Multiplicity of solutions to the Yamabe problem

Jimmy Petean

CIMAT
Guanajuato, México

Centre de Recherches Mathématiques
Montreal, July 2016

Yamabe problem: Introduction

(M, g) closed smooth Riemannian manifold

Yamabe problem: search for metrics of constant scalar curvature in the conformal class

$$[g] = \{f.g : f : M \to \mathbb{R}_{>0}\}.$$

One has to solve the Yamabe equation:

$$-a_n \Delta_g u + s_g u = \lambda \, u^{p-1}$$

$a_n = \frac{4(n-1)}{n-2}$, s_g the scalar curvature, $p = p_n = \frac{2n}{n-2}$,

$\lambda \in \mathbb{R}$ is the scalar curvature of $u^{p-2} g$.

Yamabe equation is the Euler-Lagrange equation for the Hilbert-Einstein functional restricted to $[g]$:

$$S(h) = \frac{\int_M s_h \, dvol_h}{Vol(M,h)^{\frac{n-2}{n}}}$$

The Yamabe constant of $(M,[g])$

$$Y(M,[g]) = \inf_{h\in[g]} S(h) = \inf_f \frac{\int_M a_n \|\nabla f\|^2 + s_g f^2 dvol_g}{\left(\int_M f^p dvol_g\right)^{2/p}}$$

is always achieved H. Yamabe-N. Trudinger-T. Aubin-R. Schoen
There is always at least one (volume 1) solution of the Yamabe equation.

1. Solution is unique if $Y(M,[g]) \leq 0$.
2. Solution is unique if g is Einstein, not the round metric on the sphere (M. Obata).
3. In general multiple solutions when Y(M,[g]) >0
   Examples of nonuniqueness:
   $(\mathbf{S}^n, [g_0^n])$.
   Riemannian products with constant scalar curvature $(M \times N, [g + \delta h])$, with $\delta > 0$ small cannot be a minimizer.

Every *positive* conformal class $[g]$ can be C^0-approximated by one with any large number of distinct solutions (with high energy) (D. Pollack)

There are smooth metrics, not round, on high dimensional spheres for which the space of solutions is not compact (S. Brendle)

Consider $(S^n \times S^m, g_0^n + Tg_0^m)$, $n, m \geq 2$, T small. One can obtain solutions $u : S^n \to \mathbb{R}_{>0}$ which satisfy

$$-a_{n+m}\Delta u + \lambda u = u^{p_{n+m}-1}$$

(Q. Jin-Y. Li-H. Xu ,G. Henry-Petean). There is a sequence $T_k \to 0$, at least k solutions for $T < T_k$.

$(M \times N, g + th)$ of constant positive scalar curvature, there are infinite bifurcation instants for the family (small and large) if the pair is non-degenerate (L. L. de Lima, P. Piccione, M. Zedda).

Theorem, horizontal solutions

Let $\varphi : (M^m, g(t)) \to (N, h)$, $t \in I$, be a smooth family of harmonic Riemannian submersions. Assume that the scalar curvature $\mathbf{s}_{g(t)}$ is constant for every $t \in I$. Let $t_0 \in I$ be such that there is an eigenvalue μ_i of $-\Delta_h$ with $\mathbf{s}_{g(t_0)} = (m-1)\mu_i$ and assume that $\frac{d}{dt}\mathbf{s}_{g(t)}(t_0) \neq 0$. Then t_0 is a bifurcation instant. (N. Otoba-Petean)

In the case of Riemannian product there is no nondegeneracy condition.

