The role of Seiberg-Witten theory in Riemannian geometry

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- *Can we classify the simply connected smooth 4-manifolds?*
  
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- Does a smooth manifold admit a preferred metric? We focus on Einstein metrics.

- How can we understand the differential invariants? We consider the Seiberg-Witten invariant and Yamabe invariant.
Main ingredients

$(M, g)$ compact, oriented, smooth 4–manifold, $g$ a Riemannian metric.
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Homotopy invariants

- fundamental group: $\pi_1(M)$
- Second Stiefel-Whitney class: $w_2(M) \in H^2(M, \mathbb{Z}_2)$
- Euler Characteristic: $\chi(M) = 2 - 2b_1 + b^+ + b^-$
- Signature: $\tau(M) = b^+ - b^-$

Freedman, Donaldson: Compact, smooth, simply connected 4-manifolds are classified, up to homeomorphism, by their topological invariants: $\chi(M), \tau(M)$, and the parity of the intersection form (i.e. $w_2 = 0$ or $\neq 0$).

Consequence: A simply connected, non-spin ($w_2 \neq 0$) smooth manifold is homeomorphic to $\mathbb{CP}^2 \# b\mathbb{CP}^2$ ($a = b+$, $b = b^-$).

On a non-spin manifold, we call this the canonical smooth structure, otherwise we say that the smooth structure is exotic.

We will consider manifolds with finite fundamental group, hence $b_1 = 0$, for the rest of the talk.
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Complex numerical invariants:

(M, J) (almost) complex surface, then we can consider the Chern numbers:

- $c_2(M, J) = \chi(M)$;
- $c_1^2(M, J) = 2\chi(M) + 3\tau(M) = 4 + 5b^+ - b^-$;

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- $\chi_h(M, J) = \frac{\chi + \tau}{4}(M) = \frac{1}{2}(1 + b^+)$ the holomorphic Euler characteristic, or the Todd genus;
  - always an integer if $M$ supports a (almost) complex structure.
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**Remark**

*In complex dimension 2, the complex numerical invariants are determined by the topological invariants.*
Seiberg-Witten Theory

Given \((M, g)\) and let \(\nabla_\pm\) be the spin\(^c\) structure associated to the Hermitian line bundle \(L\), \((c_1(L) \equiv w_2(M) \mod 2)\).

The Seiberg-Witten Equations:

\[
\begin{align*}
D_A \Phi &= 0 \\
F_A^+ &= i \sigma(\Phi)
\end{align*}
\]

where \(\Phi \in \Gamma(\nabla_+)\), \(A\) a connection on \(L\), \(F_A^+\) is the self-dual part of the curvature of \(A\), and where \(\sigma : \nabla_+ \rightarrow \Lambda^+\) is a natural real-quadratic map satisfying

\[
|\sigma(\Phi)| = \frac{1}{2\sqrt{2}} |\Phi|^2.
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There are large classes of manifolds for which the invariant is non-trivial: symplectic manifolds, manifolds obtained via certain gluing surgeries. (Taubes, Szabó, Morgan, etc.)
Seiberg-Witten invariant and its properties

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- There are large classes of manifolds for which the invariant is non-trivial: symplectic manifolds, manifolds obtained via certain gluing surgeries. (Taubes, Szabó, Morgan, etc.)
- \( SW_M \neq 0 \) if \( M = N_1 \# N_2 \) such that \( SW_{N_1} \neq 0 \) and \( b^+(N_2) = 0 \). In particular, if \( SW_M(L) \neq 0 \) then \( SW_{M \# \mathbb{CP}^2}(L \pm E) \neq 0 \), but \( SW_{M \# \mathbb{CP}^2}(L') = 0 \) for all \( L' \).
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- generalization of the Seiberg-Witten invariant: due to Bauer-Furuta, defines a nontrivial invariant for certain connected sums: $M = N_1 \# \ldots \# N_m, m = 2, 3, 4$,
  - necessary condition $SW_{N_j}(L_j) \equiv 1 \mod 2$. 

Weitzenb"ock formula for the Dirac operator $D_A$ in relation with the Seiberg-Witten equations:

$$0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

In particular, there are no positive scalar curvature metrics on manifolds with non-trivial S-W invariant.

Seiberg-Witten (Bauer-Furuta) invariant of $M = a\mathbb{CP}^2 \# b\mathbb{CP}^2$ vanishes if $a > 1$.

