Calculus on symplectic manifolds

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[joint work with Jan Slovák]

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de Rham complex

In 3 variables $\quad f \mapsto \mathrm{grad} f \quad \phi \mapsto \mathrm{curl}\phi \quad \psi \mapsto \mathrm{div}\psi$

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \to 0$$

Elliptic

In 4 variables

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \Lambda^4 \to 0$$

ranks $\quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

With indices

$$f \mapsto \nabla_a f$$

$$\phi_a \mapsto \nabla_{[a}\phi_{b]} \equiv \tfrac{1}{2}[\nabla_a\phi_b - \nabla_b\phi_a]$$

$$\psi_{ab} \mapsto \nabla_{[a}\psi_{bc]} \equiv \tfrac{1}{6}[\nabla_a\psi_{bc} - \nabla_b\psi_{ac} + \nabla_c\psi_{ab} - \cdots]$$

$$\theta_{abc} \mapsto \nabla_{[a}\theta_{bcd]} \equiv \tfrac{1}{24}[\nabla_a\theta_{bcd} - \nabla_b\theta_{acd} + \nabla_c\theta_{abd} - \cdots]$$

On a symplectic 4-manifold

Non-degenerate 2-form J_{ab} $\boxed{\Leftrightarrow \quad \exists\, J^{ab} \text{ s.t. } J_{ab}J^{ac} = \delta_b{}^c}$

$$\tfrac{3}{2}J^{bc}\theta_{abc} \leftarrow\!\shortmid \theta_{abc} \qquad \Lambda^1 \cong \Lambda^3 \qquad \phi_a \mapsto J_{[ab}\phi_{c]}$$

$$\tfrac{3}{8}J^{ab}J^{cd}\upsilon_{abcd} \leftarrow\!\shortmid \upsilon_{abcd} \qquad \Lambda^0 \cong \Lambda^4 \qquad f \mapsto J_{[ab}J_{cd]}f$$

2-forms split $\quad \Lambda^2 \cong \Lambda^2_\perp \oplus \Lambda^0$

$$\psi_{ab} \mapsto \underbrace{\psi_{ab} - \tfrac{1}{4}J^{cd}\psi_{cd}J_{ab}}_{J\text{-trace-free}} + \tfrac{1}{4}J^{cd}\psi_{cd}J_{ab}$$

de Rham

$$0 \to \mathbb{R} \to \Lambda^0 \to \Lambda^1 \begin{smallmatrix}\nearrow & \Lambda^2_\perp & \searrow \\ & \oplus & \\ \searrow & \Lambda^0 & \nearrow\end{smallmatrix} \Lambda^1 \to \Lambda^0 \to 0$$

Symplectic 4-manifold contd. . .

$$\begin{tikzcd}[row sep=small]
 & & \Lambda^2_\perp \arrow[dr] & & \\
\Lambda^0 \arrow[r] & \Lambda^1 \arrow[ur] \arrow[dr] & \oplus & \Lambda^1 \arrow[r] & \Lambda^0 \\
 & & \Lambda^0 \arrow[ur] & &
\end{tikzcd}$$

Smith complex (R.T. Smith 1976)

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{\text{second order operator}} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^1 \xrightarrow{d_\perp} \Lambda^0$$

Local cohomology

$$0 \to \boxed{\Lambda^0} \to \boxed{\Lambda^1} \to \Lambda^2_\perp \to \Lambda^2_\perp \to \Lambda^1 \to \Lambda^0 \to 0$$

$$\mathbb{R} \qquad\qquad \mathbb{R} = \langle [\alpha] \ \text{where} \ d\alpha = J \rangle$$