Theorem, topological condition

Let (M^n, g) be any closed Riemannian manifold, $n \geq 3$, and (N, h) be a Riemannian manifold of constant positive scalar curvature. There exists $\delta_0 > 0$ such that for any $\delta < \delta_0$ there are at least $Cat(M)$ different solutions of the Yamabe equation for $g + \delta h$ on $M \times N$.
(Petean-S. Santra)
$Cat(M)$ is the minimum number of contractible open subsets needed to cover M

Model equation-solution, $q < p_n$:

$$-\Delta U + U - U^{q-1} = 0$$

There is a unique positive solution, radial. For any $\varepsilon > 0$ $U_\varepsilon(x) = U(\varepsilon^{-1}x)$ solves

$$-\varepsilon^2 \Delta u + u - u^{q-1} = 0$$

And cu solves

$$-\Delta u + u - c^{2-p} u^{q-1} = 0$$

Consider the functional $E : H^1(\mathbb{R}^n) :\to \mathbb{R}$, $q = p_{n+m} < p_n$,

$$E(f) = \int_{\mathbb{R}^n} (1/2)|\nabla f|^2 + (1/2)f^2 - (1/q)(f^+)^q \, dx,$$

and the corresponding Nehari manifold

$$N(E) := \{u \in H^1(M) - \{0\} : \int_{\mathbb{R}^n} |\nabla u|^2 + u^2 dx = \int_{\mathbb{R}^n} (u^+)^q\}.$$

U is actually the minimizer of the functional E restricted to $N(E)$. The minimum is then

$$\mathbf{m}(E) = \min\{E(u) \ : \ u \in N(E)\} = \frac{q-2}{2q} \|U\|_q^q.$$

For any $\varepsilon > 0$ let

$$E_\varepsilon(f) = \varepsilon^{-n} \int_{\mathbb{R}^n} (\varepsilon^2/2)|\nabla f|^2 + (1/2)f^2 - (1/q)(f^+)^q \, dx$$

and

$$N_\varepsilon(E) := \{u \in H^1(M) - \{0\} : \int_{\mathbb{R}^n} \varepsilon^2 |\nabla u|^2 + u^2 dx = \int_{\mathbb{R}^n} (u^+)^q\}.$$

The function $U_\varepsilon \in N_\varepsilon$, and is a minimizer of E_ε restricted to $N_\varepsilon(E)$.
Note that the minimum is equal to $\mathbf{m}(E)$.

We look for solutions in (M, g) of $(q < p_n)$

$$-a\Delta_g u + (s_g + \delta^{-1} s_h)u = u^{q-1}$$

or

$$-\delta(a/s_h)\Delta_g u + (\delta(s_g/s_h) + 1)u = u^{q-1}$$

Consider the functional $J_\epsilon : H^1(M) \to \mathbb{R}$

$$J_\epsilon(u) = \epsilon^{-n} \int_M \left(\frac{1}{2}\epsilon^2 |\nabla u|^2 + \frac{s_g\ \epsilon^2 + a}{2a} u^2 - \frac{1}{q}(u^+)^q \right) dv_g.$$

and the *Nehari manifold* N_ϵ associated to J_ϵ:

$$N_\epsilon = \{ u : \int_M \epsilon^2 |\nabla u|^2 + ((s_g/a)\epsilon^2 + 1) u^2 dvol_g = \int_M (u^+)^q dvol_g \}.$$

Critical points of J_ϵ restricted to the Nehari manifold are positive solutions of $-\varepsilon^2 \Delta u + ((s_g/a)\epsilon^2 + 1) u = u^{q-1}$.

Let
$\mathbf{m}_\epsilon = \inf_{u\in N_\epsilon} J_\epsilon(u) = \varepsilon^{-n}(1/2 - 1/q) \inf_{u\in N_\epsilon} \int (u+)^q dvol_g > 0$

and

$\Sigma_{\epsilon,a} = \{u \in N_\epsilon : J_\epsilon(u) < a\}.$

Lusternik-Schnirelmann theory

Theorem: Let J be a C^1 functional on a complete $C^{1,1}$ Banach manifold N. Assume that J is bounded below and satisfies the Palais-Smale condition. Let $J^d = \{u \in N : J(u) < d\}$. Then J has at least $Cat(J^d)$ critical points in J^d.

