The role of Seiberg-Witten theory in Riemannian geometry

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- *Can we classify the simply connected smooth 4-manifolds?*
  We discuss the existence of a canonical and exotic structures.
- *Does a smooth manifold admit a preferred metric?*
  We focus on Einstein metrics.
- *How can we understand the differential invariants?*
  We consider the Seiberg-Witten invariant and Yamabe invariant.

Main ingredients

(M, g) compact, oriented, smooth $4-$manifold, g a Riemannian metric.

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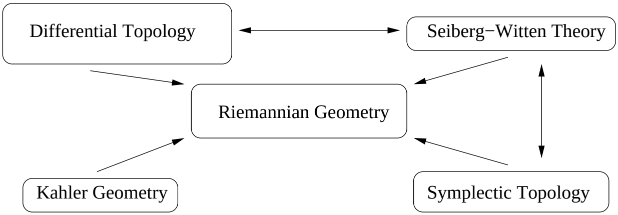

Homotopy invariants

- fundamental group: $\pi_1(M)$
- Second Stiefel-Whitney class: $w_2(M) \in H^2(M, \mathbb{Z}_2)$
- Euler Characteristic: $\chi(M) = 2 - 2b_1 + b^+ + b^-$
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- We will consider manifolds with finite fundamental group, hence $b_1 = 0$, for the rest of the talk.

Complex numerical invariants:

(M, J) (almost) complex surface, then we can consider the Chern numbers:

- $c_2(M, J) = \chi(M)$;
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Remark

In complex dimension 2, the complex numerical invariants are determined by the topological invariants.

Seiberg-Witten Theory

Given (M, g) and let $\mathbb{V}_\pm$ be the spin^c structure associated to the Hermitian line bundle $L, (c_1(L) \equiv w_2(M) \mod 2)$.

The Seiberg-Witten Equations:

$$\begin{cases} D_A\Phi = 0 \\ F_A^+ = i\sigma(\Phi) \end{cases}$$

where $\Phi \in \Gamma(\mathbb{V}_+)$, A a connection on L, F_A^+ is the self-dual part of the curvature of A, and where $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map satisfying

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

Seiberg-Witten invariant and its properties

The Seiberg-Witten Invariant, $SW_g(L)$: the number of solutions, (A, Φ), of a generic perturbation of the Seiberg-Witten monopole equations, modulo gauge transformations and counted with orientations.

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- There are large classes of manifolds for which the invariant is non-trivial: symplectic manifolds, manifolds obtained via certain gluing surgeries. (Taubes, Szabó, Morgan, etc.)
- $SW_M \neq 0$ if $M = N_1 \# N_2$ such that $SW_{N_1} \neq 0$ and $b^+(N_2) = 0$ In particular, if $SW_M(L) \neq 0$ then $SW_{M\#\overline{\mathbb{CP}^2}}(L \pm E) \neq 0$, **but** $SW_{M\#\mathbb{CP}^2}(L') = 0$ for all L'.

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- generalization of the Seiberg-Witten invariant: due to Bauer-Furuta, defines a nontrivial invariant for **certain** connected sums: $M = N_1 \# \ldots \# N_m, m = 2, 3, 4,$
  - necessary condition $SW_{N_j}(L_j) \equiv 1 \mod 2$.

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  - necessary condition $SW_{N_j}(L_j) \equiv 1 \mod 2.$
- Weitzenböck formula for the Dirac operator D_A in relation with the Seiberg-Witten equations:

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

In particular, there are no positive scalar curvature metrics on manifolds with non-trivial S-W invariant.

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- Seiberg-Witten (Bauer-Furuta) invariant of $M = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$ vanishes if $a > 1$. Lawson and Gromov showed that M supports a metric with positive scalar curvature .

Exotic structures on simply connected non-spin manifolds

Definition

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Theorem (Braungardt-Kotschick)

For every $\epsilon > 0$ there is a constant $N_\epsilon > 0$ such that every lattice point (m, n) in the first quadrant satisfying

$$n \leq (9 - \epsilon)m - N_\epsilon$$

is realized by the Chern invariants $(m, n) = (\chi_h, c_1^2)$ of infinitely many, pairwise non-diffeomorphic, non-spin, simply connected, minimal symplectic manifolds $M_{(m,n,i)}$, all of which are $\mathbb{CP}^2$-almost completely decomposable.

Geography of simply connected, minimal, symplectic 4-manifolds

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For every $\epsilon > 0$ there is a constant $N_\epsilon > 0$ such that every lattice point (a, b) satisfying the conditions

$$a \equiv 1 \mod 2$$
$$b \leq 4 + 5a$$
$$b \geq (\frac{1}{2} + \epsilon)a + N_\epsilon$$

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An exotic manifold M *is called* $S^2 \times S^2$**-almost completely decomposable** *if* $M\#(S^2 \times S^2)$ *has the canonical smooth structure.*

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Theorem (S.)