Elliptic

Else exact

On a symplectic 2n-manifold

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \rightarrow ???$$

de Rham

$$\begin{tikzcd}
 & & \Lambda^2_\perp \arrow[r] \arrow[d, phantom, "\oplus"] & \Lambda^3_\perp \arrow[d, phantom, "\oplus"] & \\
\Lambda^0 \arrow[r] & \Lambda^1 \arrow[ur] \arrow[dr] & {} \arrow[r, phantom, "\searrow"] & {} & \cdots \\
 & & \Lambda^0 \arrow[r] & \Lambda^1 &
\end{tikzcd}$$

$$d(fJ) = \boxed{\begin{matrix} 0 \\ \| \\ f\,dJ + \end{matrix}} J \wedge df$$

$$\boxed{\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp \xrightarrow{d_\perp} \cdots \xrightarrow{d_\perp} \Lambda^{n-1}_\perp \xrightarrow{d_\perp} \Lambda^n_\perp}$$

coeffective complex (T. Bouche 1990)

$$\begin{tikzcd}
\Lambda^n_\perp \arrow[r, "d_\perp"] \arrow[d, hook] & \Lambda^{n+1}_\perp \arrow[r, "d_\perp"] \arrow[d, hook] & \cdots \arrow[r, "d_\perp"] & \Lambda^{2n-2}_\perp \arrow[r, "d_\perp"] \arrow[d, hook] & \Lambda^{2n-1} \arrow[r, "d"] \arrow[d, equal] & \Lambda^{2n} \arrow[d, equal] \\
\Lambda^n \arrow[r, "d"] \arrow[d, "J\wedge_"] & \Lambda^{n+1} \arrow[r, "d"] \arrow[d, "J\wedge_"] & \cdots \arrow[r, "d"] & \Lambda^{2n-2} \arrow[r, "d"] \arrow[d, "J\wedge_"] & \Lambda^{2n-1} \arrow[r, "d"] & \Lambda^{2n} \\
\Lambda^{n+2} \arrow[r, "d"] & \Lambda^{n+3} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Lambda^{2n} & &
\end{tikzcd}$$

An elliptic complex

$\boxed{\text{NB}}$

$$\begin{array}{ccc}\Lambda^k & \xrightarrow{J\wedge J\wedge\cdots\wedge J\wedge\_} & \Lambda^{2n-k} \\ \cup & & \cup \\ \Lambda^k_\perp & \xrightarrow{\simeq} & \Lambda^{2n-k}_\perp\end{array}$$

is an <u>isomorphism</u> for $0 \leq k \leq n$

$$\begin{array}{ccccccccccccc} 0 & \to & \Lambda^0 & \to & \Lambda^1 & \to & \Lambda^2_\perp & \to & \Lambda^3_\perp & \to & \cdots & \to & \Lambda^n_\perp \\ & & & & & & & & & & & & \downarrow \\ 0 & \leftarrow & \Lambda^0 & \leftarrow & \Lambda^1 & \leftarrow & \Lambda^2_\perp & \leftarrow & \Lambda^3_\perp & \leftarrow & \cdots & \leftarrow & \Lambda^n_\perp \end{array}$$

coeffective complex

second order operator

Rumin-Seshadri

Tseng-Yau

<u>EG dim=6</u>

$$\begin{array}{ccccccccccccccc} \Lambda^0 & \to & \Lambda^1 & \to & \Lambda^2_\perp & \to & \Lambda^3_\perp & \to & \Lambda^3_\perp & \to & \Lambda^2_\perp & \to & \Lambda^1 & \to & \Lambda^0 \\ 1 & & 6 & & 14 & & 14 & & 14 & & 14 & & 6 & & 1 \end{array}$$

<u>cf</u> $1 \;\; 6 \;\; 15 \;\; 20 \;\; 15 \;\; 6 \;\; 1$ <u>de Rham</u>

Global first cohomology

Suppose $\begin{cases} M \text{ is a compact connected symplectic manifold} \\ \phi \in \ker : \Gamma(M, \wedge^1) \xrightarrow{d_\perp} \Gamma(M, \wedge^2_\perp) \end{cases}$

then

$$d\phi = fJ \quad \Rightarrow \quad 0 = df \wedge J \quad \Rightarrow \quad 0 = df \quad \Rightarrow \quad f \text{ is constant}$$

so either $\begin{cases} f \neq 0 \quad \Rightarrow \quad J = d(\phi/f) \Rightarrow \quad \int_M J \wedge \cdots \wedge J = 0 \quad ✠ \\ \text{or } f = 0 \quad \Rightarrow \quad d\phi = 0 \end{cases}$

