

Using the constructions just described we saw easily that this result generalizes to the product $S^2 \times T^2$ and the twisted product $S^2 \tilde{\times} T^2$.

Not quite as easily we were able to generalize the result to the product $S^2 \times \Sigma$ and the twisted product $S^2 \tilde{\times} \Sigma$ for Σ a compact Riemann surface of any genus.
We needed the first detour and then the rough estimate of $x_{\Sigma,n}$ to be in business.

Thank You and

Happy Birthday, Claude!

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