For any $\epsilon > 0$ there is a constant $N'_\epsilon > 0$ such that given any integer pair (a, b) in the first quadrant satisfying either one of the two conditions

$$b \geq (\frac{1}{2} + \epsilon)a + N'_\epsilon \text{ and } a \not\equiv 0 \mod 8$$

$$b \leq \frac{2}{1+2\epsilon}(a - N'_\epsilon) \text{ and } b \not\equiv 0 \mod 8,$$

the topological space $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ admits infinitely many, pairwise non-diffeomorphic, smooth structures, which are all $S^2 \times S^2$-almost completely decomposable.

Theorem (S.)

For any integer $d \geq 2$ and for any $\epsilon > 0$ there exists a constant $N''_\epsilon > 0$ such that for any point (a, b) in the region R_ϵ satisfying the divisibility conditions D_1 and D_2 the manifold $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ has the following properties:

- *M admits infinitely many, smooth, orientation preserving, free actions of the group $\mathbb{Z}_d$, which we denote by $\Gamma_{d,i}, i \in \mathbb{N}$,*
- *the actions $\Gamma_{d,i}$ are conjugate by homemorphisms but are not conjugate by the diffeomorphisms of M.*

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$$R_\epsilon = \{(a,b) | b \geq (\frac{1}{2} + \epsilon)a + N''_\epsilon\} \cup \{(a,b) | b \leq \frac{2}{1+2\epsilon}(a - N''_\epsilon)\}$$

$$\begin{aligned} D_1: &\quad a+1 \equiv 0 \mod d \text{ and } b+1 \equiv 0 \mod d, \\ D_2: &\quad \text{if } (a,b) \in R_{\epsilon,1} \text{ then } \tfrac{a+1}{d} \not\equiv 1 \mod 8 \\ &\quad \text{or if } (a,b) \in R_{\epsilon,2} \text{ then } \tfrac{b+1}{d} \not\equiv 1 \mod 8. \end{aligned}$$

Homotopy invariants from a Riemannian perspective

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$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{|\overset{\circ}{r}|^2}{2}\right) d\mu_g$$

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$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2}\int_M (\frac{s^2}{24} + 2|W^{\pm}|^2 - \frac{|\mathring{r}|^2}{2}) d\mu_g$$

A topological obstruction to existence of Einstein metrics

Einstein metric = constant Ricci curvature:

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Theorem (Hitchin-Thorpe Inequality)

If the smooth compact oriented 4-manifold M admits an Einstein metric g, then

$$(2\chi \pm 3\tau)(M) \geq 0,$$

with equality if (M, g) is finitely covered by a flat 4-torus T^4 or by the $K3$ surface with a hyper-Kähler metric or by the orientation-reversed $K3$ with a hyper-Kähler metric.

A differential obstruction to existence of Einstein metrics

Theorem (LeBrun)

Let X be a compact oriented 4-manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then

$$M = X \# k\overline{\mathbb{CP}^2}$$

does not admit Einstein metrics if $k \geq \frac{1}{3}(2\chi + 3\tau)(X)$.

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Key ingredient: curvature estimate:

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu \geq \frac{2}{3}(c_1^+(L))^2$$

where c_1^+ is the self-dual part of $c_1(L)$.

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LeBrun, LeBrun-Ishida, Kotshick & collaborators: Obstructions to the existence of Einstein metrics on manifolds with exotic structures.

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On $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 0, 1, \ldots, 8$, there exist Einstein metrics.

- on $\mathbb{CP}^2$ there is the Fubini-Study metric
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In general, the existence of an Einstein metric is unknown.

Dependence on the smooth structure

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Is the sign of the Einstein determined by the homeomorphism class of the manifold?

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Answer:

Catanese-LeBrun: **NO**.

Example: $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ and a deformation of the Barlow surface (complex surface of general type, with *ample* canonical line bundle).

Theorem (Rasdeaconu-S.)

On every $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 5, 6, 7, 8$ there exists a (exotic) smooth structure which admits an Einstein metric with $s < 0$ and infinitely many non-diffeomorphic smooth structures which do not admit Einstein metrics.

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- On $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 5, \ldots, 8$, we show that the exotic complex structures constructed by Park and collaborators, have ample canonical line bundle. Hence they admit a Kähler-Einstein metrics of negative scalar curvature, from the solution of the Calabi conjecture.

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- Due to the nature of the obstruction theorem, this bound can not be lowered.

Non-existence of invariant Einstein metrics

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Theorem (S.)