Therefore

$$\boxed{H^1(M, \blacktriangle^\bullet) \equiv \frac{\ker : \Gamma(M, \wedge^1) \xrightarrow{d_\perp} \Gamma(M, \wedge^2_\perp)}{\operatorname{im} : \Gamma(M, \wedge^0) \xrightarrow{d} \Gamma(M, \wedge^1)} \cong H^1(M, \mathbb{R})}$$

Example: complex projective space

$$\mathbb{CP}_n = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$$

Kähler manifold — complex structure $J_a{}^b$ s.t. $J_a{}^b J_b{}^c = -\delta_a{}^c$

Riemannian metric g_{ab} (Fubini-Study)

symplectic form $J_{ab} \equiv J_a{}^c g_{bc}$

$$\mathbb{CP}_n = S^{2n+1}/S^1 = \underbrace{\left(\mathrm{Sp}(2n+2,\mathbb{R})/P\right)}_{\text{contact manifold}}/S^1$$

transverse action

Global cohomology

$$H^k(\mathbb{CP}_n, \blacktriangle^\bullet) = \begin{cases} \mathbb{R} & \text{if } k=0 \text{ or } k=2n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\|$$

$$H^k(S^{2n+1},\mathbb{R})$$

Symplectic curvature

Connection $\nabla : E \to \Lambda^1 \otimes E$

Curvature $E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E \to \Lambda^3 \otimes E \to \cdots$

complex $\Leftrightarrow$ flat

Symplectic philosophy

$\boxed{\text{replace} \quad \Lambda^1 \to \Lambda^2 \quad \text{by} \quad \Lambda^1 \to \Lambda^2_\perp}$

Symplectic curvature $E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes E \to \Lambda^3_\perp \otimes E \to \cdots$

Ψ

Symplectically flat connection $\Psi = 0$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma = 2 J_{ab} \Theta \sigma \quad \text{for } \Theta : E \to E$$

In which case we find a complex

Symplectic coupling

Symplectically flat

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\sigma = 2J_{ab}\Theta\sigma \quad \text{for } \Theta : E \to E$$

$$\begin{tikzcd}
E \arrow[r, "\nabla"] & \Lambda^1 \otimes E \arrow[r, "\nabla_\perp"] & \Lambda^2_\perp \otimes E \arrow[r, "\nabla_\perp"] & \cdots \arrow[r, "\nabla_\perp"] & \Lambda^n_\perp \otimes E \arrow[dd, "\nabla_\perp \circ \nabla_\perp + \Theta"'] \arrow[dr] & \\
 & & & & & \Lambda^{n-1}_\perp \otimes E \arrow[dl] \\
E & \Lambda^1 \otimes E \arrow[l, "\nabla_\perp"'] & \Lambda^2_\perp \otimes E \arrow[l, "\nabla_\perp"'] & \cdots \arrow[l, "\nabla_\perp"'] & \Lambda^n_\perp \otimes E \arrow[l, "\nabla_\perp"'] &
\end{tikzcd}$$

complex with local cohomology only here and here

Example: complex projective space

$$\mathbb{CP}_n \quad \begin{cases} \text{Fubini-Study connection } \nabla_a \text{ on tensors} \\ \text{symplectic form } J_{ab} \text{ s.t. } \nabla_a J_{bc} = 0 \quad \text{et cetera} \end{cases}$$

$$E = \mathbb{T} \equiv \begin{array}{c} \Lambda^0 \\ \oplus \\ \Lambda^1 \\ \oplus \\ \Lambda^0 \end{array} \qquad \nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + J_{ab}\rho + g_{ab}\sigma \\ \nabla_a \rho - J_a{}^b \mu_b \end{bmatrix}$$