We will prove that for $\varepsilon > 0$, $\delta > 0$ small

$Cat(\Sigma_{\epsilon,\mathbf{m}(E)+\delta}) \geq Cat(M)$

For that we construct maps (V. Benci, G. Cerami (1991) , C. Bonanno, A. M. Micheletti (2007))

$\alpha_\varepsilon : M \to \Sigma_{\epsilon,\mathbf{m}(E)+\delta}$

$\beta_\varepsilon : \Sigma_{\varepsilon,\mathbf{m}(E)+\delta} \to M$

continuous, $\beta \circ \alpha \equiv Id_M$. This implies $Cat(\Sigma_{\epsilon,\mathbf{m}(E)+\delta}) \geq Cat(M)$

The map α

In $\mathbb{R}^n$

$\varphi_r : \mathbb{R} \to [0,1]$ such that $\varphi_r(t) = 1$ if $t \leq r$, $\varphi_r(t) = 0$ if $t \geq 2r$.

Let $\varphi_r^n : \mathbb{R}^n \to [0,1]$ be given by $\varphi_r^n(x) = \varphi_r(\|x\|)$.

Let $r_0 > 0$ be such that for any $x \in M$ the geodesic ball of radius $2r_0$ in (M,g) is strongly convex.

Let $U_{\epsilon,r} = \varphi_r^n \, U_\epsilon$, $r < r_0$.

In M

Take $r < r_0$ and for any $x \in M$ identify the geodesic ball $B(x, 2r)$ with the ball $B(0, 2r) \subset \mathbb{R}^n$, we obtain a function $U^x_{\epsilon,r} : M \to \mathbb{R}_{\geq 0}$ (supported in $B(x, 2r)$).

Let $t^x_{\epsilon,r} \in \mathbb{R}_{>0}$ be such that $t^x_{\epsilon,r} U^x_{\epsilon,r} \in N_\epsilon$. Let $\alpha_{r,\varepsilon}(x) = t^x_{\epsilon,r} U^x_{\epsilon,r}$

Theorem: Fix $\delta > 0$ and $r < r_0$. There exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ we have $\alpha_{r,\epsilon}(x) \in \Sigma_{\epsilon,\mathbf{m}(E)+\delta}$.

The map β

For a function $u \in L^1(M)$ and a positive number r the *(u,r)-concentration function* is

$$C_{u,r}(x) = \frac{\int_{B(x,r)} |u| dv_g}{\|u\|_1}$$

$C_{u,r} : M \to [0,1]$ is continuous. If $r \geq diam(M)$ then $C_{u,r} \equiv 1$.
For any $x \in M$ we have $\lim_{r\to 0} C_{u,r}(x) = 0$.
Let the *r-concentration coefficient* of u, $C_r(u)$, be the maximum of $C_{r,u}$:
$C_r(u) = \sup_{x\in M} \frac{\int_{B(x,r)} |u| dv_g}{\|u\|_1}$
Theorem: Let $r < r_0$, $\eta < 1$. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$ such that for any $\delta < \delta_0$, $\varepsilon < \varepsilon_0$ and any $u \in \Sigma_{\varepsilon,\mathbf{m}(E)+\delta}$ we have that $C_r(u^q) > \eta$.

Idea: As $\varepsilon \to 0$ $J_\varepsilon(1) \to \infty$.

The function then must have peaks.

To keep $\|\nabla u\|$ small the peaks must be concentrated in small regions.

In the small regions is like Euclidean space, then we can have only one peak.

Let $u \in L^1(M)$ be nonnegative. Consider the function continuous $P_u : M \to \mathbb{R}$

$$P_u(x) = \int_M (d(x,y))^2 u(y) dvol_g(y).$$

If r is small and the support of u is contained in $B(x,r)$ then H. Karcher, K. Grove defined the Riemannian center of mass of the function u as the unique global minimum of the function P_u

For any $\mu \in (0,1)$ let $L^1_{r,\mu}(M,g) = \{u \in L^1(M) : C_r(u) > \mu\}$.
$L^1_{r,\mu}(M,g) = L^1_{r,\mu}(M)$

For any $\eta \in (1/2, 1)$ consider the piecewise linear continuous function $\varphi_\eta : \mathbb{R} \to [0,1]$ such that $\varphi_\eta(t) = 0$ if $t \leq 1 - \eta$, $\varphi_\eta(t) = 1$ if $t \geq \eta$ and it is linear and increasing in $[1-\eta, \eta]$.