For any integer $d \geq 2$ *and any* $\epsilon > 0$ *there exists a* $N(\epsilon) > 0$ *such that for any integer lattice point* (n, m)*, satisfying:*

- $n > 0$
- $n \equiv 0 \mod d, m \equiv 0 \mod d$
- $n < (6 - \epsilon)m - N(\epsilon)$

there exist infinitely many free, non-equivalent smooth $\mathbb{Z}_d-$*actions on canonical smooth manifold* M *with* $c_1^2(M) = n$*,* $\chi_h(M) = m$ *such that there is no Einstein metric on* M *invariant under any of the* $\mathbb{Z}_d-$*actions.*

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there exist infinitely many free, non-equivalent smooth $\mathbb{Z}_d-$actions on canonical smooth manifold M with $c_1^2(M) = n$, $\chi_h(M) = m$ such that there is no Einstein metric on M invariant under any of the $\mathbb{Z}_d-$actions.

- The existence of Einstein metrics is topologically unobstructed: Hitchin-Thorpe inequality, $n > 0$, is satisfied.
- Admissibility condition for a free action: $d/n, d/m$

- the region $n < (6-\epsilon)m - N(\epsilon)$ is determined by the geography of simply connected, symplectic manifolds
- If we denote by $\Gamma_i, i \in \mathbb{N}$, the actions of $\mathbb{Z}_d$ on M, then the quotient manifolds M/Γ_i are homeomorphic, but pairwise non-diffeomorphic.
  - $$M/\Gamma_i = M_{(m,n,i)} \# k\overline{\mathbb{CP}^2} \# S_d$$

    where S_d is a rational homology sphere, $\pi_1(S_d) = \mathbb{Z}_d$,

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- If we denote by $\Gamma_i, i \in \mathbb{N}$, the actions of $\mathbb{Z}_d$ on M, then the quotient manifolds M/Γ_i are homeomorphic, but pairwise non-diffeomorphic.
  - $$M/\Gamma_i = M_{(m,n,i)} \# k\overline{\mathbb{CP}^2} \# S_d$$

    where S_d is a rational homology sphere, $\pi_1(S_d) = \mathbb{Z}_d$, $\widetilde{S_d} = \#(d-1)(S^2 \times S^2)$.

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Proposition (S.)

There exists an involution σ acting freely on the manifold $M = 15\mathbb{CP}^2 \# 77\overline{\mathbb{CP}^2}$, and $15\mathbb{CP}^2 \# 77\overline{\mathbb{CP}^2}$ does not admit a $\sigma-$invariant Einstein metric.

Here $n = c_1^2(M) = 2, m = \chi_h(M) = 8$.

Existence of invariant Einstein metrics

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Theorem (S.)

Given an integer $d \geq 2$, there are infinitely many compact, smooth, simply connected, non-spin manifolds $M_i, i \in \mathbb{N}$, whose topological invariants satisfy $c_1^2(M_i) = n_i > 0$ and $c_1^2(M_i) < 5\chi_h(M_i)$, and have the following properties:

- *There is at least one free, smooth, $\mathbb{Z}_d$ action on M_i,*
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$$M \xrightarrow{\pi_2} N \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

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- The $\mathbb{Z}_d$ action on M is induced by a diagonal action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ $\rho^d = 1, \rho([z_1 : z_2]) = [\rho z_1 : z_2]$, which can be extended to a free action on M if the branch locus is $\mathbb{Z}_d$-invariant and certain numerical conditions are satisfied.

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Proposition

The iterated branched covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$, branched along smooth curves of positive self-intersections which are pairwise transverse, are almost completely decomposable.

The Yamabe invariant:

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- Yamabe invariant is positive for $M = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$ (Gromov-Lawson) and it is not computed, in general.
- $Y(S^4) = 8\sqrt{6}\pi$ (Aubin), $Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$ (LeBrun).

Yamabe invariant of Kähler surfaces

Theorem (LeBrun)

Let M be a Kähler surface of Kodaira dimension $0, 1$ or 2, and X its minimal model, then:

$$Y(M) = -4\pi\sqrt{2c_1^2(X)}.$$

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  Moreover, equality is obtained iff g is a Kähler metric of negative constant scalar curvature.

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Theorem (S.)

For every point on the half-Noether line there is a simply connected, minimal, symplectic 4−manifold of general type M, for which the Yamabe invariant is

$$Y(M) = -4\pi\sqrt{2c_1^2(M)}. \tag{1}$$

Moreover, if $M' = M\#\ell\overline{\mathbb{CP}^2}$ then $Y(M') = Y(M)$.

Key ideas:

- M is obtained from an elliptic surface $E(n)$ which contains a chain of $n-3$ rational curves of self-intersection $(-n, -2, \cdots, -2)$ by doing a rational blow down surgery.

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Open question

Is there a minimal symplectic manifold of general type for which the Yamabe invariant does not equal $-4\pi\sqrt{2c_1^2}$ *?*

Thank you!

Happy Birthday Claude!