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- Seiberg-Witten (Bauer-Furuta) invariant of \( M = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2} \) vanishes if \( a > 1 \). Lawson and Gromov showed that \( M \) supports a metric with positive scalar curvature.
Exotic structures on simply connected non-spin manifolds

Definition

An exotic manifold $M$ is called $\mathbb{CP}^2$-almost completely decomposable if $M \# \mathbb{CP}^2$ has the canonical smooth structure.

Theorem (Braungardt-Kotschick)

For every $\epsilon > 0$ there is a constant $N_{\epsilon} > 0$ such that every lattice point $(m, n)$ in the first quadrant satisfying $n \leq (9 - \epsilon) m - N_{\epsilon}$ is realized by the Chern invariants $(m, n) = (\chi, c_2)$ of infinitely many, pairwise non-diffeomorphic, non-spin, simply connected, minimal symplectic manifolds $M(m, n, i)$, all of which are $\mathbb{CP}^2$-almost completely decomposable.
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Geography of simply connected, minimal, symplectic 4-manifolds

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For every $\epsilon > 0$ there is a constant $N_\epsilon > 0$ such that every lattice point $(a, b)$ satisfying the conditions

\[
a \equiv 1 \pmod{2}
\]
\[
b \leq 4 + 5a
\]
\[
b \geq \left(\frac{1}{2} + \epsilon\right) a + N_\epsilon
\]

is realized by the Betti two invariants $(a, b) = (b_2^+, b_2^-)$ of infinitely many, pairwise non-diffeomorphic, non-spin, simply connected, minimal symplectic manifolds $M_{(a, b, i)}$, all of which are $\mathbb{CP}^2$-almost completely decomposable.
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Theorem (S.)

For any $\epsilon > 0$ there is a constant $N'_\epsilon > 0$ such that given any integer pair $(a, b)$ in the first quadrant satisfying either one of the two conditions

\[ b \geq \left( \frac{1}{2} + \epsilon \right) a + N'_\epsilon \text{ and } a \not\equiv 0 \mod 8 \]
\[ b \leq \frac{2}{1 + 2\epsilon} (a - N'_\epsilon) \text{ and } b \not\equiv 0 \mod 8, \]

the topological space $M = \# a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$ admits infinitely many, pairwise non-diffeomorphic, smooth structures, which are all $S^2 \times S^2$-almost completely decomposable.
For any integer $d \geq 2$ and for any $\epsilon > 0$ there exists a constant $N''_\epsilon > 0$ such that for any point $(a, b)$ in the region $R_\epsilon$ satisfying the divisibility conditions $D_1$ and $D_2$ the manifold $M = \# a\mathbb{CP}^2 \# b\mathbb{CP}^2$ has the following properties:

- $M$ admits infinitely many, smooth, orientation preserving, free actions of the group $\mathbb{Z}_d$, which we denote by $\Gamma_{d,i}, i \in \mathbb{N}$,
- the actions $\Gamma_{d,i}$ are conjugate by homemorphisms but are not conjugate by the diffeomorphisms of $M$. 

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\[
R_\epsilon = \{ (a, b) | b \geq (\frac{1}{2} + \epsilon)a + N''_\epsilon \} \cup \{ (a, b) | b \leq \frac{2}{1 + 2\epsilon}(a - N''_\epsilon) \}
\]

- \( D_1 : \quad a + 1 \equiv 0 \mod d \) and \( b + 1 \equiv 0 \mod d \),
- \( D_2 : \quad \) if \((a, b) \in R_{\epsilon,1} \) then \( \frac{a+1}{d} \not\equiv 1 \mod 8 \)
- or if \((a, b) \in R_{\epsilon,2} \) then \( \frac{b+1}{d} \not\equiv 1 \mod 8 \).
Homotopy invariants from a Riemannian perspective

$g$ a Riemannian metric: $s$, $W^\pm$, $\circ$ are the scalar, Weyl, trace free Ricci curvatures and $\mu_g$ the volume form
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$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{1}{2} \frac{\circ r^2}{2} \right) d\mu_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) d\mu_g$$
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$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W^\pm|^2 - \frac{1}{2} \frac{\mathring{r}^2}{2} \right) d\mu_g$$
A topological obstruction to existence of Einstein metrics

**Einstein metric** = constant Ricci curvature:

\[ r_{jk} = \lambda \ g_{jk} \text{ or } \tilde{r} = 0. \]
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**Theorem (Hitchin-Thorpe Inequality)**

If the smooth compact oriented 4-manifold \( M \) admits an Einstein metric \( g \), then

\[ (2\chi \pm 3\tau)(M) \geq 0, \]

with equality if \( (M, g) \) is finitely covered by a flat 4-torus \( T^4 \) or by the K3 surface with a hyper-Kähler metric or by the orientation-reversed K3 with a hyper-Kähler metric.
A differential obstruction to existence of Einstein metrics

**Theorem (LeBrun)**

Let $X$ be a compact oriented 4-manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then

$$M = X \# k\mathbb{CP}^2$$

does not admit Einstein metrics if $k \geq \frac{1}{3}(2\chi + 3\tau)(X)$. 