[E-Goldschmidt, Zero-energy fields…, JDG 2013]

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} = 2J_{ab} \begin{bmatrix} \rho \\ J_c{}^d \mu_d \\ -\sigma \end{bmatrix}$$

symplectically flat

Fedosov structures

Symplectic structure J_{ab} and $\boxed{\text{torsion-free } \nabla_a}$ s.t. $\boxed{\nabla_a J_{bc} = 0}$

Curvature $\boxed{(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = R_{ab}{}^c{}_d X^d}$

$$\boxed{R_{ab}{}^c{}_d = -R_{ba}{}^c{}_d \qquad R_{[ab}{}^c{}_{d]} = 0 \qquad R_{ab}{}^c{}_d J_{ce} = R_{ab}{}^c{}_e J_{cd}}$$

Decompose $R_{ab}{}^c{}_e J_{cd} \quad \ydiagram{3,1} = \ydiagram{3,1}_{\perp} \oplus \ydiagram{2} \quad \rightsquigarrow \quad V_{ab}{}^c{}_d \oplus \Phi_{ab}$

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + J_{ab}\rho + \Phi_{ab}\sigma \\ \nabla_a \rho - \Phi_{ab} J^{bc} \mu_c + S_a \sigma \end{bmatrix}$$

tractor connection

where $\boxed{\Phi_{ab} \equiv \frac{1}{2(n+1)} R_{ab}}$ and $\boxed{S_a \equiv \frac{1}{2n+1} J^{bc} \nabla_c \Phi_{ab}}$

Tractor curvature

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix}\sigma\\ \mu_d\\ \rho\end{bmatrix} = \begin{bmatrix}0\\ -V_{ab}{}^c{}_d\mu_c + Y_{abd}\sigma\\ J^{cd}\big(Y_{abd}\mu_c + \tfrac{1}{2n}(\nabla_d Y_{abc} + V_{ab}{}^e{}_c\Phi_{de})\sigma\big)\end{bmatrix}$$

$$+\,2J_{ab}\begin{bmatrix}\rho\\ \Phi_{cd}J^{ce}\mu_e - S_d\sigma\\ J^{cd}\big(S_c\mu_d + \tfrac{1}{2n}(\nabla_c S_d - J^{ef}\Phi_{ce}\Phi_{df})\sigma\big)\end{bmatrix}$$

where $Y_{abd} \equiv \frac{1}{2n+1}\nabla_c V_{ab}{}^c{}_d$.

Therefore

$$\boxed{\text{the tractor connection is symplecticaly flat} \Leftrightarrow V_{ab}{}^c{}_d = 0}$$

On $\mathbb{CP}_n$: $\quad V_{ab}{}^c{}_d = 0 \qquad \Phi_{ab} = g_{ab} \qquad S_a = 0$

Consequences on $\mathbb{CP}_n$

Kähler $+$ $V_{ab}{}^c{}_d = 0 \Rightarrow$ Constant holomorphic sectional curvature $\Rightarrow \mathbb{CP}_n$ or ...

$$\nabla : \Lambda^2_\perp \mathbb{T} \to \Lambda^1 \otimes \Lambda^2_\perp \mathbb{T}$$

skew

$$\begin{bmatrix} X_b \\ K_{bc} \\ \vdots \end{bmatrix} \xmapsto{\nabla_a} \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \vdots \end{bmatrix} \leftrightsquigarrow X_b \mapsto \nabla_{(a} X_{b)}$$