Fix $r < (1/2) r_0$ and for any $u \in L^1_{r,\eta}(M)$ let

$$\Phi_{r,\eta}(u)(x) = \varphi_\eta(C_{u,r}(x))\, u(x).$$

For any $u \in L^1_{r,\eta}(M)$ the support of $\Phi_{r,\eta}(u)$ is contained in a geodesic ball of radius $2r$ (centered at a point of maximal concentration).

For any $r < (1/2)r_0$ and $\eta > 1/2$ there exists a continuous function $\mathbf{Cm}(r,\eta) : L^1_{r,\eta}(M) \to M$, such that if $x \in M$ verifies that $C_{r,u}(x) > \eta$ then $\mathbf{Cm}(r,\eta)(u) \in B(x, 2r)$

Definition ; Any function $\mathbf{Cm}(r,\eta)(u)$ as above will be called a *(r,η)-Riemannian center of mass* of u.

Define $\beta_\varepsilon(u) = \mathbf{Cm}(r,\eta)(u^q)$

If (M, g_h) is a closed hyperbolic 3-manifold we obtain 4 different solutions of the Yamabe equation on $(M \times \mathbf{S}^2, g_h + \delta g_0^2)$ for $\delta > 0$ small. These solutions are constant on any slice $\{x\}\mathbf{S}^2$.

Moreover the solutions built in the theorem are non-constant so counting the constant solution we have 5 solutions.

The 4 solutions given by the theorem concentrate around a point and are certainly cantidates to be minimizers for the Yamabe constant. But it seems interesting to understand if for δ small enough one could get solutions concentrating around any point in M.

Bifurcation:

We have a family of Riemannian metrics g_t, where $t \in I \subset \mathbb{R}$. the scalar curvature $\mathbf{s}_{g_t}$ is constant and positive for all $t \in I$. We are interested in finding new solutions to the g_t-Yamabe equation:

$-a\Delta_{g_t} u + \mathbf{s}_{g_t}(u - u^{p-1}) = 0$

(close to 1). We look at it as an operator equation and use the Implicit Function Theorem.

Let $\lambda_i^{t_0}$ be the eigenvalues of $-\Delta_{g_t}$. It follows that if for all i

$$\lambda_i^t \neq \frac{\mathbf{s}_{g_{t_0}}}{m-1}$$

then t_0 is not a bifurcation instant for the family

Now consider a family of CSC metrics g_t on M with Riemannian submersions $\pi : (M, g_t) \to (N, g)$. Consider t_0 such that for some i, $\lambda_i^{t_0}(m-1) = \mathbf{s}_{g_{t_0}}$ and study if t_0 is actually a bifurcation instant.

We will do this by studying solutions of the Yamabe equation which are constant along the fibers, we will call such functions *horizontal*. $F \in C^{2,\alpha}(M)$ is a horizontal function if there exists $f \in C^{2,\alpha}(N)$ such that $F = f \circ \pi$.

Laplacians of g_t and g commute with the projection $\pi : (M, g_t) \to (N, g)$ of the Riemannian submersion if π is a harmonic map. So for any $f \in C^{2,\alpha}(N)$ we have that $\Delta_{g_t}(f \circ \pi) = \Delta_g(f) \circ \pi$.

The Theorem is proved by studying bifurcation for the restricted functional.

If all $\lambda_i^{t_0}$-eigenfunctions are horizontal, by uniqueness in the Implicit Function Theorem one obtains that all local solution of the Yamabe equation are horizontal (N. Otoba, Petean).

For a Riemmanian submersion with totally geodesic fibers, and the family obtained by shrinking the fibers, without restriction (R. Bettiol, P. Piccione) one obtains bifurcation instants under curvature conditions ($Ricci_F \geq (n-1)k$, $s_F < n(m-1)k$, $Ricci_{g_t} \geq (m-1)l$, $s_B \leq m(m-1)l$) which assure jump in the Morse index.

If the typical fiber is stable with respect to the Yamabe functional and t is close to 0 , then every eigenfunction is horizontal.

Thank you !!