**Key ingredient:** curvature estimate:

$$\frac{1}{4}\pi^2 \int_M (s^2 + 2|\mathcal{W}|^2) \, d\mu \geq 2\beta(\mathcal{L})^2$$

where $\beta + 1$ is the self-dual part of $\beta^1$.
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LeBrun, LeBrun-Ishida, Kotshick & collaborators: Obstructions to the existence of Einstein metrics on manifolds with exotic structures.
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On $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 0, 1, \ldots, 8$, there exist Einstein metrics.

- on $\mathbb{CP}^2$ there is the Fubini-Study metric
- on $\mathbb{CP}^2 \# 3, \ldots, 8\overline{\mathbb{CP}^2}$ there are Kähler-Einstein metrics (Tian-Yau, Siu)
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In general, the existence of an Einstein metric is unknown.
Dependence on the smooth structure

Question (Besse)

*Is the sign of the Einstein determined by the homeomorphism class of the manifold?*
Dependence on the smooth structure

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**Answer:**

Catanese-LeBrun: NO.

Example: $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$ and a deformation of the Barlow surface (complex surface of general type, with *ample* canonical line bundle).
Theorem (Rasdeaconu-S.)

On every $M = \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k = 5, 6, 7, 8$ there exists a (exotic) smooth structure which admits an Einstein metric with $s < 0$ and infinitely many non-diffeomorphic smooth structures which do not admit Einstein metrics.
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- On $M = \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k = 5, \ldots, 8$, we show that the exotic complex structures constructed by Park and collaborators, have ample canonical line bundle. Hence they admit a Kähler-Einstein metrics of negative scalar curvature, from the solution of the Calabi conjecture.
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- Starting with exotic smooth structures on \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \) (due to Akhmedov, Baykur and Park) we construct infinitely many exotic smooth structures on \( M \) which don’t admit an Einstein metric. All these exotic smooth structures have negative Yamabe invariant.

- Due to the nature of the obstruction theorem, this bound can not be lowered.
Non-existence of invariant Einstein metrics

Theorem (S.)

For any integer $d \geq 2$ and any $\epsilon > 0$ there exists a $N(\epsilon) > 0$ such that for any integer lattice point $(n, m)$, satisfying:

- $n > 0$
- $n \equiv 0 \mod d$
- $m \equiv 0 \mod d$
- $n < (6 - \epsilon) m - N(\epsilon)$

there exist infinitely many free, non-equivalent smooth $\mathbb{Z}_d$-actions on canonical smooth manifold $M$ with $c_2^1(M) = n$, $\chi_h(M) = m$ such that there is no Einstein metric on $M$ invariant under any of the $\mathbb{Z}_d$-actions.

The existence of Einstein metrics is topologically unobstructed: Hitchin-Thorpe inequality, $n > 0$, is satisfied.

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Seiberg-Witten theory and geometry

Montreal, July 6th
Non-existence of invariant Einstein metrics

Theorem (S.)

For any integer $d \geq 2$ and any $\epsilon > 0$ there exists a $N(\epsilon) > 0$ such that for any integer lattice point $(n, m)$, satisfying:

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the region \( n < (6 - \varepsilon)m - N(\varepsilon) \) is determined by the geography of simply connected, symplectic manifolds.

If we denote by \( \Gamma_i, i \in \mathbb{N} \), the actions of \( \mathbb{Z}_d \) on \( M \), then the quotient manifolds \( M/\Gamma_i \) are homeomorphic, but pairwise non-diffeomorphic.

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M/\Gamma_i = M_{(m,n,i)} \# k\overline{\mathbb{C}P^2} \# S_d
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where $S_d$ is a rational homology sphere, $\pi_1(S_d) = \mathbb{Z}_d$, $\widetilde{S_d} = \#(d - 1)(S^2 \times S^2)$. 
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Proposition (S.)

There exists an involution $\sigma$ acting freely on the manifold $M = 15\mathbb{CP}^2 \# 77\overline{\mathbb{CP}^2}$, and $15\mathbb{CP}^2 \# 77\overline{\mathbb{CP}^2}$ does not admit a $\sigma$–invariant Einstein metric.

Here $n = c_1^2(M) = 2$, $m = \chi_h(M) = 8$. 
Existence of invariant Einstein metrics

Theorem (S.)