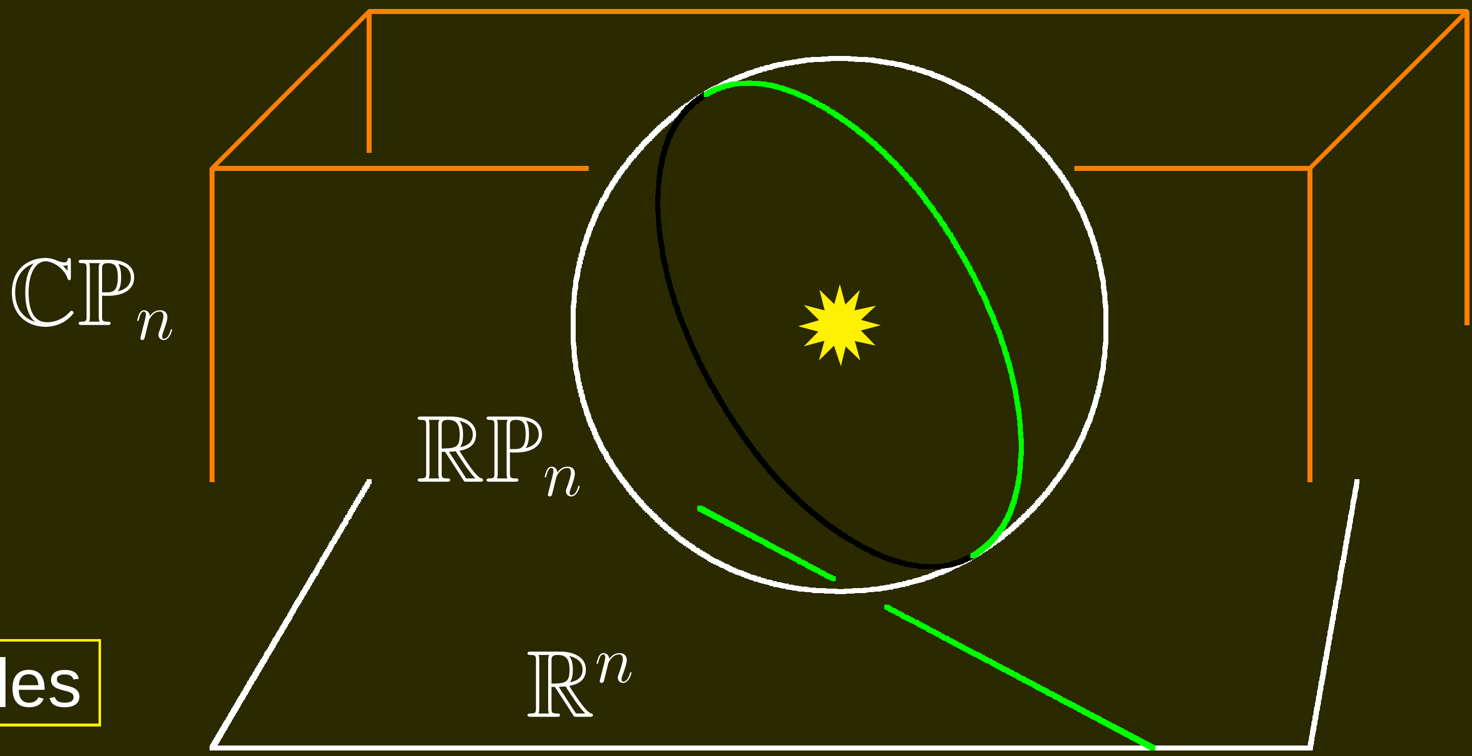
Killing operator

$\rightsquigarrow$ Integrability conditions for range of Killing operator !

$$h_{ab} = \nabla_{(a} X_{b)} \Leftrightarrow \boxed{\pi_\perp(\nabla_{(a}\nabla_{c)} h_{bd} + g_{ac} h_{bd}) = 0}$$

$\perp =$ trace free part w.r.t. J_{ab}, e.g. $\boxed{\boxplus = \boxplus_\perp \oplus \boxminus_\perp \oplus \mathbb{R}}$

Integral geometry on $\mathbb{CP}_n$



Thales

Fourier

Radon

Funk

Blaschke

LeBrun-Mason

Penrose

HAPPY BIRTHDAY CLAUDE!