Given an integer \( d \geq 2 \), there are infinitely many compact, smooth, simply connected, non-spin manifolds \( M_i \), \( i \in \mathbb{N} \), whose topological invariants satisfy

\[
\text{c}^2(M_i) = n_i > 0 \quad \text{and} \quad \text{c}^2(M_i) < 5 \chi(M_i),
\]

and have the following properties:

There is at least one free, smooth, \( \mathbb{Z}_d \) action on \( M_i \), \( M_i \) admits an Einstein metric which is invariant under the above \( \mathbb{Z}_d \) action, \( M_i \) is \( \mathbb{CP}^2 \)-almost completely decomposable.
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- \( M_i \) are complex surfaces with ample canonical line bundle, and admit Kähler-Einstein metrics.

\( M_i = \pi_2 \rightarrow N \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \)

where \( \pi_1, \pi_2 \) are branched covers of orders \( d \) and \( p \) (\( p = 2, 3 \)), branched on transverse curves of positive self-intersection.

The \( \mathbb{Z}_d \) action on \( M \) is induced by a diagonal action on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), \( \rho_d(\mathbb{Z}_d) \rightarrow \mathbb{Z}_d \), which can be extended to a free action on \( M \) if the branch locus is \( \mathbb{Z}_d \)-invariant and certain numerical conditions are satisfied.

**Proposition**

The iterated branched covers of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), branched along smooth curves of positive self-intersections which are pairwise transverse, are almost completely decomposable.
- $M_i$ are complex surfaces with ample canonical line bundle, and admit Kähler-Einstein metrics.
- Construct $M_i = M$ as a bi-cyclic branched cover:

$$M \xrightarrow{\pi_2} N \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

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- Yamabe invariant of a conformal class

\[ [g] = \{ \tilde{g} = e^f g \mid f : M \to \mathbb{R} \text{ smooth} \} : \]

\[ Y[g] = \inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}} \, d\mu_{\tilde{g}}}{\Vol(\tilde{g})^{1/2}} \]

Yamabe invariant (O. Kobayashi, Schoen):

\[ Y(M) = \sup [g] Y[g]. \]

Yamabe invariant is positive for

\[ M = a\mathbb{CP}^2 \# b\mathbb{CP}^2 \text{ (Gromov-Lawson)} \]

and it is not computed, in general.

\[ Y(S^4) = 8\sqrt{6}/\pi \text{ (Aubin)}, \]

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Yamabe invariant of Kähler surfaces

Theorem (LeBrun)

Let $M$ be a Kähler surface of Kodaira dimension 0, 1 or 2, and $X$ its minimal model, then:

$$Y(M) = -4\pi \sqrt{2c_1^2(X)}.$$
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- if $M$ is a symplectic manifold, if $b^+ = 1$ assume $c_1(X) \cdot [\omega] < 0$ ($M$ of general type), then
  $$\int_M s_g^2 d\mu_g \geq 32\pi^2 (c_1^+)^2 \quad (*)$$
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$$\int_M s_g^2 d\mu_g \geq 32\pi^2 (c_1^+)^2 $$ (*)

Moreover, equality is obtained iff $g$ is a Kähler metric of negative constant scalar curvature.
$Y(M) \leq -4\pi \sqrt{2c_1^2(X)}.$
If $M$ is a surface of general type, by starting with the Kähler-Einstein metric on the canonical model, LeBrun constructs a family of metrics which shows that the bound is optimal.

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There was a strong indication that on a symplectic manifold of general type, the Yamabe bound is optimized only when the manifold is of Kähler type.

**Theorem (S.)**

*For every point on the half-Noether line there is a simply connected, minimal, symplectic 4–manifold of general type \( M \), for which the Yamabe invariant is*

\[ Y(M) = -4\pi \sqrt{2c_1^2(M)}. \]  \hspace{1cm} (1)

*Moreover, if \( M' = M \# \ell \mathbb{C}P^2 \) then \( Y(M') = Y(M) \).*
Key ideas:

- $M$ is obtained from an elliptic surface $E(n)$ which contains a chain of $n - 3$ rational curves of self-intersection $(-n, -2, \ldots, -2)$ by doing a rational blow down surgery.
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- $Y(M) = Y_{orb}(\hat{M})$.  

Open question

Is there a minimal symplectic manifold of general type for which the Yamabe invariant does not equal $\frac{-4\pi}{\sqrt{2}c_2^1}$?
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Is there a minimal symplectic manifold of general type for which the Yamabe invariant does not equal $-4\pi \sqrt{2c_1^2}$?
Thank you!
Happy Birthday Claude!